

Fuzzy Optimality and Evolutionary Multiobjective Optimization

M. Farina and P. Amato

STMicroelectronics Srl,
Via C. Olivetti, 2, 20041, Agrate (MI), IT.
`marco.farina@st.com`, `paolo.amato@st.com`

Abstract. Pareto optimality is somehow ineffective for optimization problems with several (more than three) objectives. In fact the Pareto optimal set tends to become a wide portion of the whole design domain search space with the increasing of the numbers of objectives. Consequently, little or no help is given to the human decision maker. Here we use fuzzy logic to give two new definitions of optimality that extend the notion of Pareto optimality. Our aim is to identify, inside the set of Pareto optimal solutions, different “degrees of optimality” such that only a few solutions have the highest degree of optimality; even in problems with a big number of objectives. Then we demonstrate (on simple analytical test cases) the coherence of these definitions and their reduction to Pareto optimality in some special subcases. At last we introduce a first extension of (1+1)ES mutation operator able to approximate the set of solutions with a given degree of optimality, and test it on analytical test cases.

1 Introduction

The application of Pareto optimum definition to optimization problems with a high number of objectives is somewhat unsatisfactory. This happens for two reasons. First, when there are more than three objectives the visualization of Pareto front must be carefully considered. Second, the set of solution classified as Pareto optimal can be a relevant fraction of the whole objective search space. Consequently, there we have little help in our effort to find the solution which is most suitable for the given problem. Many real life optimization problems have several (more than three) objectives; examples may be:

- the optimal design of electromagnetic devices where electrical efficiency, weight, cost and electric or magnetic field properties have to be considered [1, 4],
- combustion process and engine optimization where efficiency, NO_x emissions, soot emissions, noise are considered [10, 17]
- the aerodynamic shape optimization of supersonic wings where transonic and supersonic drag coefficients together with bending and twisting momenta [7]
- the paper machine optimization where up to five or six objective can be considered [11]

Due to the number of examples that may be considered and due to the unsolved difficulties that are encountered when more than three objectives are considered, the treatment of several objective problems is probably one of the most actual open issues in practical multiobjective evolutionary optimization.

In [8] and [13] the authors proposed a hierarchy of optimality definitions extending the Pareto one. These definitions are based on fuzzy logic [12] a key tool for the treatment of uncertainty and partial truth, and for the processing of large class of data.

The main idea behind the given definitions is to introduce different degree of optimality (each degree defining its own front). The lowest degree (let's say 0) corresponds to Pareto optimality. The highest degree (let's say 1) corresponds to a strong definition of optimality. The set of points classified as optimal by the latter are, thereafter, a small subset (eventually a single point) of the Pareto front. This classification is obtained by considering in the dominance relation (i) how many objectives a solution improves with respect to the other solution, and (ii) the size of each improvement. In fact, in this way, it is possible not only to discriminate between dominated and non-dominated (optimal) solution, but also between "less" and "more" optimal solutions.

In this work we show how these definitions can be exploited in continuous optimization problems. In particular we introduce an evolution strategy algorithm for approximating the optimal front associated to a certain degree of optimality.

2 Limits and drawbacks of Pareto optima definition

The following multi-objective optimization problem is considered [14, 9, 15]:

Definition 1 (Multi-criteria optimization problem). *Let $\mathbf{V} \subseteq K_1 \times K_2 \times \dots \times K_N$ and $\mathbf{W} \subseteq O_1 \times O_2 \times \dots \times O_M$ be vector spaces, where the K_i, O_j (with $i = 1, \dots, N$ and $j = 1, \dots, M$) are (continuous or finite) fields and $N, M \in \mathbb{N}$, and let $\mathbf{g} : \mathbf{V} \mapsto \mathbb{R}^p, \mathbf{h} : \mathbf{V} \mapsto \mathbb{R}^q$ and $\mathbf{f} : \mathbf{V} \mapsto \mathbf{W}$ be three mappings, where $p, q \in \mathbb{N}$. A Non-linear constrained multi-criteria (minimum) optimization problem with M objectives is defined as:*

$$\min_{\mathbf{v} \in \mathbf{V}} \mathbf{f} \triangleq \{f_1(\mathbf{v}), \dots, f_M(\mathbf{v})\} \quad \text{subject to} \quad \begin{cases} \mathbf{g}(\mathbf{v}) \leq 0 \\ \mathbf{h}(\mathbf{v}) = 0. \end{cases}$$

Definition 2 (Design and objective search space). *We call Design domain search space Ω and objective domain search space Ω_O the following two set:*

$$\Omega = \{\mathbf{v} \in \mathbf{V} \mid \mathbf{g}(\mathbf{v}) \leq 0 \wedge \mathbf{h}(\mathbf{v}) = 0\}, \quad \Omega_O = \{\mathbf{f}(\mathbf{v}) \in \mathbf{W} \mid \mathbf{v} \in \Omega\}.$$

We consider multi-objective non-linear constrained optimization in a continuous search space are considered; Ω and Ω_O are thus continuous spaces.

The Pareto definition of optimality in a multi-criteria decision making problem can be unsatisfactory due to essentially two reasons:

P1 the number of improved or equal objective values is not taken into account,
P2 the (normalized) size of improvements is not taken into account.

These issues are essential decision elements when looking for the best solution, and they are implicitly included in the common-sense notion of optimality.

The limit of Pareto definition when the first issue (P1) is considered can be viewed in the schema shown in figure 1. Since the Pareto dominance gives a partial order of solutions in criteria space¹, when a vector (a candidate for optimal solution) in the criteria space is considered, all other possible solutions can belong to one of the following three different sets: better solutions, worse solutions and equivalent² solutions.

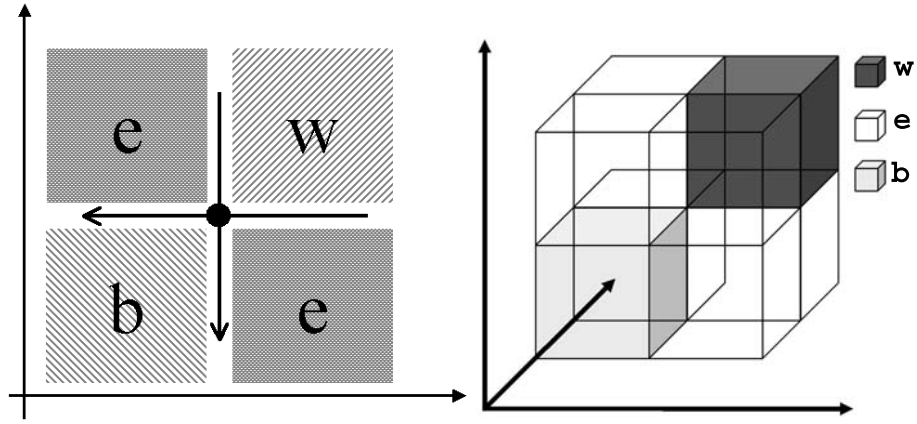


Fig. 1. Schematic view of Pareto-dominance based partial order in 2D and 3D problems when a candidate solution is considered (●): equal (e), better (b) and worse (w) solution regions are shown.

Figure 1 shows such sets for 2 and 3 criteria problems. Let e the portion of the M -dimensional criteria domain space containing all the points that the P-dominance concept classify as equivalent to a given one. The portion e increases with the increasing of the number M of objectives as follows:

$$e = \frac{2^M - 2}{2^M} \quad (1)$$

Thus when M tends to infinity, e tends to 1 (i.e., it is the whole space). This fact is general and problem independent and when a single problem is considered and Ω and Ω_O are introduced, always e tends to 1 when M tends to infinity,

¹ Indeed the partial ordering induced by Pareto dominance is a weak ordering. In fact, in general, an algebraic structure equipped with this partial order is not a lattice.

² Since Pareto dominance does not induce an equivalent relation, it should be better to call them “indifferent” solutions.

eventually with a behavior different from equation 1. From this it derives that Pareto definition is ineffective for a large number of objectives, even without considering the second aforementioned issue.

In the following sections we will give two more general definition of optimum for a multi-criteria decision making problem, taking into account one issue at a time. As we shall see, Pareto optimum definition is a special case of both definitions.

3 Two generalizations of Pareto definitions

In this section we give a brief description of two optimality definitions that soundly extend the Pareto one. For a deeper discussion about these definitions see [8, 13].

3.1 Taking into account the number of improved objectives: *k*-optimality

In Pareto optimality definition two candidate solutions \mathbf{v}_1 and \mathbf{v}_2 are equivalent if at least in one objective the first solution is better than the second one, and at least in one objective the second one is better than the first one (or if they are equal in all the objectives). Indeed a more general definition, able to cope with a wider variety of problems, should take into account in how many objectives the first candidate solution is better than the second one and viceversa. To do so, we introduce the following functions which associate to every couple of points in Ω a natural number.

$$\begin{aligned} n_b(\mathbf{v}_1, \mathbf{v}_2) &\triangleq |\{i \in \mathbb{N} | i \leq M \wedge f_i(\mathbf{v}_1) < f_i(\mathbf{v}_2)\}| \\ n_e(\mathbf{v}_1, \mathbf{v}_2) &\triangleq |\{i \in \mathbb{N} | i \leq M \wedge f_i(\mathbf{v}_1) = f_i(\mathbf{v}_2)\}| \\ n_w(\mathbf{v}_1, \mathbf{v}_2) &\triangleq |\{i \in \mathbb{N} | i \leq M \wedge f_i(\mathbf{v}_1) > f_i(\mathbf{v}_2)\}| \end{aligned}$$

For every couple of points $\mathbf{v}_1, \mathbf{v}_2 \in \Omega$, the function n_b computes the number of objectives in which \mathbf{v}_1 is better than \mathbf{v}_2 , n_e computes the number of objectives in which they are equal, and n_w the number of objectives in which \mathbf{v}_1 is worse than \mathbf{v}_2 . To lighten the mathematical notation, from now on we will consider a generic couple of points and we will write simply n_b, n_e and n_w instead of $n_b(\mathbf{v}_1, \mathbf{v}_2), n_e(\mathbf{v}_1, \mathbf{v}_2)$ and $n_w(\mathbf{v}_1, \mathbf{v}_2)$.

A moment's reflection tell us that the following inequalities holds:

$$n_b + n_w + n_e = M \quad 0 < n_b, n_w, n_e < M$$

We are now able to give a first new definition of dominance and optimality namely *k*-dominance and *k*-optimality:

Definition 3 (*k*-dominance).

\mathbf{v}_1 is said to k -dominate \mathbf{v}_2 if and only if:

$$\begin{cases} n_e < M \\ n_b \geq \frac{M - n_e}{k + 1}, \end{cases} \quad (2)$$

where $0 \leq k \leq 1$.

As can be easily seen, definition 3 with $k = 0$ corresponds to Pareto-dominance. Ideally k can assume any value in $[0, 1]$, but because n_b has to be a natural number only a limited number of optimality degree need to be considered. In fact in equation (2) the second inequality is equivalent to $n_b \geq \left\lceil \frac{M - n_e}{k + 1} \right\rceil$. With this new dominance definition the following new optimality can be defined:

Definition 4 (k -optimality). \mathbf{v}^* is k -optimum if and only if there is no $\mathbf{v} \in \Omega$ such that \mathbf{v} k -dominates \mathbf{v}^*

The terms “ k -dominance” and “ k -optimality” derives respectively from the fact that the former is a loose version of Pareto dominance (1-dominance), while the latter is a strong version of Pareto optimality (0-optimality). We can now easily extend concepts of \mathcal{S}_P and \mathcal{F}_P in the following way:

Definition 5 (k -optimal set and front).

We call k -optimal set (\mathcal{S}_k) and k -optimal front (\mathcal{F}_k) the set of k -optimal solutions in design domain and objective domain respectively.

Several \mathcal{S}_k sets and \mathcal{F}_k fronts are thus introduced, one for each value of k . Let us refer to Pareto optimal front as \mathcal{F}_P , and to Pareto optimal set as \mathcal{S}_P . Then it is evident that $\mathcal{S}_0 = \mathcal{S}_P$ and $\mathcal{F}_0 = \mathcal{F}_P$.

3.2 Taking into account the size of improvements: Fuzzy optimality

A natural way of extending the notion of k -dominance and k -optimality is to introduce fuzzy relations instead of crisp ones. As first step, to take into account to which degree in each objective function a point \mathbf{v}_1 is different from (or equal to) a point \mathbf{v}_2 , we will consider fuzzy numbers and fuzzy arithmetic. As second step, we will consider the dominance relation itself as a fuzzy relation.

Fuzzy numbers A standard way to introduce fuzzy arithmetic on a given universe (here the objective domain search space Ω_O), is to associate to each of its point a triple of fuzzy sets — one for equality (fuzzy number), one for “greater than” and one for “less than”. Figures 2 and 3 shows two possible definitions of the fuzzy sets for “equal to 0”, “greater than 0” and “less than 0”. For coherence with the terminology used so far, we refer to their respective membership function as μ_e , μ_w (where w means “worst”, remember that we are talking about minimization problems) and μ_b .

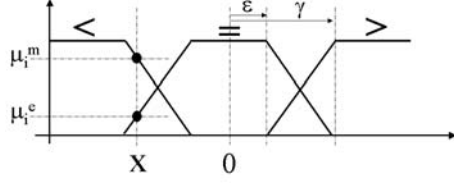


Fig. 2. Linear membership for i -th objective to be used in equation 3 for a fuzzy definition of $=$, $<$ and $>$: ε and γ are parameters to be chosen by the decision maker.

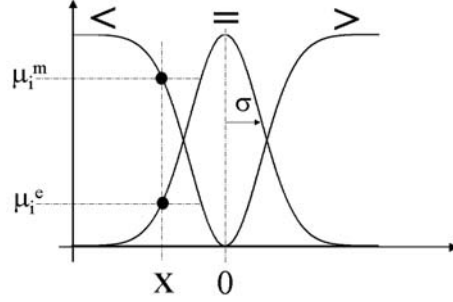


Fig. 3. Gaussian membership for i -th objective to be used in equation 3 for a fuzzy definition of $=$, $<$ and $>$: σ is a parameter to be chosen by the decision maker.

The fuzzy definition of n_b , n_w and n_e (now with superscript F) are the following:

$$\begin{aligned} n_b^F(\mathbf{v}_1, \mathbf{v}_2) &\triangleq \sum_{i=1}^M \mu_b^{(i)}(f_i(\mathbf{v}_1) - f_i(\mathbf{v}_2)) \\ n_w^F(\mathbf{v}_1, \mathbf{v}_2) &\triangleq \sum_{i=1}^M \mu_w^{(i)}(f_i(\mathbf{v}_1) - f_i(\mathbf{v}_2)) \\ n_e^F(\mathbf{v}_1, \mathbf{v}_2) &\triangleq \sum_{i=1}^M \mu_e^{(i)}(f_i(\mathbf{v}_1) - f_i(\mathbf{v}_2)) \end{aligned}$$

In order n_b^F , n_w^F and n_e^F to be a sound extension of n_b , n_w and n_e , the membership functions $\mu_b^{(i)}$, $\mu_w^{(i)}$ and $\mu_e^{(i)}$ must satisfy Ruspini condition (i.e, in each point they must sum up to 1) [2]. In fact, under this hypothesis the following holds:

$$n_b^F + n_e^F + n_w^F = \sum_{i=1}^M (\mu_b^{(i)} + \mu_w^{(i)} + \mu_e^{(i)}) = M$$

In the figures 2 and 3, two possible different membership shapes are considered: linear and gaussian. Both of them are characterized by parameters defining the shape, ε_i and γ_i for the linear one and σ_i for the gaussian one. Although the definition of such parameters is to be carefully considered, their intended meaning is clear. They thus can be derived from the human decision-maker knowledge on the problem.

- ε_i defines in a fuzzy way the practical meaning of equality and it can thus be considered the tolerance on the i -th objective, that is the interval within which an improvement on objective i is meaningless.
- γ_i can be defined as a relevant but not big size improvement for objective i .

- σ_i evaluation requires a combination of the two aforementioned concepts of maximum imperceptible improvement on objective i (ε_i) and γ_i ; the following formula for the membership μ_e can be used:

$$\mu_e^{(i)} = \exp\left(-\frac{\ln \chi}{\varepsilon_i^2}(f_1^i - f_2^i)^2\right) \quad (3)$$

where χ is an arbitrary parameter ($0.8 < \chi < 0.99$) and where μ_b and μ_w can be computed univocally in order to satisfy Ruspini's condition.

With such a fuzzy definition of n_b^F , n_e^F and n_w^F , both k -dominance and k -optimality definition can be reconsidered.

k-optimality with fuzzy numbers of improved objectives (fuzzy optimality) A first extension of the definitions of k -dominance and k -optimality can be given if n_b , n_e and n_w are replaced by n_b^F , n_e^F and n_w^F in definitions 3 and 4 as follows:

Definition 6 (k_F -dominance). \mathbf{v}_1 is said to k_F -dominate \mathbf{v}_2 if and only if:

$$\begin{cases} n_e^F < M \\ n_b^F \geq \frac{M - n_e^F}{k_F + 1}, \quad 0 \leq k_F \leq 1 \end{cases} \quad (4)$$

Definition 7 (k_F -optimality). \mathbf{v}^* is k_F -optimum if and only if there is no $\mathbf{v} \in \Omega$ such that \mathbf{v} k_F -dominates \mathbf{v}^*

The parameter k (now called k_F) has the same meaning as in the previous case ($0 \leq k_F \leq 1$) but now a continuous degree of optimality and dominance are introduced (k_F -dominance and k_F -optimality). An extension of \mathcal{S}_k and \mathcal{F}_k can be defined as follows:

Definition 8 (k_F -optimal set and front). We call \mathcal{S}_{k_F} and \mathcal{F}_{k_F} the set of k_F -optimal solutions in design domain and objective domain respectively.

Each \mathcal{S}_{k_F} can be viewed as the k_F -cut³ of a fuzzy set O for the notion of “optimality in design domain search space”. Then the whole membership function for this fuzzy set is implicitly defined by its k_F -cuts. In fact let \mathbf{V} be the search space, then for all $\mathbf{v} \in \mathbf{V}$ its degree of optimality is given by $O(\mathbf{v}) = \sup_{\alpha} \{\alpha \in [0, 1] | \mathbf{v} \in \mathcal{S}_{\alpha}\}$. An example of fuzzy set for optimality is given in figure 5. The same reasoning can be applied to define the fuzzy sets for optimality in objective space and the fuzzy sets for the dominance relation.

³ Remember that an α -cut of a fuzzy set A on an Universe U is the crisp set $\{u \in U | A(u) \geq \alpha\}$.

Remark: It is possible to give a further extension of the notion of optimality. In fact a more general procedure can be introduced by fuzzifying not only the quantities n_b^F , n_e^F and n_w^F , but also the dominance relation itself. The resulting notion of optimality has been introduced and discussed by the authors in [13]. It is this last definition that is properly referred to as *fuzzy optimality*. However in the rest of the paper we will sometime use this term to indicate k_F -optimality.

4 Test cases

This last section is devoted to two examples showing the validity of the introduced definitions. They both are continuous constrained multiobjective optimization problems ($\mathbb{R}^N \rightarrow \mathbb{R}^M$).

4.1 A first test case: 6 objectives

In the first test case (figure 4 and 5) a 6D problem with only two design variables is considered with parabolic function located in an asymmetric way on a rhombus borders.

$$\begin{cases} \min_{\mathbf{v} \in \mathbb{R}^2} & \mathbf{f} = \{f_1(\mathbf{v}), \dots, f_j(\mathbf{v}), \dots, f_6(\mathbf{v})\} \\ & f_j(\mathbf{v}) = (v_1 - c_{1,j})^2 + (v_2 - c_{2,j})^2 \\ \text{s. t.} & -1 \leq v_1, v_2 \leq 1 \end{cases} \quad (5)$$

The coordinates for points $(c_{1,j}, c_{2,j})$ are marked with black bullets and the

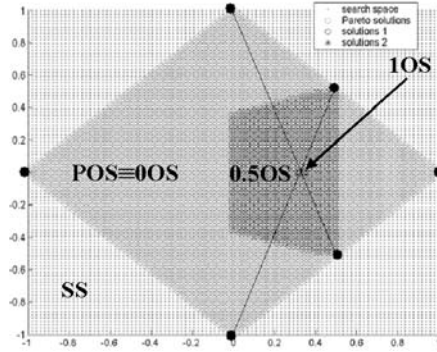


Fig. 4. k -optimality classification on problem (5). Six parabolic functions are centered on \bullet and three different degree of k -optimal solutions are shown together with the whole search space.

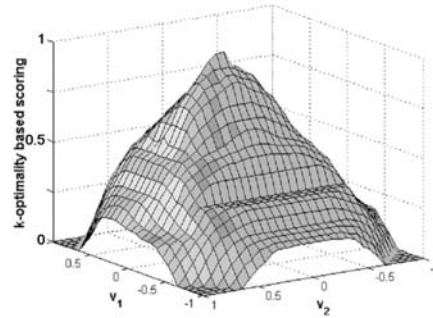


Fig. 5. *fuzzy* optimality classification on problem (5) with linear membership

search space is shown with bright gray dots. We apply k -optimality definition;

as can be seen the region of Pareto optimal solution ($\mathcal{S}_P = \mathcal{S}_0$) is quite big with respect to the search space. If k is increased ($k=.5$) a smaller region is selected up to a single optimum for $k=1$. We thus have four regions, one included in the other: $SSP \subset \mathcal{S}_P \equiv \mathcal{S}_0 \subset \mathcal{S}_{\frac{1}{2}} \subset \mathcal{S}_1$ where SSP is the search space. If we move from a solution f^1 to a solutions f^2 both belonging to \mathcal{S}_P we can expect one of the two to be better than the other in at least one objective; if on the other hand we move from a solution f^1 to a solutions f^2 both belonging to $\mathcal{S}_{\frac{1}{2}}$ we can expect one of the two to be better than the other in at least two objective. The same may in general hold for \mathcal{S}_1 , but in this example \mathcal{S}_1 is a single solution. When fuzzy-optimality is considered on the same test problem a continuous classification of solution can be obtained and is shown in figure 5; the sets \mathcal{S}_{k_F} are infinite and corresponds to any real value of $k_F \in [0 : 1]$ (any k_F -cut of μ_O surface).

4.2 Second test case: M objectives

As a second test case (figure 6 and 7) the classification ability of *fuzzy*-dominance and *fuzzy*-optimality is shown on the following more general $\mathbb{R}^N \rightarrow \mathbb{R}^M$ multiobjective optimization problem on a box-like constrained search space:

$$\begin{cases} \min_{\mathbf{v} \in \mathbb{R}^N} & \mathbf{f} = \{f_1(\mathbf{v}), \dots, f_i(\mathbf{v}), \dots, f_M(\mathbf{v})\} \\ & f_i(\mathbf{v}) = \sum_{k=1}^{np} \exp(-c_{i,k} \sum_{j=1}^N (v_j + p_{k,j})^2) \\ & i = 1 : M \\ s. t. & L_j \leq v_j \leq U_j \quad j = 1 : N \end{cases} \quad (6)$$

In this example, the number of objectives is increased with respect to the previous one and the shape is a little bit more complex. A case with $M=12$ and $N=2$ is first considered. A sampling of 400 points on the search space (candidate solutions) is considered and classified. It is easy to see that almost all points in the search space (a box $\in \mathbb{R}^2$) are Pareto-optimal (see figure 6 where the \mathcal{S}_P is shown on the ground level). No (or at most very poor) decision could thus be taken with Pareto optimality.

Classically multiobjective search problems are tackled *via* an equivalent scalar function such as weighted sum of objectives [14],[5]. The limitations of such an approach are the following: only one special Pareto-optimal solution can be computed, preference and tolerance on objectives are difficult to be expressed in a clear way. As an example, figure 7 shows a classification of solutions in the search space for problem (6) with a normalized sum of objectives. As can be seen the peak value identifies a special Pareto-optimal solution and there is no way to identify a bigger subset of the Pareto-optimal set.

On the other hand the proposed optimality definition can give a continuous classification of solution, and consequently any number of Pareto optimal solutions with a clearly defined degree of stronger (with respect to P-optimality) optimality can be easily computed. This possibility is clearly shown in figure 6

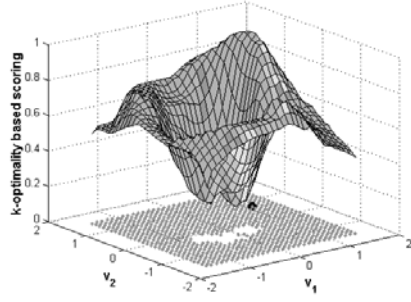


Fig. 6. *fuzzy*-optimality based classification with linear membership and corresponding maximum point (●); linear membership with $\varepsilon_i = 0.01$ and $\gamma_i = 0.2$, $1 < i < 11$ have been used.

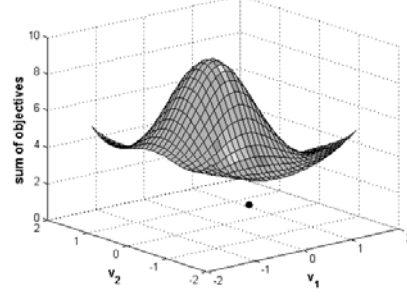


Fig. 7. Classical normalized sum of objectives on problem (6) and corresponding minimum point (●)

where a continuous classification similar to the one in figure 5 is obtained. Moreover, the point corresponding to the minimum of the sum function is coincident with the maximum of the *fuzzy*-optimality-based classification; both points are shown under 3D surfaces. The normalized number of solution $s(k_F)$ satisfying *fuzzy*-optimality at degree k_F is plotted against k_F in figure 8 for problem (6). As can be seen the $s(k_F)$ behavior depends on the problem complexity. In case of

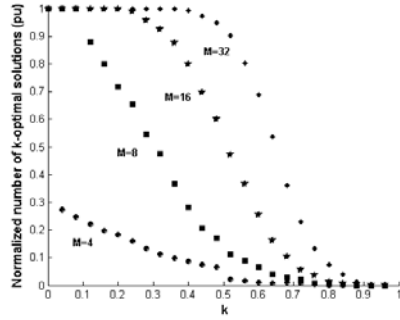


Fig. 8. Normalized number of *fuzzy*-optimal solutions versus the optimality degree k_F for different number of objectives.

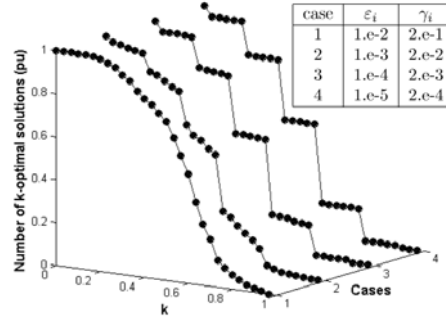


Fig. 9. From *fuzzy* $_{\alpha}$ -optimality to k -optimality for a 4-objectives problem (6) ; membership parameters for the four cases are listed in the table.

a problem with 32 objectives an high value of k_F is required in order to find some solutions belonging to \mathcal{S}_{k_F} . We now point out with an example (problem (6),

M=6) that the crisp k -optimality can be seen as the limit of *fuzzy*-optimality when $\varepsilon_i, \gamma_i, \sigma_i \rightarrow 0$, that is when crisp memberships are considered. This can be easily seen from figure 9 where four different $s(k)$ functions are plotted for the values of membership parameters shown in the table. From a continuous *fuzzy*-optimality based classification (case 1), a crisp k -optimality based classification (case 4) can be obtained with membership parameters close to zero; some intermediate levels are also shown.

An example of fuzzy-dominance membership function with respect to one fixed solution is shown in figure 10; as can be seen the region of the most fuzzy-dominated solution with respect to the fixed one (that is fuzzy-dominated with the highest value of k_F) correspond to the maximum region of the surface. An example of fuzzy-optimality membership building through k_F -cuts can be shown in figure 11; as can be seen the obtained maximum optimality degree is 0.5 and not 1. This is a numerical effect due to the poor sampling of the search space (a coarse sampling has been considered for better figure rendering purposes).

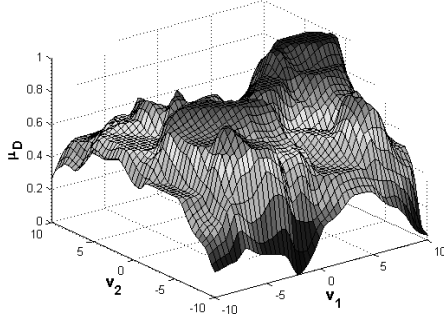


Fig. 10. Membership $\mu_D(v_1, v_2)|_{\tilde{v}}$ for dominance degree with respect to a fixed solution \tilde{v} .

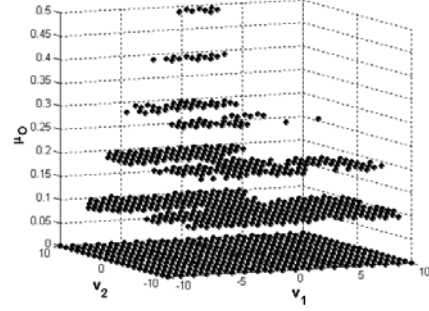


Fig. 11. *fuzzy*-optimality membership building through k_F -cuts, a sampling of 400 solutions in the search space is considered.

A comparison of two k_F -cuts (corresponding to two different degrees of optimality) and the crisp Pareto optimal front is also shown in figure 12. As can be seen the fuzzy optimality definition is able to select properly, among Pareto optimal solutions, some “more optimal” solutions corresponding to the degree k_F of optimality. Moreover for low values of k_F , the crisp Pareto Optimal front (which is obtained via a different procedure for comparison and checking purposes) can be properly reconstructed.

5 Is *fuzzy*-optimality useful for 2D and 3D cases?

Though being specifically developed for treatment of many (> 3) objectives problems, *fuzzy*-optimality is meaningful even in case of 1 or 2 objectives only.

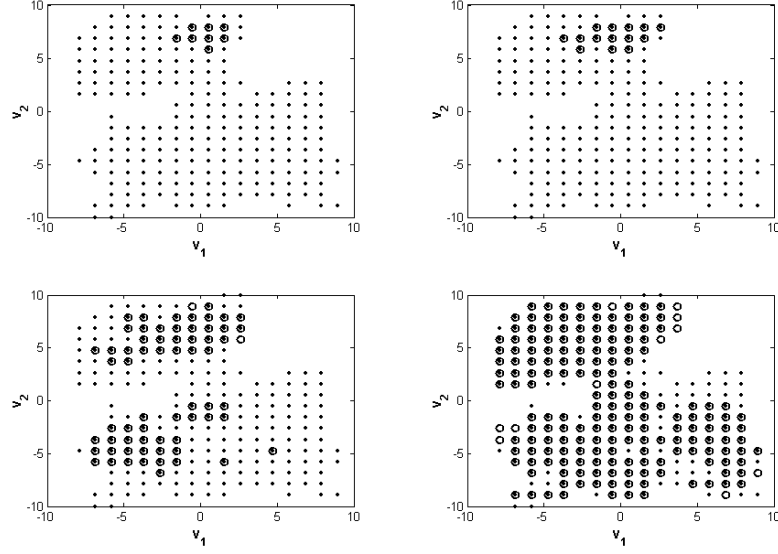


Fig. 12. Comparison of k_F -cuts (\circ) for a fuzzy-optimality membership at different k_F values and crisp Pareto Optimal front (\cdot), a sampling of 400 solutions in the search space is considered.

In such cases, no reduction of Pareto Optimal front is possible because only one degree of optimality can be considered ($\lceil M/2 \rceil < 2$) and k -optimality coincides with Pareto optimality. Nevertheless *fuzzy*-optimality can be used for the evaluation of a larger front taking into account the tolerances on objectives; an example is shown in figures 13 and 14.

As can be seen, when including tolerances (coming from the decision maker's perception) in the objectives comparison between candidate solution, the crisp Pareto optimal set and front are a smaller part of the fuzzy sets both in objective and design space. The size and the shape of the \mathcal{S}_{k_F} set and \mathcal{F}_{k_F} front (where $k_F = 0$) depends on ε and γ values. We point out with this example that $\mathcal{S}_P = \mathcal{S}_0$ if k -optimality is used but $\mathcal{S}_P \subseteq \mathcal{S}_0$ if fuzzy optimality is considered.

We stress that here we consider the objectives function as crisp, (i.e. that there is no uncertainty (fuzziness) in each objectives function). Thus the fuzzy front in objective space in figure 13 descends only from the fuzzification of the comparison between objective values. Of course it is possible to obtain different fuzzy front by considering fuzzy objective functions.

6 Toward *fuzzy*-optimal sets approximation via evolutionary algorithms

We have so far shown that, given a MCDM or a MO problem we can classify candidate solutions with fuzzy-optimality and build N-dimensional memberships

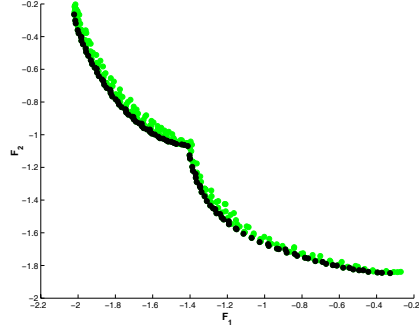


Fig. 13. *Fuzzy* front \mathcal{F}_{k_F} in objective space ($k_F = 0$) (green \bullet) and Pareto optimal front (black \bullet) for a 2 objective 2 variable problem (6) .

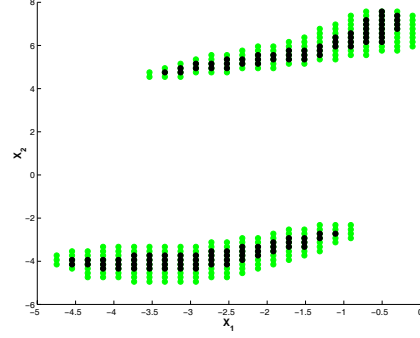


Fig. 14. *Fuzzy* set \mathcal{S}_{k_F} in design space ($k_F = 0$) (green \bullet) and Pareto optimal set (black \bullet) for a 2 objective 2 variable problem (6).

for optimality and dominance (once one solution is fixed). From a practical point of view we now need a tool that can give us a satisfactory approximation of an *fuzzy*-optimal set once a desired value for k_F is fixed, without classifying all solutions in the search space. A huge variety of methods is available in literature for the approximation via evolutionary computation of the Pareto optimal front. From a general point of view any method based on Pareto-dominance [16] can be extend including fuzzy-dominance. As an example we consider a (1+1)ES algorithm for Pareto optimal front approximation [3, 6] and we modify as follows the mutation operator in order the algorithm to converge towards the *fuzzy*-optimal set \mathcal{S}_{k_F} and *fuzzy*-optimal front \mathcal{F}_{k_F} at a given value of k_F . Given the objective values \mathbf{f}_P for the parent P and for the sun S (\mathbf{f}_S): **previous mutation operator**

- if $n_b(f_P, f_S) = M$ accept the sun as a new parent else discard, it
- go to next generation.

new mutation operator

- if $n_b(f_P, f_S) \geq M - k_F M$ accept the sun as a new parent else discard it,
- go to next generation.

Several algorithms (n) with this modification are then run in parallel in order to have the population evolution. The introduced larger acceptance criterion for mutation leads to the convergence of the algorithm towards the desired k_F -optimal set (smaller than the Pareto-optimal one). An example is given in figure 15 where a comparison of k_F -optimal set from sampling (\bullet), k_F -optimal set from evolutionary algorithm (\circ) and Pareto Optimal set (\cdot) is given in the design space. As can be seen the k_F -optimal set is correctly sampled though being a disconnected set.

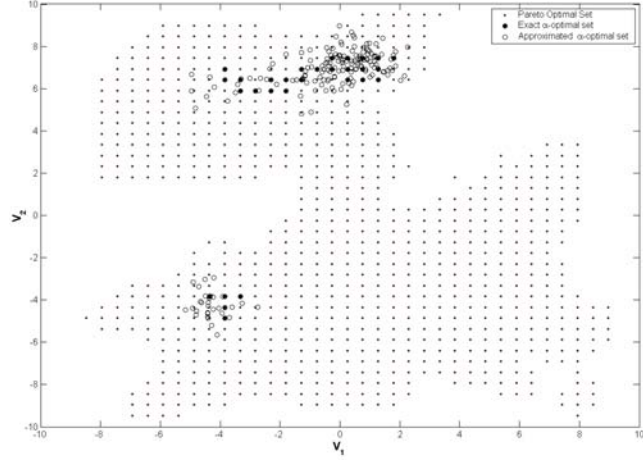


Fig. 15. Comparison of *fuzzy*-optimal set from sampling (\bullet), *fuzzy* $_{\alpha}$ -optimal set from evolutionary algorithm (\circ) and Pareto Optimal set (\cdot).

7 Conclusion

Via simple and general examples, in this paper we have shown that some possible weakness of Pareto optimality (weakness becoming evident in the application to optimization problems with more than three objectives) can be avoided by considering other more general definitions.

In particular it seems that fuzzy logic can be profitably used in the generalization of the notion of Pareto optimality. In fact it easily and naturally allows considering the number of improved objectives (not only if at least one objectives is improved without worsening the others) and the size of improvement in each objective.

Though being developed for more than three criteria the proposed fuzzy optimality definition is meaningful for 2D and 3D cases for different reasons that are discussed in some details. In order to apply the definition to real-life decision making problems, the use of evolutionary algorithms for the computation of *fuzzy*-optimality based subset has also been shown.

References

1. P. Alotto, A. V. Kuntsevitch, Ch. Magele, C. Paul G. Molinari, K. Preis, M. Repetto, and K. R. Richter. Multiobjective optimization in magnetostatics: A proposal for benchmark problems. *Technical report, Institut fr Grundlagen und Theorie Electrotechnik, Technische Universitt Graz, Graz, Austria, <http://www-igte.tu-graz.ac.at/team/berl01.htm>, 1996.*
2. P. Amato and C. Manara. Relating the theory of partitions in mv-logic to the design of interpretable fuzzy systems. In J. Casillas, O. Cordon, F. Herrera, and

- L. Magdalena, editors, *Trade-off between Accuracy and Interpretability in Fuzzy Rule-Based Modeling*. Springer-Verlag, Berlin, to be published.
3. P. Di Barba, M. Farina, and A. Savini. Multiobjective Design Optimization of Real-Life Devices in Electrical Engineering: A Cost-Effective Evolutionary Approach. In Eckart Zitzler, Kalyanmoy Deb, Lothar Thiele, Carlos A. Coello Coello, and David Corne, editors, *First International Conference on Evolutionary Multi-Criterion Optimization*, pages 560–573. Springer-Verlag. Lecture Notes in Computer Science No. 1993, 2001.
4. P. Di Barba and M. Farina. Multiobjective shape optimisation of air cored solenoids. *COMPEL International Journal for computation and mathematics in Electrical and Electronic Engineering*, 21(1):45–57, 2002. in press.
5. C. Carlsson and R. Fuller. Multiobjective optimization with linguistic variables. *Proceedings of the Sixth European Congress on Intelligent Techniques and Soft Computing (EUFIT98, Aachen, 1998, Veralg Mainz, Aachen,, 2:1038–1042*, 1998.
6. Lino Costa and Pedro Oliveira. An Evolution Strategy for Multiobjective Optimization. In *Congress on Evolutionary Computation (CEC'2002)*, volume 1, pages 97–102, Piscataway, New Jersey, May 2002. IEEE Service Center.
7. Masashi Morikawa Daisuke Sasaki, Shigeru Obayashi, and Kazuhiro Nakahashi. Constrained Test Problems for Multi-objective Evolutionary Optimization. In Eckart Zitzler, Kalyanmoy Deb, Lothar Thiele, Carlos A. Coello Coello, and David Corne, editors, *First International Conference on Evolutionary Multi-Criterion Optimization*, pages 639–652. Springer-Verlag. Lecture Notes in Computer Science No. 1993, 2001.
8. M. Farina and P. Amato. A fuzzy definition of “optimality” for many-criteria decision-making and optimization problems. *submitted to IEEE Trans. on Sys. Man and Cybern.*, 2002.
9. T. Hanne. *Intelligent strategies for meta multiple criteria decision making*. Kluwer Academic Publishers, Dordrecht, THE NETHERLANDS, 2001.
10. Ryoji Homma. Combustion process optimization by genetic algorithms - reduction of co emission via optimal post-flame process. *Energy and Environmental Technology Laboratory, Tokyo Gas Co.,Ltd.*, 9, 1999.
11. P. Tarvainen J. Hamalainen, R.A.E Makinen. Optimal design of paper machine headboxes. *International Journal of Numerical Methods in Fluids*, 34:685–700, 2000.
12. G. J. Klir and Bo Yuan editors. *Fuzzy Sets, Fuzzy Logic and Fussy Systems, selected papers by L.A. Zadeh*, volume 6 of *Advances in Fuzzy Systems - Applications and Theory*. World Scientific, Singapore, 1996.
13. M.Farina and P. Amato. On the optimal solution definition for many-criteria optimization problems. In *Proceedings of the NAFIPS-FLINT International Conference 2002, New Orleans, June, 2002*, pages 233–238. IEEE Service Center, June 2002.
14. K. Miettinen. *Nonlinear Multiobjective Optimization*. Kluwer Academic Publishers, Dordrecht, THE NETHERLANDS, 1999.
15. P.P. Chakrabarti P. Dasgupta and S. C. DeSarkar. *Multiobjective Heuristic Search*. Vieweg, 1999.
16. N. Srinivas and Kalyanmoy Deb. Multiobjective Optimization Using Nondominated Sorting in Genetic Algorithms. *Evolutionary Computation*, 2(3):221–248, Fall 1994.
17. <http://energy.bmstu.ru/e02/diesel/d11eng.htm>.