# A Multiobjective Evolutionary Algorithm
# The Study Cases


To Thanh Binh
Institute for Automation and Communication (**ifak**),
Barleben, Germany



**Abstract**

In the few last years, among other tools a multiobjective evolutionary algorithm (MOBEA) for successfully solving many multicriteria optimization problems (MOPs) was proposed. However, there is a lack of the systematically testing our approach with other benchmark MOPs that may cause the algorithm very difficult to achieve its performance: robustness of the convergence to the true pareto-optimal surface, uniform distribution of the population on it. In this work after briefly discussing a concept for our approach we illustrate its effectiveness for solving some difficult MOPs and propose some basic way to improve it in the future.


## 1 MOPs and their challenges

Our MOBEA system is implemented for solving the multiobjective optimization problems with linear and nonlinear constraints as follow:

$$\min_{\vec{x}} \ \vec{f}(\vec{x}) = \min_{\vec{x}} \ (f_1(\vec{x}), f_2(\vec{x}), \cdots, f_{\mathrm{N}}(\vec{x}))$$

where $\vec{x} = (x_1, x_2, \cdots, x_{\mathrm{n}})^T \in \mathcal{F} \subseteq \mathcal{S} \subseteq \mathcal{R}^{\mathrm{n}}$.
$N$ and $n$ are the number of objective functions and the number of variables, respectively.
The *search space* $\mathcal{S}$ is a region of the $n$-dimensional space $\mathcal{R}^n$, for example, a $n$-dimensional rectangle defined by lower and upper bounds of variables:

$$x_{\mathrm{i}}^{(\mathrm{lower})} \leq x_{\mathrm{i}} \leq x_{\mathrm{i}}^{(\mathrm{upper})}, \ \forall i = \overline{1,n}$$

whereas the *feasible region* $\mathcal{F} \subseteq \mathcal{S}$ is defined by a set of $m$ additional linear and nonlinear constraints ($m \geq 0$)[1]:

$$g_{\mathrm{j}}(\vec{x}) \quad \leq \quad 0, \ \forall j = \overline{1,m}$$

The solution of the multicriteria optimization problem is to find all feasible trade-offs among the multiple, often conflicting objectives, known as a set $\mathcal{P}_{\mathrm{x}}$ of *global feasible pareto-optimal solutions* ($\mathcal{P}_{\mathrm{x}} \subseteq \mathcal{F}$) in the variable space [4, 1, 3]. Because the set of pareto-optimal solutions can be easily recognized in the objective function space using the definition of the pareto-optimality [4] it is reasonable to discover and represent this set in the objective function space.

According to the discussion in [3] there are some challenging problems that multiobjective evolutionary algorithms have to be faced with:

- Like single-objective optimization problems, some (multi-modal) multi-objective optimization problems may have both *local* and *global* feasible pareto-optimal solutions. A MOBEA should have the possibility to overcome the *local* solutions and robustly to find the *global* ones.

[1] Here only constraints in terms of inequations are considered, because constraints in terms of equalities can easily be converted into that of inequations

- Different to single-objective optimization, which has one single optimal solution, a multi-objective optimization problem usually contains a set of pareto-optimal or non-dominated solutions. The size and the shape of the pareto-optimal fronts often depend on the number of objective functions, properties of objective functions and interactions among the individual objective functions. A pareto-optimal front may be a point (when all objectives are cooperating) or a multidimensional hyper-surface (when objectives are conflicting to each other), while as the result MOBEAs provide a finite number of solutions (the population size). For this reason, in the solving of MOPs the MOBEAs should not only make the current population to evolve towards the global feasible pareto-optimal region, but also maintain the population diversity in the current non-dominated front.
- A pareto-optimal front may be convex or non-convex, continuous or discontinuous.
- The presence of the constraints may cause for the MOBEAs further difficulties in converging to the true feasible pareto-optimal region and in maintaining a diverse pareto-optimal solutions.

In this paper, we attempt to give a brief survey about our MOBEA system in the next section and to contruct some difficult optimization problems corresponding to the above mentioned challenges of MOPs for testing our system. Finally, the results of tests are devoted to analyse the performances and to improve this system in the future.

# 2 Overview of the MOBEA system

In order to make our MOBEA possible to solve challenging MOPs mentioned above, the following algorithms are implemented in this system [2]:

- we use the $\mathcal{C}$-Fitness (degree of (in)feasibility) for effectively searching for the feasible region.
- Some operators on the $\mathcal{C}$-Fitness are designed both to make the diversity of the infeasible population and to maintain a mixed population (containing infeasible and feasible individuals) in every generation. It is advisable for the optimization cases, where the feasible region is concave so that it is very difficult to generate a feasible individual from the feasible parent using normal evolutionary mechanisms.
- For maintaining of the population diversity on the current non-dominated front,a special preselection method is implemented.

# 3 The study cases

Keeping the challenging problems of MOPs in mind, and based on the idea proposed recently by Prof. Deb [3] we try to extent test problems for the cases with linear/nonlinear constraints and to test our MOBEA system with them in this section. The use of constrained MOPs allows us to study how our methodology for handling the constraints works correctly.

For all test cases we use the configuration as belows:

- The population size is 30. We choose a small population size because it enables us to show and to learn how the MOBEA can explore other region of the pareto-optimal set and distribute a population on it.
- The true pareto-optimal front is plotted by a thin solid line and the current one is marked by '*'.

## 3.1 Test case 1

The problem is minimizing $(f_1(\vec{x}), f_2(\vec{x}))$:

$$\begin{aligned} f_1(x_1, x_2) &= 4x_1^2 + 4x_2^2 \\ f_2(x_1, x_2) &= (x_1 - 5)^2 + (x_2 - 5)^2 \end{aligned}$$

with the starting point at [-5, 10]. The search space is defined $-5 \leq \vec{x} \leq 10$.

This problem has two interesting properties:

- The true pareto-optimal front is convex.
- Two objectives are in conflict so that a small improvement with respect to the second objective can only be achieved with a big change of the first one into the worse direction. This problem can often be met in the real-world problems, where the cost on the implementation, for example, has to be rapidly increased for having a small improvement of the system performances. The MOBEAs should have the possibility to good approximate the true pareto-optimal curve by the current population, instead of the uniformly distributing the population on it (see the figure 1).

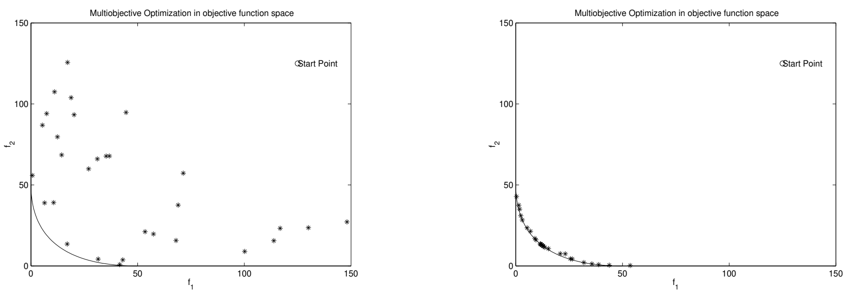


Figure 1: A population at gen. 0 (on the left) and at gen. 40 (on the right)

### 3.2 Test case 2

The problem is minimizing $(f_1(\vec{x}), f_2(\vec{x}))$ [6]:

$$
\begin{aligned}
f_1(x_1, x_2) &= (x_1 - 2)^2 + (x_2 - 1)^2 + 2 \\
f_2(x_1, x_2) &= 9x_1 - (x_2 - 1)^2
\end{aligned}
$$

subject to non-linear constraints:

$$
\begin{aligned}
x_1^2 + x_2^2 - 225 &\leq 0 \\
x_1 - 3x_2 + 10 &\leq 0
\end{aligned}
$$

and bounds: $-20 \leq x_i \leq 20$, $\forall i = 1, 2$. In this case the feasible region is convex and can be easily found after some first generations. The evolution process can be shown in the figure 2.

### 3.3 Test case 3

The problem is based on the minimization of the two competing objectives [?]:

$$
\begin{aligned}
f_1(\vec{x}) &= 1 - exp(-\sum_{i=1}^{n}(x_i - \frac{1}{\sqrt{n}})^2) \\
f_2(\vec{x}) &= 1 - exp(-\sum_{i=1}^{n}(x_i + \frac{1}{\sqrt{n}})^2)
\end{aligned}
$$

The pareto-optimal set is concave (in the objective function space) and corresponds to all points on the line defined by

$$
(x_1 = x_2 = \cdots = x_n) \ \wedge \ (-\frac{1}{\sqrt{n}} \leq x_i \leq \frac{1}{\sqrt{n}})
$$

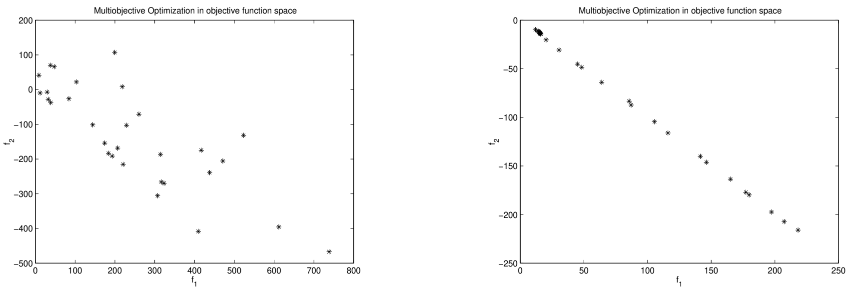


Figure 2: A population at gen. 0 (on the left) and at gen. 50 (on the right)

in the parameter space. The convergence to the pareto-optimal set and the maintenance of the diversity of the current population on it become more difficult as the number of variables increases while maintaining the population size fixed. Our experiment showed that the population evolves more slowly towards the true pareto-optimal set for the cases $n \geq 10$.

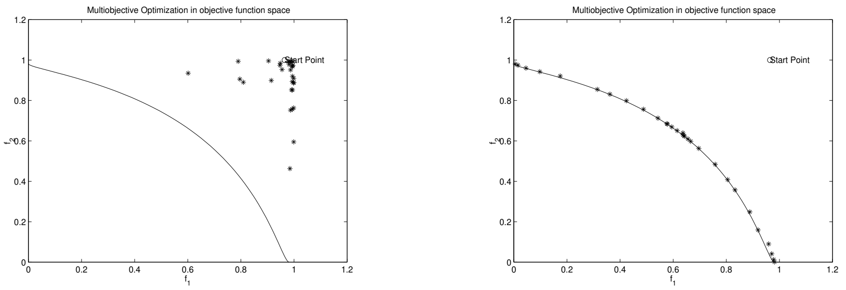


Figure 3: A population at gen. 0 (on the left) and at gen. 1000 (on the right) ($n = 8$)

### 3.4 Test case 4

The problem is maximizing $(f_1(\vec{x}), f_2(\vec{x}))$ [5]:

$$\begin{aligned} f_1(x_1, x_2) &= -x_1^2 + x_2 \\ f_2(x_1, x_2) &= 0.5x_1 + x_2 \ + \ 1 \end{aligned}$$

subject to linear constraints:

$$\begin{aligned} \frac{x_1}{6} + x_2 - 6.5 &\leq 0 \\ 0.5x_1 + x_2 - 7.5 &\leq 0 \\ 5x_1 + x_2 - 30 &\leq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

The evolution process can be shown in the figure 4.

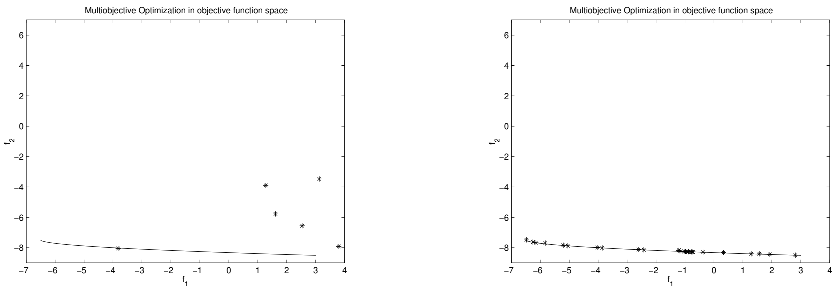


Figure 4: A population at gen. 0 (on the left) and at gen. 100 (on the right)

### 3.5 Test case 5

The specific of this problem is the multi-modality of the objective function [3]:

$$\begin{aligned} f_1(x_1, x_2) &= x_1 \\ f_2(x_1, x_2) &= \frac{g(x_2)}{x_1} \end{aligned}$$

where

$$g_1(x_2) = 2 - exp\{-(\frac{x_2 - 0.2}{0.004})^2\} - 0.8exp(-(\frac{x_2 - 0.6}{0.4})^2\}$$

is multi-modal with one local $x_2^{(local)} \approx 0.6$ and one global $x_2^{(global)} \approx 0.2$ minimum and the search space is defined by $0.1 \leq x_1 \leq 1$ and $0 \leq x_2 \leq 1$. The problem has, therefore, also local and global pareto-optimal solutions corresonding to solutions $(x_1, x_2^{(local)})$ and $(x_1, x_2^{(global)}) \forall x_1 \in [0.1, 1]$, respectively. In the function space the pareto-optimal curve can be discribed by a hyperbola $f_1 f_2 = c$, where $c = g(x_2^{(local)}) \approx 1.2$ for the local solution and $c = g(x_2^{(global)}) \approx 0.7057$ for the global one. In our experiment, all individuals of the current population get attracted to the global pareto-optimal front after the 50th generation and will be uniformly distributed on it after the 200th generation for all runs (see figure 5).

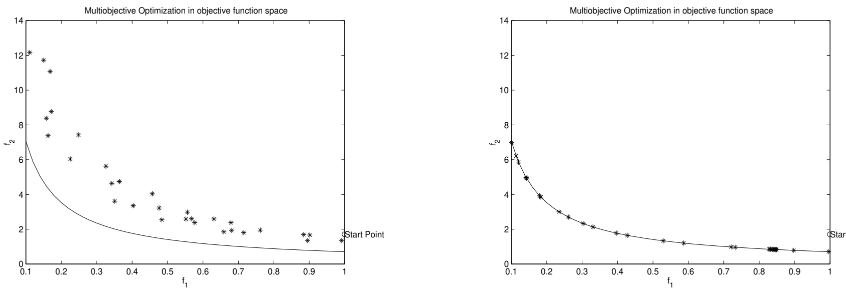


Figure 5: Test 5: A population at gen. 0 (on the left) and at gen. 200 (on the right)

### 3.6 Test case 6

Here we would like to construct a more difficult test problem, where the function $g(\vec{x})$ has only one global minimum in a small and relative flat valley so that it also makes difficulty to find the global minimum of $g(\vec{x})$ even using the scalar optimization. As an example, we choose the banana function:

$$g_3(x_4) \quad = \quad 100(x_4 - x_3^2)^2 + (x_3 - 1)^2 + 2$$

for a multiobjective optimization problem:

$$\begin{aligned} f_1(\vec{x}) &= \sqrt{x_1^2 + x_2^2 + 1} \\ f_2(\vec{x}) &= \frac{g(x_3, x_4)}{f_1(\vec{x}} \end{aligned}$$

where $\vec{x} = (x_1, x_2, x_3, x_4)$. The evolution process can be shown in the figure 6.

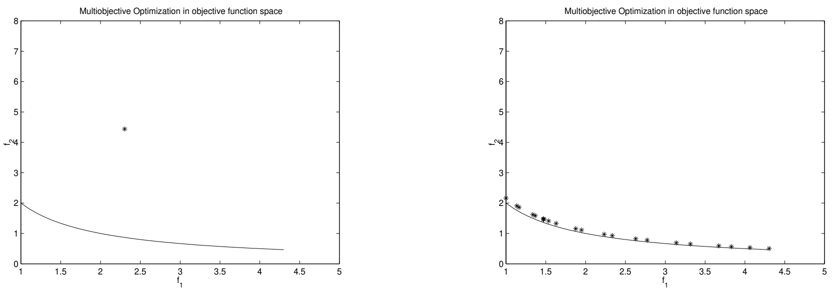


Figure 6: Test 6: A population at gen. 0 (on the left) and at gen. 200 (on the right)

### 3.7 Test case 7

Unlike the Test case 5, where both the local and global pareto-optimal front are convex, it is meaningfull here to study how the EAs to adopt to a different kind of the pareto-optimal front while evolving from local (concave front) to global (convex front) pareto-optimal front. The problem can be described as the minimizing of $\vec{f}(\vec{x})$ in the search region defined by $0 \leq x_i \leq 1,\ \forall i = \overline{1,2}$:

$$\begin{aligned} f_1(\vec{x}) &= 4x_1 \\ f_2(\vec{x}) &= 1 - (\frac{f_1}{\beta g})^\alpha \end{aligned}$$

where

$$g(x_2) = \begin{cases} 4 - 3exp(-(\frac{x_2-0.2}{0.02})^2), & \text{if } 0 \leq x_2 \leq 0.4 \\ 4 - 2exp(-(\frac{x_2-0.7}{0.2})^2), & \text{if } 0.4 \leq x_2 \leq 1 \end{cases} \qquad \alpha = 0.25 + 3.75\frac{g(x_2) - g^{**}}{g^* - g^{**}}$$

($g^*$ and $g^{**}$ are the weekest locally optimal and the globally optimal function value of $g$, respectively) [3]. An analyse in [3] showed the difficulty in the finding the global pareto-optimal solutions. But it is not in the case with our experiment. For all runs, the population after generation 30 is shown to have found solutions in the global pareto-optimal front and relativ uniformly distributed on this front after generation 50 (see Fig. 7).

### 3.8 Test case 8

In this case, a biobjective minimization problem of

$$\begin{aligned} f_1(x_1, x_2) &= x_1 + x_2 \\ f_2(x_1, x_2) &= 1 - exp(-4x_1)sin^4(5\pi x_1) \end{aligned}$$

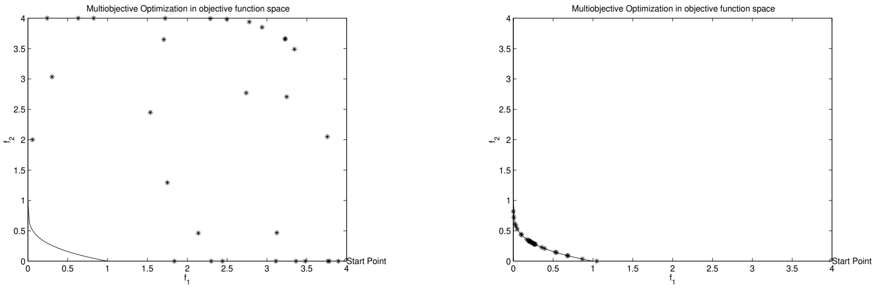


Figure 7: A population at gen. 0 (on the left) and at gen. 50 (on the right)

in the region $0 \leq x_i \leq 1, \forall i = \overline{1,2}$ is considered. This problem is used for the comparing the performance of the evolutionary algorithms with parameter-space niching and with function-space niching. A difficuty to solve it is the finding solutions with a good diversity both in the function space and in the parameter space. Because our system is based on the function-space niching, the final solutions should be uniformly distributed on the global pareto-optimal front of the function space, but not in the parameter space. However, if the performance in both spaces is simultanously monitored, it is very easy to get a good trade-off between them in a generation of the evolution process (see Fig. 8).

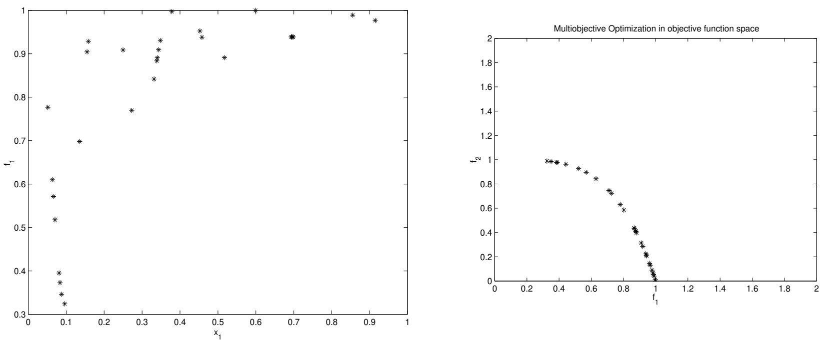


Figure 8: A pareto-optimal front in the parameter space (on the left) and in the function space (on the right) at gen. 80

## 3.9 Test case 9

This test shows that the MOBEA system has also the posibility to find all subregions of a discontinuous pareto-optimal front by minimizing:

$$\begin{aligned} f_1(\vec{x}) &= x_1 \\ f_2(\vec{x}) &= g(\vec{x})h(\vec{x}) \end{aligned}$$

where

$$\begin{aligned} g(\vec{x}) &= 1 + 10x_2; \\ h(\vec{x}) &= 1 - (\frac{f_1}{g})^2 - \frac{f_1}{g} sin(8\pi f_1). \end{aligned}$$

The evolution process can be shown in the figure 9.

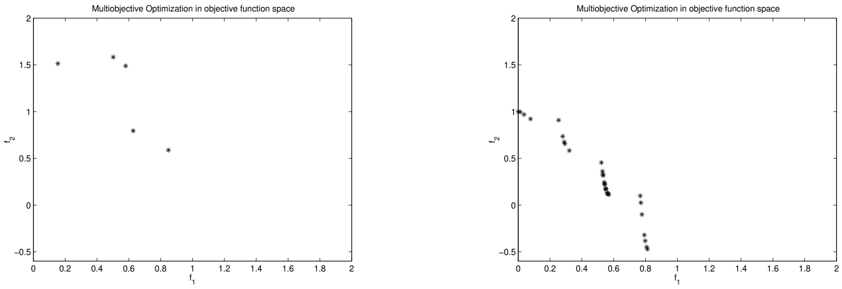


Figure 9: A population at gen. 0 (on the left) and at gen. 100 (on the right)

# 4 Conclusions

In this paper, the performance of our MOBEA is successfully tested for solving some very difficult multiobjective optimization problems. For all runs, the MOBEA was showed to be able to make the population quicky to evolve towards the feasible region and then towards the global pareto-optimal front.

# References

[1] T. Binh and U. Korn. Multicriteria control system design using an intelligent evolution strategy. *Proc. of Conf. for Control of Industrial Systems (CIS97), France*, 2:242–247, 1997.

[2] T. Binh and U. Korn. Multiobjective Evolution Strategy with Linear and Nonlinear Constraints. *Proc. of the* $15^{th}$ *IMACS World Congress on Scientific Computation, Modelling and Applied Mathematics*, pages 357–362, 1997.

[3] K. Deb. Multiobjective genetic algorithms: Problem difficulties and construction of test problems. Technical report, Indian Institute of Technology Kanpur, 1998.

[4] D. Goldberg. *Genetic algorithms in Search, Optimization, and Machine Learning*. Addision–Wesley Publishing Company, Inc., New York, England, Bonn, Tokyo, 1. edition, 1989.

[5] Hajime, Y. Yabumoto, N. Mori, and Yoshikazu. Multiobjective optimization by means of the thermodynamical genetic algorithm. *The* $4^{th}$ *Int. Conf. on Parallel Problem Solving from Nature*, pages 504–512, 1996.

[6] A. Osyczka and H. Tamura. Pareto set distribution method for multicriteria optimization using genetic algorithm. *The* $2^{nd}$ *International Conference Mendel96 in Brno, Juni 1996*, pages 97–102, 1996.