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A Minimal Cost Hybrid Strategy for Pareto Optimal Front Approximation

Research paper

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Abstract. A strategy is proposed for coarse grained Pareto Optimal Front approximation. It is devoted to industrial design optimization problems when the number of objective function calls that can be afforded in a practical time is much lower than the number required for convergence of available and powerful MOEAs. An hybrid evolutionary-deterministic and global-local search is applied on a movable preference function derived from L norm in objective domain. Both convergence and diversity of solution is tested on several analytical functions.

1. Introduction

Thanks to a decade of research work after the early methods, evolutionary multiobjective optimization is now a mature computational research area [2]. Several evolutionary methods are available ensuring full convergence toward the Pareto Optimal Front (POF) both in terms of precision and diversity of solutions (e.g. SPEA2, NSGAII). These methods have been widely and deeply tested and compared on many different test functions and some convergence measuring criteria are available, being specifically developed for multiobjective optimization problems [3, 4, 5]. Nevertheless the application of such a wide variety of methods to multiobjective optimization problems arising from industrial design (we refer to electromagnetic devices shape design [6] and fuzzy controllers design), is still not fully straightforward due to the computational cost of objective function evaluation (being often non-linear or coupled FEM in the first case and a long time-domain full-system simulation in the second case). Three alternative (to available MOEAs) approaches have been introduced, being specifically devoted to the reduction of objective function calls. They are useful and meaningful when on one hand the number of objective function calls that can be afforded from the point of view of an industrially practical computational cost is much smaller than the threshold number required for convergence of available powerful MOEAs, but on the other hand the designer nevertheless wants the few affordable solutions to be convergent towards the POF and to be diverse each other. This means that solutions are to be distributed all along the POF. The main feature of such approaches can be summarized as follows:

- Build specific MOEAs for tiny populations.
- Adapt Generalised Response Surfaces Methods (GRSM), a well established technique for single-objective optimization, to POF approximation.
- Reconsider a particular preference function method with hybrid global-evolutionary and local-deterministic search in an innovative way.

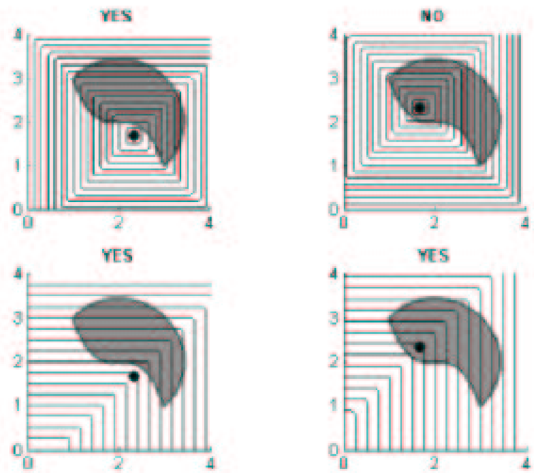

Figure 1. Contour lines on objective space of $\widetilde{f}_\infty$ (first two plots) and $\widetilde{f}$ (second two plots) preference function for two different P_i locations ( ) on a schematic objective domain search space Ù$_O$

Regarding the first approach, a non-dominated sorting based MOEA for tiny population (NSESA) was developed and applied to several electromagnetic shape design optimization problems [7, 8]. Regarding the second approach a preliminary study on POF approximation from NN Interpolated objective functions (both analytical and real-life) is reported in [9], but work is still in progress in order to build and study an iterative optimize-update interpolation procedure being a real extension to multiobjective optimization problems of GRSM methods that are classical in single-objective optimization [10, 11]. This paper is devoted to the third approach. After a general description of Multi-objective Optimization Problems (MOP) defining used terminology in section 1, the strategy is described and several results on test function are shown.

```
BEGIN
Compute random starting individuals.
FOR I=1:M
          minimize f_i with (1+1)ES + GDA or NMA
END
compute U and M̃
FOR J=1:n_s
          compute P_j from P_{k=1:j-1} and S_{k=1:j-1}
          minimize f_j with (1+1)ES + GDA or NMA
          obtain S_j
END
END
```

Figure 2. Pseudo-code of the proposed strategy with iterative choice of centers P_i, n_s being the number of desired solutions

2. Multi-objective optimization problems and related concepts

In order to define the terminology we'll use throughout the paper we give a concise formulation of multiobjective optimization problem (MOP) and some related concepts. For a detailed and rigorous mathematical theory of MOP and for theorems and proofs we refer to [12] and [13]. The following nonlinear constrained MOP will be considered throughout the paper assuming, without loss of generality, that all objective are to be minimized:

$$\begin{cases} \min_{\mathbf{x}\in R^N} & \mathbf{f} = \{f_1(\mathbf{x}),\ldots,f_M(\mathbf{x})\} \\ \textit{subject to} & \mathbf{g}(\mathbf{x}) \le 0 \\ & \mathbf{h}(\mathbf{x}) = 0 \end{cases} \tag{1}$$

Problem 1 give rice to the following subspaces known as design domain and objective domain search spaces respectively:

$$\text{Ù} : \{\mathbf{x} \in \mathrm{R}^N \textit{ s.t. } \mathbf{g}(\mathbf{x}) \le 0 \textit{ and } \mathbf{h}(\mathbf{x}) = 0\}$$

$$\text{Ù}_O : \{\mathbf{f}(\mathbf{x}) \in \mathrm{R}^M \textit{ s.t. } \mathbf{x} \in \text{Ù}\} \tag{2}$$

Ù_O being the image of Ù through function **f**. In order problem 1 to be non-trivial the following condition is to be imposed:

$$\exists\, \mathbf{x}_U \in \Omega \textit{ s.t. } f_i(\mathbf{x}_U) = \min_{x\in\Omega} f_i(\mathbf{x})\ \forall i = 1 : \mathrm{M} \tag{3}$$

that is a real contrast among objectives has to exists and no points in - minimize at the same time all objectives (no cooperative objectives). We consider the following very common, though non unique, (see Nash optimum definition and its applications in [1]) definition of solutions (Pareto-optimal solutions) for problem 1:

Def 2.1 ***x**** *$\in \Omega$ is Pareto-optimal (PO) if $\exists\, \mathbf{x} \in \Omega$ s.t. $f_i(\mathbf{x}) \le f_i(\mathbf{x}^*)$ $\forall i = 1 : M$ and $f_j(\mathbf{x}) < f_j(\mathbf{x}^*)$ for at least one $j \in [1 : M]$.*

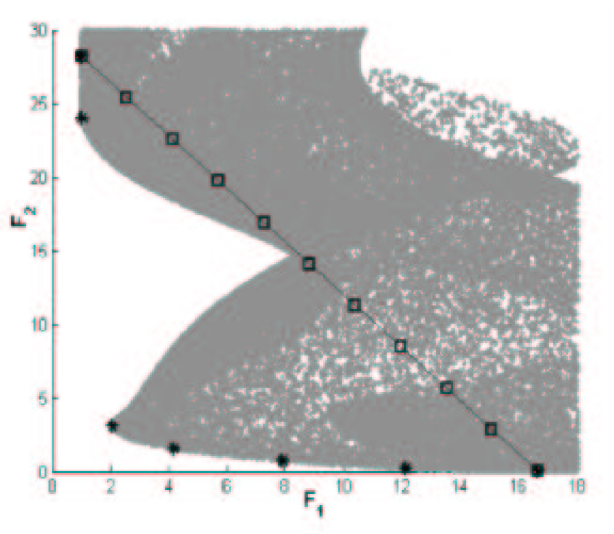

Figure 3. POL test case: results in objectives space of a run with 11 centers ( ),
6 diverse solution (•) over an exhaustive sampling (gray dots)

As a consequence of definition 2.1 the number of solution for problem 1 is infinite and we call Pareto-optimal set (POS) and Pareto-optimal front (POF) the following two subspaces respectively:

$$\begin{aligned} POS &= \{\mathbf{x}^* \in \Omega \; s.t. \, \tilde{\mathbf{x}} \; is \; PO\} \\ POF &= \{\mathbf{f}(\mathbf{x}^*) \in \Omega_O \; s.t. \, \mathbf{x}^* \in POS\} \end{aligned} \tag{4}$$

The following two points in the objective domain, giving some very preliminary information about Ω_O and known as Utopia and Distopia point respectively, can be evaluated:

$$\mathbf{U} = \left[\min_{\mathbf{x} \in \Omega} f_i\right] \quad i = 1 : M \tag{5}$$

$$\mathbf{D} = \left[\max_{\mathbf{x} \in \Omega} f_i\right] \quad i = 1 : M \tag{6}$$

As a consequence of statement 3 the inverse image of $\mathbf{U}$ does not belong to Ω and $\mathbf{U}$ does no belong to Ω_O.

Some primary information on the POF may be obtained from the evaluation of the following matrix

$$[\tilde{\mathbf{M}}] = \begin{cases} U_i & i = j, \\ f_j(\tilde{\mathbf{x}}) \; when \; f_i(\tilde{\mathbf{x}}) = U_i & otherwise. \end{cases} \tag{7}$$

when matrix $[\tilde{\mathbf{M}}]$ is computed the Nadir point R can be computed as follows:

$$R_i = \max_{j=1:M} [\tilde{M}_{i,j}] \tag{8}$$

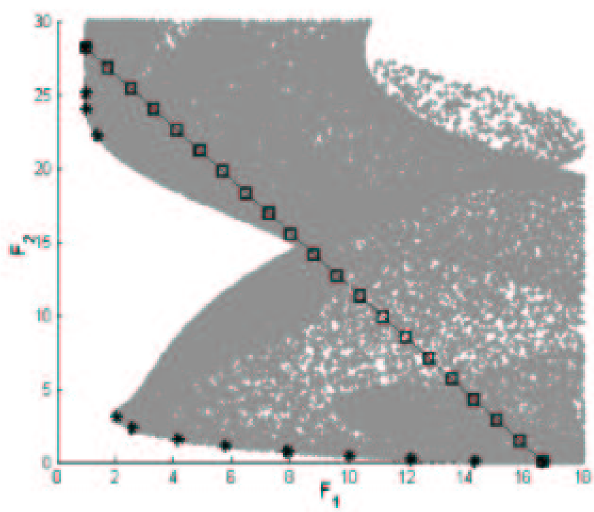

Figure 4. POL test case: results in objectives space of a run with 21 centers ( ), 13 diverse solution (•) over an exhaustive sampling (gray dots)

As will be seen in next section the Utopia point computation is essential and preliminary for implementing the proposed strategy. From a practical point of view we remark that the computation of M x M matrix $[\widetilde{\mathbf{M}}]$ and thus **R** and **U**, only requires M single-objective optimization.

3. Proposed hybrid strategy

Let us consider the following L^{∞}-norm preference function:

$$\widetilde{f}_{\infty} = \max_{i+1:M}\left[\frac{c_i\left|f_i - P_i\right|}{N_i}\right] \quad \sum_{i=1}^{M} c_i = 1 \tag{9}$$

where N_i are normalization values and P_i will be called search centers and c_i weights. Convergence of such a formulation toward the POF is assured by theorems (see [12]) also in case of non-convex problems. The degrees of freedom in formulation 9 affecting the location of optimal solution on the POF are weights c_i and P_i. As it is well known (see [13]) a good POF approximation requires both good convergence and good diversity of solutions. From the point of view of diversity, the strategy consist on choosing a suitable distribution of points P_i in order to have a good equi-spacing of solution on the POF; the values of weights c_i are fixed. In order this preference function to be effective in converging towards the POF for all possible choices of P_i (internal point of the objective domain search space Ω_O or external), the sign in the L^{∞}-distance is to be considered, that is the modulus in 9 is to be removed. This can be easily understood looking at figure 1 (schematic case) where two different P_i locations are considered. Contour lines of preference function $\widetilde{f}_{\infty}$ are plotted in the two upper cases while in the lower two cases preference function $\widetilde{f}_{\infty}$ without modulus is plotted; as can be seen the L^{∞} norm preference function works only when the center P_i is outside Ω_O; if P_i is internal to Ω_O the search algorithm would converge towards P_i and not towards the POF.

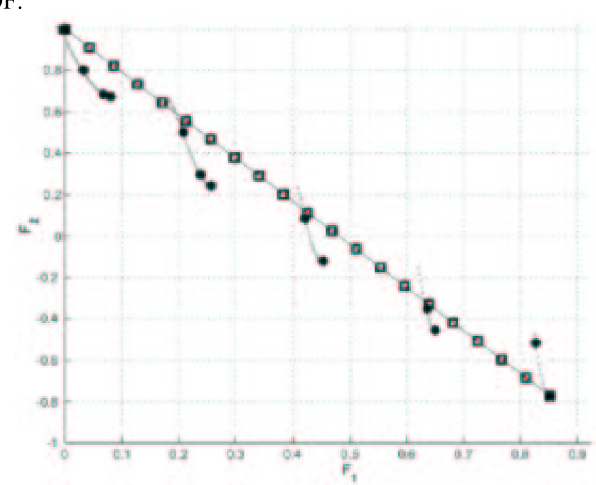

Figure 5. DTZ3 test case: results in the objective space of a run with 21 centers ( ), 13 diverse solution (•) over the analytical POF (gray dots) with linear P_i 's choice

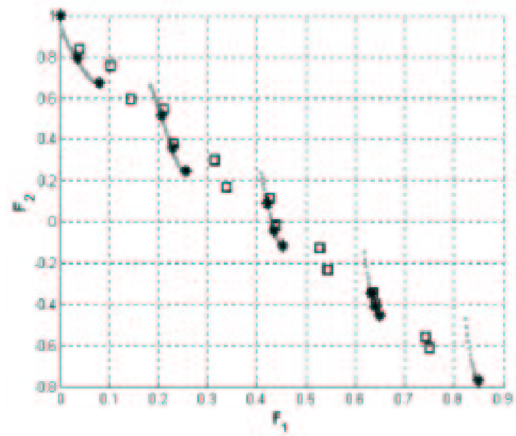

Figure 6. DTZ3 test case: results in the objective space of a run with 17 centers ( ), 13 diverse solution (•) over the analytical POF (gray dots) with iterative P_i 's choice

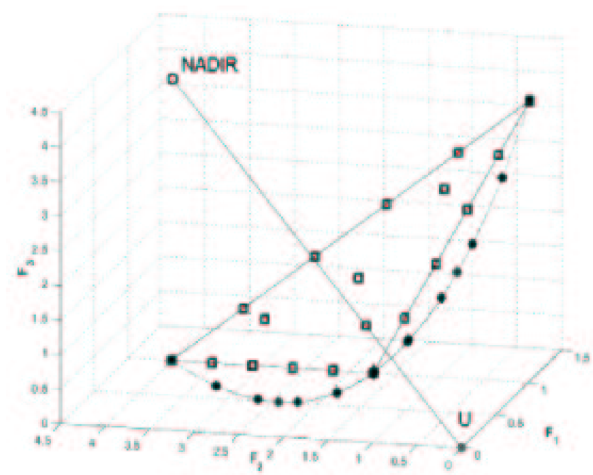

Figure 7. F3Db+ test case: results in the objective space of a run with 19 centers ( ), 13 diverse solution (•) over the analytical POF curve (gray dots) with linear P_i 's choice

We thus at the end deal with the following preference function:

$$\tilde{f} = \max_{i=1:M}\left[\frac{\frac{1}{M}(f_i - P_i)}{R_i - U_i}\right] \tag{10}$$

where diversity of solution is obtained by means of centers P_i variation with fixed weights (defining parallel search directions in objective domain).

An example of a problem where both internal and external P_i are used is shown in figure 10. As can be seen approximation quality does not depend on the position of centers and also a convex POF can be approximated with equally spaced solutions.

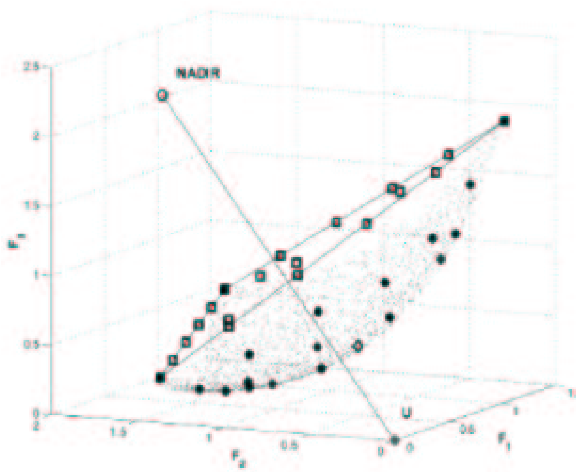

Figure 8. F3Db- test case: results in the objective space of a run with 19 centers ( ), 19 diverse solution (•) over the analytical POF surface (gray dots) with linear P_i 's choice

3.1 Search algorithm

Let us now consider the task of ensuring convergence to the POF for each solution with a small number of objective function calls, that is consider an effective search algorithm with a compromise between convergence properties, local minima avoiding ability and computational cost in terms of objective function calls. Once the preference function is built the minimization process is tackled via a minimal cost hybrid evolutionary deterministic strategy. For the evolutionary part (global search) a (1+1) ES ([14]) is considered; so doing only one individual evolution is required for each solution. Once the (1+1) ES is at the early convergence stage, the optimizer is switched to a gradient based algorithm (GBA) or a simplex Nelder Mead search algorithm (NMA). As switching criteria the convergence index (distance in design space between solution at iteration t and t+1) of (1+1) ES is considered together with a minimum dispersion index value. Because of (1+1) ES is used as a global search it is stopped at the early convergence stage; the final refinement of solution is left to the deterministic search. An average computational cost (number of objective function calls) may be evaluated as follows:

$$cost = (ni + ne)\ ns \tag{11}$$

where n_i is the average number of iteration for the global search, n_e the average objective function evaluation for the deterministic local search and n_s the number of desired solution. The starting point for each optimization run, wherever the center P_i is located, is randomly chosen in the whole design domain search space Ω. The pseudo-code of such a strategy is shown in figure 2; the main advantage of such a strategy is that the quality of each solution in terms of convergence properties does not depend on the number of solutions.

3.2 Different strategies for centers P_i choice

The task of choice criteria for P_i is now considered. The most immediate P_i choice strategy is a linear one and it may be seen on figure 3,4 or 5,9 for a two-objectives problem and in

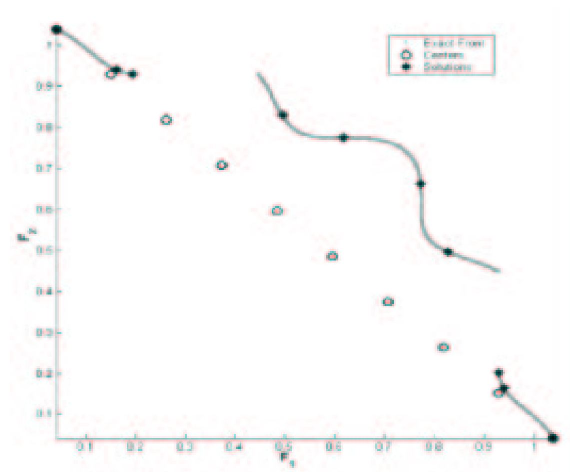

Figure 9. TNK test case: results in the objective space of a run with 10 centers ( ), 10 diverse solution (•) over the analytical POF (gray dots) with linear P_is choice

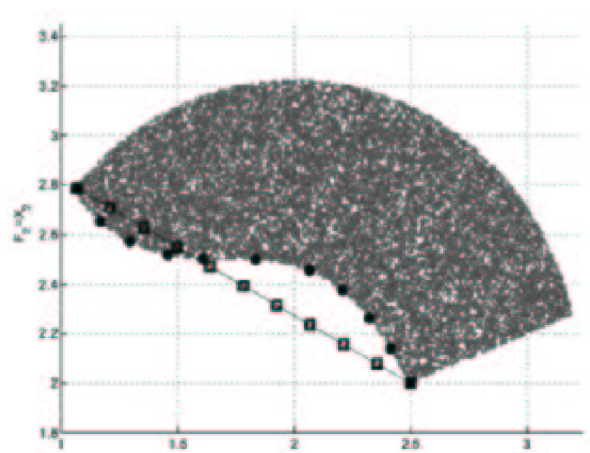

Figure 10. F2D test case: results in the objective space of a run with 11 centers ( ), 11 diverse solution (•) over exhaustive sampling (gray dots), with linear P_is choice

figure 7,8 for a three-objectives one. Once the utopia point U, and the matrix $\tilde{\mathbf{M}}$ are computed the line (2-objective problems) or the triangle (three-objectives problems) between the extremal point of the POF is considered and an uniformly distributed set of points is built on it as shown in figure 7,8 or 11,12. For both 2D and 3D cases the following formulas can be used for computing P_i:

$$P_i = \sum_{j=1}^{M} k_{i,j}\tilde{\mathbf{M}}(j,:) \quad \sum_{j=1}^{M} k_{i,j} = 1 \quad i = 1 : ns \tag{12}$$

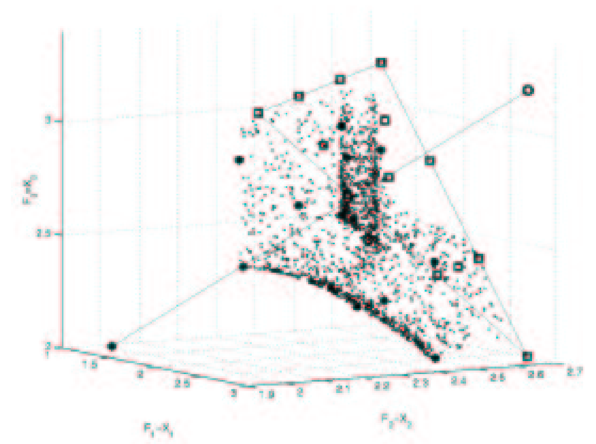

Figure 11. F3Da test case 3D view: results in the objective space of a run with 13 centers ( ), 12 diverse solution (•) over POF from exhaustive sampling (gray dots), with linear P_is choice

where *M(j; :)* is the j-th line of matrix **M**. Both convergence towards the POF and diversity of solution are satisfactory when considering the small computational time that is required for such POF sampling. As can be seen from figures also when dealing with 3-objectives problems the proposed strategy gives a satisfactorily uniform sampling of the POF both if the POF is a surface (figure 8) or a curve (figure 7). The strategy works in terms of diversity of solutions because in most cases the uniform distribution of points P_i on the line (2D case) or triangle (3D case) between extreme points of the POF leads to a quasiuniform distribution of solutions. This is particularly true for POL (figure 3,4) and TNK (figure 9); the drawback comes with DTZ3 (figure 5) where several solutions are overlapping with a significant loss of diversity and loss of computational effort. In order to overcome this problem an iterative choice of centers P_i can be considered (see pseudo-code in figure 2). The procedure is shown schematically in figure 13. Starting from points $\widetilde{M}_1$ and $\widetilde{M}_2$ a first center P_1 is considered and a first solution S_1 (asterisk) is obtained. After that, centers P_2 and P_3 are computed as mid points of lines $\widetilde{M}_1 - S_1$ and $\widetilde{M}_2 - S_1$. With this new centers solutions S_2 and S_3 are obtained. Examples of such a strategy are shown in figure 6 and 13. In case of three-objectives problems a similar iterative strategy can be considered.

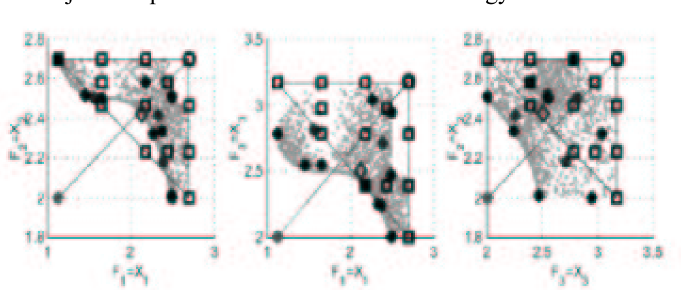

Figure 12. F3Da test case 2D view: results in the objective space of a run with 13 centers ( ), 12 diverse solution (•) over POF from exhaustive sampling (gray dots), with linear P_is choice

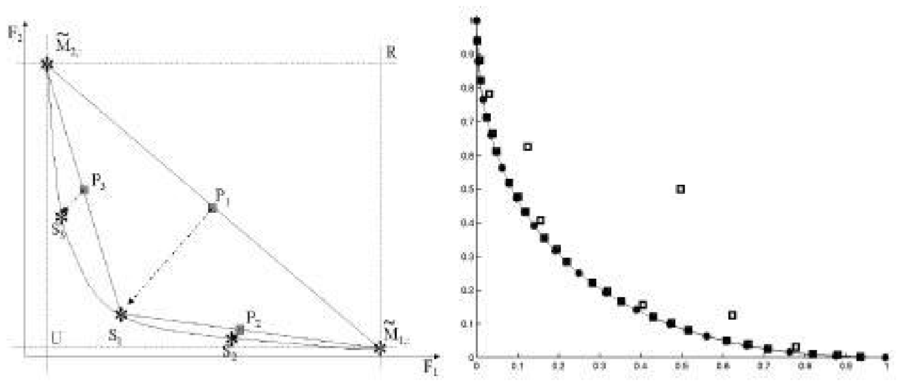

Figure 13. Schematic view of the proposed iterative choice of centers P_i and Shaffer's 2D test case with iterative centers choice

When iterative distribution of centers is considered, nodes and solutions gets iteratively closer, the number of lost solutions is reduced and diversity (at a fixed computational effort) is increased.

4. Conclusion

A strategy for POF coarse grained but precise approximation in industrial design optimization problems is proposed. The main advantage of such a strategy is that it can give convergent and diverse solutions even when classical MOEAs are unaffordable, but nevertheless the designer's aim is to have few but equally spaced solutions on the POF. The hybrid minimal cost global-local and evolutionary-deterministic search algorithm seems to be particularly suited for the proposed strategy. Both two-objective and three-objective analytical test cases are considered showing the validity of the strategy for convex or non-convex constrained non-linear problems.

From the applicative point of view, future works will consists in tackling industrial problems with the proposed strategy both in the field of Electromagnetic devices shape design and fuzzy controllers; on the other hand from a methodological point of view some more "intelligent" choice of centers can be developed, iteratively using some information coming from diversity of solution so far computed.

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Appendix

Test cases equations

Among all $R^2 \Rightarrow R^2$, $R^2 \Rightarrow R^3$ and $R^3 \Rightarrow R^3$ analytical problems that have been considered throughout the paper to validate the proposed strategy, some of them (POL, TNK, DTZs) are described in [13]; we give here formulas for the other introduced problems:

F2D

$$\begin{cases} \min & \mathbf{f} = (f_1(\mathbf{x}), f_2(\mathbf{x})) = (x_1, x_2) \\ \textit{subject to} & x_2 < -(x_1 - 1.7)^3 + 2.5 \\ & (x_1 - 2)^2 + (x_2 - 2)^2 > 1.5 \\ & x_1 - 4x_1 < 1 \end{cases} \tag{13}$$

F3Da

$$\begin{cases} \min & \mathbf{f} = (f_1(\mathbf{x}), f_2(\mathbf{x}), f_3(\mathbf{x})) = (x_1, x_2, x_3) \\ \textit{subject to} & x_j < -(x_i - 1.7)^3 + 2.5 \\ & (x_i - 2)^2 + (x_j - 2)^2 > 1.5 \\ & x_j - .4x_i < 1 \\ & (x_i - 0.5)^2 + (x_j - 2)^2 < .2 \end{cases} \tag{14}$$

F3Db±

$$\begin{cases} \min & f_1(\mathbf{x}) = x_1^2 + x_2^2 \\ \min & f_2(\mathbf{x}) = (x_1 - \frac{\sqrt{2}}{2})^2 + (x_2 - \frac{\sqrt{2}}{2})^2 \\ \min & f_3(\mathbf{x}) = (x_1 \pm \frac{\sqrt{2}}{2})^2 + (x_2 + \frac{\sqrt{2}}{2})^2 \\ \textit{subject to} & -3 < (x_1, x_2) < 3 \end{cases} \tag{15}$$