Including Preferences into a Multiobjective Evolutionary Algorithm to Deal with Many-objective Engineering Optimization Problems

Antonio López-Jaimes[a], Carlos A. Coello Coello[a,]

[a] *CINVESTAV-IPN, Computer Science Department, Mexico City, MEXICO*

Abstract

In this paper, we introduce a new preference relation based on a reference point approach. This relation offers an easy approach to integrate decision maker's preferences into a Multiobjective Evolutionary Algorithm (MOEA) without modifying its basic structure. Besides finding the optimal solution of the achievement scalarizing function, the new preference relation allows the decision maker to find a set of solutions around that optimal solution. Then, a MOEA equipped with the proposed preference relation can be integrated into an interactive optimization method. One of the main advantages of the new method is that setting its parameters is an intuitive task to the decision maker. The other advantage is that, since our preference relation induces a finer order on vectors of objective space than that achieved by the Pareto dominance relation, it is appropriate to cope with problems having a high number of objectives.

Keywords: Evolutionary computation, Multi-objective optimization, Many-objective optimization problems, Interactive optimization methods,

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1. Introduction

MOEAs rely on preference relations to identify high-potential regions of the search space in order to approximate the optimal solution set. A preference

[*]Corresponding author, telephone: +81-50-3362-5271, fax:+81-42-759-8461
Email addresses: `tonio.jaimes@gmail.com` (Antonio López-Jaimes), `ccoello@cs.cinvestav.mx` (Carlos A. Coello Coello)

relation is a mean to decide if a solution $\mathbf{x}$ is preferable over another solution $\mathbf{y}$ in the search space.

In single-objective optimization, the determination of the optimum among a set of given solutions is clear. However, in the absence of preference information, in multiobjective optimization, there does not exist a unique preference relation to determine if a solution is better than other. The most common preference relation adopted is known as the *Pareto dominance relation* [35], which leads to the best possible trade-offs among the objectives. Thus, by using this relation, it is normally not possible to obtain a single optimal solution (except when there is no conflict among the objectives), but instead, a set of good solutions can be produced. This set is called the *Pareto optimal set* and its image in objective space is known as the *Pareto optimal front*.

Multiobjective optimization involves three stages: model building, search, and decision making (preference articulation). Having a good approximation of the Pareto optimal set does not completely solve a multiobjective optimization problem. The decision maker (DM) still has the task of choosing the most preferred solution out of the approximation set. This task requires preference information from the DM. Following this need, there are several methodologies available for defining how and when to incorporate preferences from the DM into the search process. These methodologies can be classified in the following categories [33, 7]:

1. Prior to the search (*a priori* approaches).
2. During the search (interactive approaches).
3. After the search (*a posteriori* approaches).

Although interactive approaches for incorporating preferences have been widely used for a long time in Operations Research (see e.g., [6, 33]), it was only until very recently that the inclusion of preference information into MOEAs started to attract a considerable amount of interest among researchers (see for example, [7, 2]).

On the other hand, as noted by several researchers [28, 26, 50, 36, 37, 29, 47], the Pareto dominance relation has an important drawback when it is applied to multiobjective optimization problems with a high number of objectives (these are the so-called *many-objective problems*, e.g., [30]). That is, the deterioration of its ability to discern between good and bad solutions as the number solutions increases. A widely accepted explanation for this problem is that the proportion of nondominated solutions (i.e., incomparable solutions according to the Pareto dominance relation) in a population increases rapidly with the number of objectives (see e.g., [1, 20]).

Being aware of the need of integrating MOEAs into interactive methods in many-objective optimization problems, in this paper, we present a new preference relation based on an achievement scalarizing function [53]. The main purpose of the new preference relation is to offer a simple approach to integrate decision maker's preferences into a MOEA without modifying the original structure of the MOEA.

There are other proposed schemes to incorporate user's preferences into a MOEA. However, the proposed preference relation, although can be applied for a general Multiobjective Optimization Problem (MOP), it is specially suited to deal with many-objective problems since it has some particular features: *i*) the location and size of the region of interest can be easily controlled during the search of a MOEA, *ii*) the new relation is scalable with respect to the number of objectives in terms of effectiveness, computational efficiency and amount of information required from the DM. As shown in Section 3, in other preference relations the number of questions asked to the DM depends on the number of objectives, which these techniques difficult to use with many-objectives problems. In addition, in a general sense, our approach successfully overcomes some of the drawbacks of similar methods (Section 3).

The new preference relation divides the objective function space into two subspaces. The solutions in one of these subspaces are compared using the usual Pareto dominance relation, while the others are compared using the achievement scalarizing function. By means of a reference point, the proposed preference relation allows the decision maker to guide the search towards a certain region of the Pareto optimal front. Each component of the reference point represents the aspiration levels that the decision maker requires for each objective. Later on, the new preference relation is embedded into an interactive optimization scheme in which a sample of the current approximation of the Pareto front is presented, at each interaction point, to the DM in order to change the reference point and the size of the region of interest.

Since, by using an achievement scalarizing function, the developed preference relation induces a finer order on vectors of the objective space than that achieved by the Pareto dominance relation, we believe that the use of the new preference relation is a promising approach to deal with many-objective problems. Additionally, by using an interactive optimization technique we can avoid the generation of millions or even billions of nondominated points in many-objective problems.

The main contributions of this work can be summarized as follows:

- A new preference relation to incorporate decision maker's preferences into a MOEA without modifying the original structure of the MOEA.
- A variant of the new preference relation which is able to naturally converge towards the central part of the Pareto front with no need of DM's information.
    - Both variants of the preference relation can be used just by replacing the dominance-checking procedure in a given Pareto-based MOEA.
    - The preference relations proposed are not affected if the DM provides an infeasible reference point. Furthermore, the relations take into account the magnitude by which a solution over- or under-attains the reference point.
    - Since the relations are based on a reference point, unlike other methods, the amount of information required from the DM is low even for more than 3 objectives.
    - In addition, these relations have a lower time complexity than that

of the Pareto dominance relation and other existing relations that perform component-wise comparisons.

- An interactive optimization scheme using the proposed preference relation.
- Experimentation of the interactive scheme using an airfoil design problem with 6 objectives.

The results show that the new preference relation is able to guide the search towards the region defined by the reference point given by the decision maker, even if the reference point is infeasible. In addition, the experiments show that the proposed relation improves notably the convergence ability of a MOEA in problems with a high number of objectives.

The remainder of this paper is structured in the following manner. The next section presents some basic concepts and the notation adopted throughout the paper. Section 3 shortly describes some previous proposals to incorporate preferences into MOEAs. Section 4 introduces the new preference relation and the interactive optimization method. In Section 5 we present the evaluation of the interactive method using three instances of an airfoil shape design optimization problem, namely with 2, 3 and 6 objectives. Finally, in Section 6 we present our conclusions and some potential paths for future research.

2. Basic Concepts and Notation

In this section, we will introduce the concepts and notation that will be used throughout the rest of the paper.

2.1. Multiobjective Optimization Problems

Definition 1 (Multiobjective optimization problem). *A MOP is defined as:*

$$\begin{aligned} \text{“Minimize”} \quad & \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})]^T \\ \text{subject to} \quad & \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{1}$$

The vector $\mathbf{x} \in \mathbb{R}^n$ is formed by n *decision variables* representing the quantities for which values are to be chosen in the optimization problem. The *feasible set* $\mathcal{X} \subseteq \mathbb{R}^n$ is implicitly determined by a set of equality and inequality constraints. The vector function $\mathbf{f} : \mathcal{X} \to \mathbb{R}^k$ is composed by $k \geq 2$ scalar *objective functions* $f_i : \mathcal{X} \to \mathbb{R}$ $(i = 1, \ldots, k)$. In multiobjective optimization, the sets $\mathbb{R}^n$ and $\mathbb{R}^k$ are known as *decision variable space* and *objective function space*, respectively. The image, $\mathcal{Z} = \mathbf{f}(\mathcal{X})$, of $\mathcal{X}$ is referred to as the *feasible set in the objective function space*.

In order to define precisely the multiobjective optimization problem stated in Definition 1 we have to establish the meaning of minimization in $\mathbb{R}^k$. That is to say, we need to define how vectors $\mathbf{z} = \mathbf{f}(\mathbf{x}) \in \mathbb{R}^k$ have to be compared for different solutions $\mathbf{x} \in \mathbb{R}^n$. In single-objective optimization the relation "less than or equal" $(\leq)$ is used to compare the scalar objective values. By using this relation there may be many different optimal solutions $\mathbf{x} \in \mathcal{X}$, but only one optimal value $f^{\min} = \min\{f(\mathbf{x}) \,|\, x \in \mathcal{X}\}$ since the relation $\leq$ induces a total order in $\mathbb{R}$ (i.e., every pair of solutions is comparable, and thus, we can sort solutions from the best to the worst one). In contrast, in multiobjective optimization problems, there is no canonical order on $\mathbb{R}^k$, and thus, we need weaker definitions of order to compare vectors in $\mathbb{R}^k$. In multiobjective optimization, the *Pareto dominance relation* is usually adopted [18, 35].

Definition 2 (Pareto dominance relation). *We say that a vector* $\mathbf{z}^1$ *dominates vector* $\mathbf{z}^2$*, denoted by* $\mathbf{z}^1 \prec_{pareto} \mathbf{z}^2$*, if and only if:*

$$\forall i \in \{1, \ldots, k\} : z_i^1 \leq z_i^2 \;\; \textit{and} \;\; \exists i \in \{1, \ldots, k\} : z_i^1 < z_i^2. \tag{2}$$

Thus, to solve a MOP we have to find those solutions $\mathbf{x} \in \mathcal{X}$ whose images, $\mathbf{z} = \mathbf{f}(\mathbf{x})$, are not dominated by any other vector in the feasible space. It is said that two vectors, $\mathbf{z^1}$ and $\mathbf{z^2}$, are mutually nondominated vectors if $\mathbf{z^1} \not\prec_{\text{pareto}} \mathbf{z^2}$ and $\mathbf{z^2} \not\prec_{\text{pareto}} \mathbf{z^1}$.

Definition 3 (Pareto optimality). *A solution* $\mathbf{x}^* \in \mathcal{X}$ *is Pareto optimal if* *there does not exist another solution* $\mathbf{x} \in \mathcal{X}$ *such that* $\mathbf{f}(\mathbf{x}) \prec_{pareto} \mathbf{f}(\mathbf{x}^*)$.

Definition 4 (ρ-properly Pareto optimality). *A solution* $\mathbf{x}^* \in \mathcal{X}$ *and its corresponding vector* $\mathbf{z}^* \in \mathcal{Z}$ *are ρ-properly Pareto optimal, in the sense of Wierzbicki [53], if*

$$(\mathbf{z}^* - \mathbb{R}^k_\rho \setminus \{0\}) \cap \mathcal{Z} = \emptyset,$$

where $\mathbb{R}^k_\rho = \{\mathbf{z} \in \mathbb{R}^k | \max_{i=1,\dots,k} z_i + \rho \sum_{i=1}^k z_i \geq 0\}$, *and ρ is some scalar.*

Definition 5 (Pareto optimal set). *The Pareto optimal set, P_{opt}, is defined as:*

$$P_{opt} = \{\mathbf{x} \in \mathcal{X} \mid \nexists\, \mathbf{y} \in \mathcal{X} : \mathbf{f}(\mathbf{y}) \prec_{pareto} \mathbf{f}(\mathbf{x})\}. \tag{3}$$

Definition 6 (Pareto front). *For a Pareto optimal set P_{opt}, the Pareto front, PF_{opt}, is defined as:*

$$PF_{opt} = \{\mathbf{z} = (f_1(\mathbf{x}), \dots, f_k(\mathbf{x})) \mid \mathbf{x} \in P_{opt}\}. \tag{4}$$

In practice, the goal of *a posteriori* MOEAs is finding the "best" approximation set of the Pareto optimal front. An approximation set is a finite subset of $\mathcal{Z}$ composed of mutually nondominated vectors and is denoted by PF_{approx}. Currently, it is well accepted that the best approximation set is determined by the closeness to the Pareto optimal front, and the spread over the entire Pareto optimal front [16, 56, 7].

In interactive optimization methods it is useful to know the lower and upper bounds of the Pareto front. The *ideal point* represents the lower bounds and is defined by $z_i^\star = \min_{z \in \mathcal{Z}} z_i \; \forall i = 1, \dots, k$. In turn, the upper bounds are defined by the *nadir point*, which is given by $z_i^{\text{nad}} = \max_{z \in PF_{\text{opt}}} z_i \; \forall i = 1, \dots, k$. In order to avoid some problems when the ideal and nadir points are equal or very close, a point strictly better than the ideal point is usually defined. This point is called the *utopian point* and is defined by $z_i^{\star\star} = z_i^\star - \epsilon \; \forall i = 1, \dots, k$, where $\epsilon > 0$ is a small scalar.

As we mentioned before, Pareto dominance is the most common preference relation used in multiobjective optimization. However, it is only one of the possible preference relations available.

2.2. The Reference Point Approach and the Achievement Scalarizing Function

The proposed preference relation is based on the achievement scalarizing function approach proposed by Wierzbicki [52, 53]. An achievement scalarizing function uses a reference point to capture the desired values of the objective functions.

Definition 7 (Achievement scalarizing function). *An achievement scalarizing function (or achievement function for short) is a parameterized function $s_{\mathbf{z}^{ref}}(\mathbf{z}) : \mathbb{R}^k \to \mathbb{R}$, where $\mathbf{z}^{ref} \in \mathbb{R}^k$ is a reference point representing the decision maker's aspiration levels. Thus, the multiobjective problem is transformed into the following scalar problem:*

$$\begin{aligned} &\textit{Minimize} \quad s_{\mathbf{z}^{ref}}(\mathbf{z}) \\ &\textit{subject to} \quad \mathbf{z} \in \mathcal{Z}. \end{aligned} \tag{5}$$

A common achievement function is based on the Chebyshev distance (L_∞ metric), see e.g., [33, 19].

Definition 8 (Chebyshev distance). *For two vectors $\mathbf{z}^1, \mathbf{z}^2 \in \mathbb{R}^k$ the Chebyshev distance is defined by*

$$d_\infty(\mathbf{z}^1, \mathbf{z}^2) = ||\mathbf{z}^1 - \mathbf{z}^2||_\infty = \max_{i=1,\ldots,k} |z_i^1 - z_i^2|. \tag{6}$$

We will now define an appropriate achievement function.

Definition 9 (Weighted achievement function). *The weighted achievement function (or achievement function for short) is defined by*

$$s_\infty(\mathbf{z}, \mathbf{z}^{ref}) = \max_{i=1,\ldots,k} \{\lambda_i(z_i - z_i^{ref})\} + \rho \sum_{i=1}^{k} \lambda_i(z_i - z_i^{ref}), \tag{7}$$

where $\mathbf{z}^{ref}$ is a reference point, $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_k]$ is a vector of weights such that $\forall i$ $\lambda_i \geq 0$ and, for at least one i, $\lambda_i > 0$, and $\rho > 0$ is an augmentation *coefficient sufficiently small. The main role of ρ is to avoid the generation of weakly Pareto optimal solutions.*

We should note that, unlike the Chebyshev distance, the achievement function does not use the absolute value in the first term. This small difference allows the achievement function to correctly assess solutions that improve the reference point.

The achievement function has some convenient properties over other scalarizing functions. As proved in [45, 33, 19], the minimum of (7) is a Pareto optimal solution and we can find any ρ-properly Pareto optimal solution (see def. 4).

In most of the reference point methods, the exploration of the objective space is made by moving the reference point at each iteration (for example, [31]). In turn, the weights are kept unaltered during the interactive optimization process. That is, weights do not define preferences, but they are mainly used for normalizing each objective function. Usually, the weights are set for all $i = 1, \ldots, k$ as

$$\lambda_i = \frac{1}{z_i^{\mathrm{nad}} - z_i^{\star\star}}. \tag{8}$$

It is important to mention that the DM can provide both feasible and infeasible reference points, or more precisely, $\mathbf{z}^{\mathrm{ref}} \in \mathcal{Z} + \mathbb{R}_+^k$ or $\mathbf{z}^{\mathrm{ref}} \notin \mathcal{Z} + \mathbb{R}_+^k$, where $\mathbb{R}_+^k$ is the nonnegative orthant of $\mathbb{R}^k$. On the one hand, if $\mathbf{z}^{\mathrm{ref}} \in \mathcal{Z} + \mathbb{R}_+^k$, then the minimization of (7) subject to $\mathbf{z} \in \mathcal{Z}$ should represent the maximization of the surplus $\mathbf{z} - \mathbf{z}^{\mathrm{ref}} \in \mathbb{R}^k$. On the other hand, if $\mathbf{z}^{\mathrm{ref}} \notin \mathcal{Z} + \mathbb{R}_+^k$, the minimization of (7) subject to $\mathbf{z} \in \mathcal{Z}$ minimizes the distance between the reference point and the Pareto optimal set.

3. Previous Related Work

In this section we describe similar approaches to incorporate DM's preferences into a MOEA. The descriptions emphasize the drawbacks of those approaches that our method overcomes. Table 1 summarize other proposals and their main features which include the kind of information provided by the DM to articulate preferences, the optimality guaranteed by the method, i.e., (weakly, properly) Pareto optimal solutions. Whether the method is applicable to non-convex Pareto fronts, or whether feasible reference points can be used in the method.

Among the earliest attempts to incorporate preferences in a MOEA, we can find the preferability relation proposed by Fonseca and Fleming [22, 24]. This relation accommodates goal information (equivalent to a reference point in other methods) and priorities in a single preference relation. The DM should define goal values and group objectives according to their priority. Using the preferability relation two solutions are first compared starting with the highest priority group. If the objectives of both solutions meet all their goal values or, contrarily, violate some or all of their goal values in a similar way, the next priority objective group is considered. One disadvantage of this relation is that is affected by the feasibility of the goal provided by the decision maker. If the given goal is far away from the feasible region, then the solutions will be mainly compared in terms of the objective priorities, reducing the relation to the lexicographic relation. In addition, if two solutions either do or do not meet their goals, the relation does not take into account the degree of under- or over-attainment.

Deb [11] proposed a technique to transform goal programming problems into multiobjective optimization problems which are then solved using a MOEA. In goal programming the DM has to assign goals that wishes to achieve for each objective, and these values are incorporated into the problem as additional constraints. Unfortunately, as in the previous method, this approach is sensitive to the feasibility of the goal values. If the goal is contained in the feasible space, it could prevent the generation of a better solution. On the other hand, if the goal is located far away from the feasible space, the effect of the method is practically inexistent.

Branke et al. [5] proposed an approach called Guided MOEA (G-MOEA) which models DM's preferences using the trade-off between pairs of objectives[1].

[1]Here, a trade-off is the amount of units of objective f_i the DM is willing to trade-off in

By setting appropriate trade-offs it is possible to focus the search to any subregion of the Pareto front. The main drawback of this approach is the difficulty to determine the trade-offs as the number of objectives increases, since the DM has to provide $k(k-1)$ trade-offs. Furthermore, this method is only applicable in problems with a convex Pareto front.

Cvetković and Parmee [8, 9] proposed the use of binary preference relations expressed qualitatively (i.e., objectives are classified into "less important" or "don't care" classes). These preferences are translated to quantitative terms (i.e., weights) to guide the search towards certain region of interest of the Pareto front. The weights generated can be used with a simple aggregating approach (i.e., a sum of weights) or with Pareto ranking. There may be some practical issues to take into account if this approach is used interactively, since the DM is asked a considerably high number of questions to make it possible to translate qualitative preferences into quantitative values.

More recently, Deb et al. [15] proposed the Reference-Point-Based NSGA-II (R-NSGA-II). They introduced a modification in the crowding distance operator in order to select from the last nondominated front the solutions that would take part of the new population. They used the Euclidean distance to sort and rank the population accordingly (the solution closest to the reference point receives the best rank). The drawback of this scheme is that it only guarantees weak Pareto optimality since, besides Pareto optimal solutions, the method might generate some weakly Pareto optimal solutions, particularly in MOPs with disconnected Pareto fronts. Likewise, in [38] a preference relation based on Euclidean distance was presented. Using the aspiration levels defined by a reference point, Molina et al. [34] proposed a method that classifies the solutions into two types, namely *i*) those that either fail to satisfy all the aspiration levels or fulfill all of them, and *ii*) solutions that satisfy only some of the aspiration levels. The former type of solutions are preferred over those of the latter type. As a consequence, dominated solutions are preferred over solutions

exchange of one unit of objective f_j, and vice versa.

improving some of the aspiration levels. Another approach was also proposed by Deb and Kumar [13], in which the light beam search procedure [27] was incorporated into the Nondominated Sorting Genetic Algorithm II (NSGA-II) [14]. Similar to R-NSGA-II, DM's preferences are articulated into the crowding operator. This algorithm finds a subset of solutions around the optimum of the achievement function adopting the usual outranking relation[2]. In [27] three kinds of thresholds are defined to determine if one solution outranks another one, namely, indifference, preference, and veto threshold. This relation depends on the crowding comparison operator. In contrast, the new preference relation presented in this work does not depend on external methods, and, therefore, it can be used in any Pareto-based MOEA.

Thiele et al. [48] proposed a variant of the Indicator-Based Evolutionary Algorithm (IBEA) [54], in which preference information is incorporated by means of an achievement scalarizing function. The basic idea is to divide the original indicator value (which is to be maximized) by the achievement value (which is to be minimized). Thus, solutions with a smaller achievement value will be preferred since the modified indicator value is larger. This approach is similar to the one proposed in this paper. However, our approach can be used both in IBEAs and Pareto-based MOEAs. In a further paper, the new IBEA introduced in [48] was used by Figueira et al. [21] in order to approximate the entire Pareto front by defining several reference points.

Recently, Sindhya et al. [41] proposed an interactive framework in which an Evolutionary Algorithm (EA) is used to solve a single-objective optimization problem defined by an achievement function. The weights and the reference point of this function are provided by the DM at each iteration to incorporate his preferences. The drawback of this approach is that only shows a single point to the DM since the MOP is converted into a single-objective problem.

Another recent method is the interactive Decomposition Based Multiobjective Evolutionary Algorithm (MOEA/D) proposed by Gong et al. [25] In this

[2] A vector $\mathbf{z}^1$ outranks vector $\mathbf{z}^2$ if $\mathbf{z}^1$ is considered to be at least as good as $\mathbf{z}^2$.

Technique	**Preference information**	**Optimality**	**Non-convex**	**Feasible $\mathbf{z}^{\mathbf{ref}}$ allowed**
Preferability relation [24]	RP, classes	Pareto†	Yes	No*
Goal Prog. NSGA [11]	RP	Pareto†	Yes	No
G-MOEA [5]	Tradeoffs	Pareto	No	–
Preference MOGA [8]	Classes	Pareto	Yes	–
R-NSGA-II [15]	RP	Weakly	Yes*	Yes
g-dominance [34]	RP	Weakly	Yes	Yes
r-dominance [38]	RP	Weakly	Yes*	Yes
Light Beam NSGA-II [13]	RP, thresholds	Pareto	Yes	Yes
iMOEA/D [25]	RP	Properly‡	Yes	–
Preference IBEA [48]	RP	Properly	Yes	Yes
PIE [41]	RP, classes	Properly	Yes	Yes
Chebyshev relation	RP	Properly	Yes	Yes

RP: Reference point. †When reference point is unfeasible.
‡If Chebyshev achievement function is used. *Preferences might have no effect.

Table 1: Summary of the features of other MOEA with preference incorporation.

method the standard MOEA/D is initially applied. Later, at each interaction stage, the DM selects the most preferred solution and the weight vectors are redistributed inside a circular neighborhood around the selected solution. Unfortunately, in this approach is not possible to revisit regions of interest.

4. Chebyshev Relation to Guide the Search

Our preference relation was designed with two goals in mind. First, we aimed to provide an easy way to integrate preferences into different types of MOEAs requiring only slight modifications to their structure. The second goal was to investigate the use of achievement functions when dealing with many-objective problems.

In the following sections, we introduce the new preference relation and its use as an interactive technique for multi- and many-objective optimization.

4.1. User Reference Point Chebyshev Preference Relation

In this paper, we propose combining the Pareto dominance relation and the achievement function to compare solutions in objective function space. The achievement function will allow the incorporation of DM's preferences using a feasible or an infeasible reference point.

We can easily define a simple preference relation using the achievement function. For example, we could say that a vector $\mathbf{z}^1$ will be preferred to $\mathbf{z}^2$ if and only if $s_\infty(\mathbf{z}^1, \mathbf{z}^{\text{ref}}) < s_\infty(\mathbf{z}^2, \mathbf{z}^{\text{ref}})$. However, by doing so, we would obtain only one Pareto optimal solution, which we will denote by $\mathbf{z}^\star_\infty = \arg\min_{\mathbf{z}\in\mathcal{Z}} s_\infty(\mathbf{z}, \mathbf{z}^{\text{ref}})$. In order to find a set of solutions around the point $\mathbf{z}^\star_\infty$, we will allow a threshold, δ, in the preference relation. That is, we want to find the set of points, $\mathbf{z}$, such that $s_\infty(\mathbf{z}, \mathbf{z}^{\text{ref}}) \leq s^{\min} + \delta$, where $s^{\min} = \min_{\mathbf{z}\in\mathcal{Z}} s_\infty(\mathbf{z}, \mathbf{z}^{\text{ref}})$ (or in different terms, $s(\mathbf{z}^\star_\infty, \mathbf{z}^{\text{ref}})$). All these points would be located in the dark region shown in Figure 1.

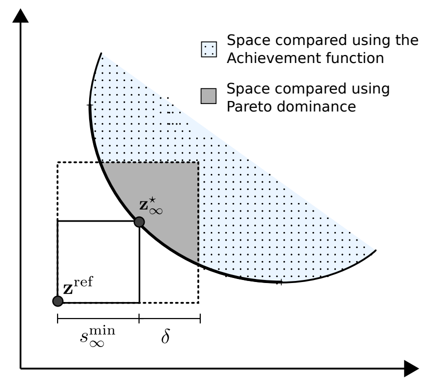


Figure 1: Illustration of how the objective space is divided, and how the vectors in each subspace are compared.

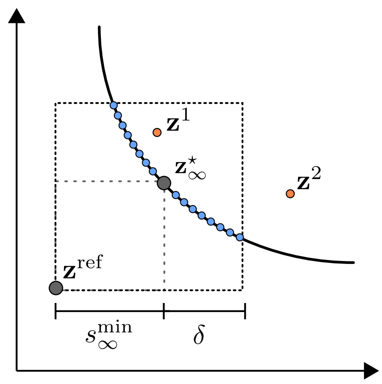


Figure 2: Nondominated solutions with respect to the Chebyshev relation.

Nevertheless, as we can see in the figure, we obtain both Pareto and dominated solutions. In order to obtain exclusively Pareto solutions we compare those solutions using the Pareto dominance relation. By doing so, only the Pareto nondominated solutions in the square region shown in Figure 2 are considered as the nondominated solutions with respect to the new preference relation developed. In some sense, we can consider that the new relation divides the feasible objective space in two parts as can be seen in Figure 1. The larger part of the feasible objective space is compared with the achievement function, while the remainder of the space is compared adopting the usual Pareto dominance relation. For the sake of simplicity, we will refer to this new relation as the *Chebyshev preference relation.* Now, we can give a formal definition of the Chebyshev preference relation.

Definition 10 (Chebyshev preference relation). *A solution* $\mathbf{z}^1$ *is preferred to solution* $\mathbf{z}^2$ *with respect to the Chebyshev relation (*$\mathbf{z}^1 \prec_{cheby} \mathbf{z}^2$*), if and only if:*

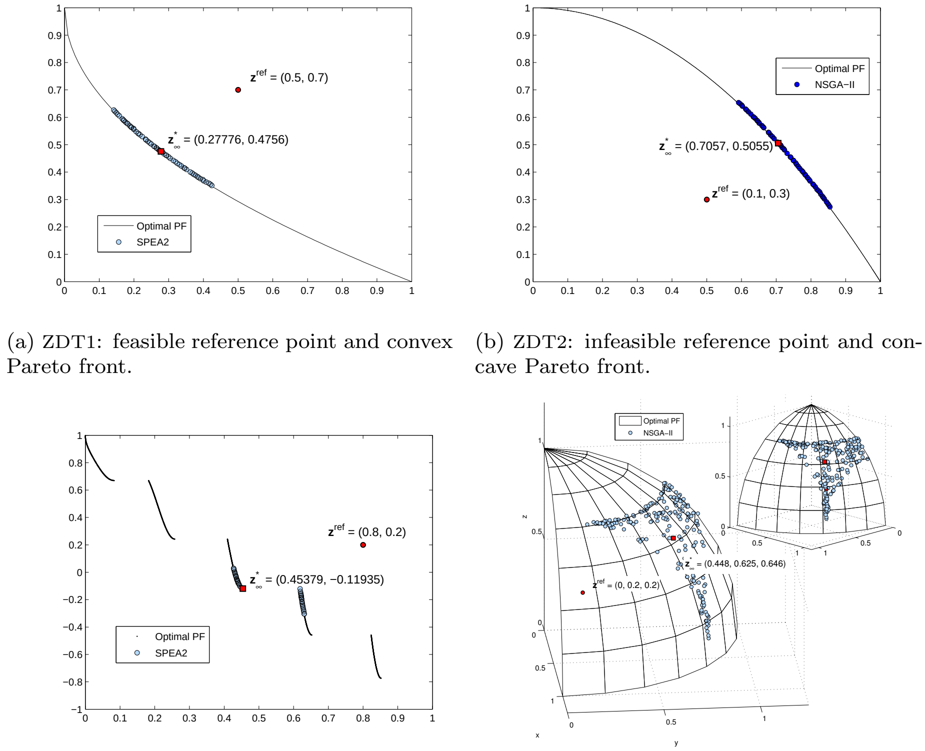
1. $s_\infty(\mathbf{z}^1, \mathbf{z}^{ref}) < s_\infty(\mathbf{z}^2, \mathbf{z}^{ref}) \;\wedge\; \{\mathbf{z}^1 \notin N(\mathbf{z}^{ref}, \delta) \vee \mathbf{z}^2 \notin N(\mathbf{z}^{ref}, \delta)\}$*, or,*
2. $\mathbf{z}^1 \prec_{pareto} \mathbf{z}^2 \wedge \{\mathbf{z}^1, \mathbf{z}^2 \in N(\mathbf{z}^{ref}, \delta)\}$*,*

where $N(\mathbf{z}^{ref}, \delta) = \{\mathbf{z} \,|\, s_\infty(\mathbf{z}, \mathbf{z}^{ref}) \leq s^{\min} + \delta\}$*. That is, the set* $N(\mathbf{z}^{ref}, \delta)$ *is composed of vectors with an achievement value better than* $s^{\min} + \delta$ *with respect to the vector of aspiration levels* $\mathbf{z}^{ref}$*.*

As an illustration of the preference relation, consider solutions $\mathbf{z}^1$ and $\mathbf{z}^2$ presented in Figure 2. Since $\mathbf{z}^2 \notin N(\mathbf{z}^{\mathrm{ref}}, \delta)$ and $s_\infty(\mathbf{z}^1, \mathbf{z}^{\mathrm{ref}}) < s_\infty(\mathbf{z}^2, \mathbf{z}^{\mathrm{ref}})$, then $\mathbf{z}^1 \prec_{\mathrm{cheby}} \mathbf{z}^2$.

Figure 3 shows the use of the Chebyshev preference relation in NSGA-II [14] and Strength Pareto Evolutionary Algorithm 2 (SPEA2) [55]. As we can see in Figure 3, unlike some distance metrics, the achievement function (Eq. (7)) allows a MOEA to find points in problems with nonconvex Pareto fronts. Moreover, the figure shows how the DM can provide both feasible and infeasible reference points. Also, we have to note the result obtained in problem DTLZ2. If we had used the Euclidean distance to define the preference relation, with $\mathbf{z}^{\mathrm{ref}} = 0$ we had obtained nondominated solutions over the entire Pareto front. The reason for this, is that all the vectors in DTLZ2's Pareto optimal front are situated on a sphere of radius 1.

In order to incorporate the Chebyshev relation into the two previously mentioned MOEAs we only have to change the usual Pareto dominance checking procedure by the function that implements the new relation. In order to have an efficient procedure, the evaluation of the achievement function was computed



(a) ZDT1: feasible reference point and convex Pareto front.

(b) ZDT2: infeasible reference point and concave Pareto front.

(c) ZDT3: feasible reference point and non-convex Pareto front.

(d) DTLZ2: infeasible reference point and concave Pareto front.

Figure 3: Illustration of the Chebyshev preference relation incorporated into NSGA-II and SPEA2, using feasible and infeasible reference points. $\mathbf{z}^{\star}_{\infty}$ is the optimum of the achievement function with respect to the current population of the MOEA. In all the examples, we used a threshold $\delta = 0.2$.

and stored for each solution before each ranking process. This way, the comparisons required to rank the current population use the stored values of the achievement function.

In practice, it might be difficult to set a value for the parameter δ since it does not have an upper bound that is known *a priori*. In order to have a better control of this parameter during the search, we can set it in terms of the proportion of the current range of the achievement function values (namely, the difference between the minimum and maximum achievement with respect to a given solution set P). If $\tau \in [0,1]$ is that proportion, then $\delta = \tau \cdot (s^{\max} - s^{\min})$, where $s^{\max} = \max_{\mathbf{z}\in P} s_\infty(\mathbf{z},\mathbf{z}^{\mathrm{ref}})$ and $s^{\min} = \min_{\mathbf{z}\in P} s_\infty(\mathbf{z},\mathbf{z}^{\mathrm{ref}})$. As a

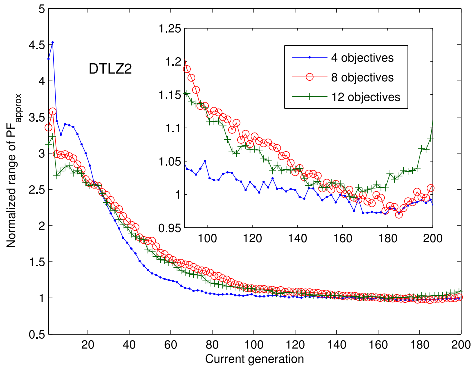


Figure 4: Ratio of the extension of PF_{approx} and PF_{opt} in terms of the Chebyshev distance in problem DTLZ2.

consequence, for $\tau = 0$ we would only find the minimum of the achievement function, whereas if $\tau = 1$, then we would get the usual Pareto dominance relation since for every solution $\mathbf{z} \in P$, $\mathbf{z} \in N(\mathbf{z}^{\text{ref}}, \delta)$. The idea of the parameter τ is to determine the size of the region of interest in terms of the extension of the current approximation set. Usually, in the first iterations the extension of the approximation set is wide, however even in that case, the value of τ is useful to bound the exploration area. On the other hand, as the search progresses, the extension of the approximation set tends to equal the extension of the Pareto optimal front. That means that when the approximation of the Pareto front is close to the Pareto optimal front, the value of τ represents the extension of the Pareto optimal front. Figure 4 shows the ratio of the extension of PF_{approx} and PF_{opt} in terms of the achievement function value. As can be seen, after generation 100, the extension of PF_{approx} is almost equal to that of the Pareto optimal front.

In that way, for example, if the DM sets $\tau = 0.2$, then the region of interest will cover around 20% of the extension of the Pareto optimal front. Therefore, the DM can use the value of τ for adjusting the size of the region of interest around solution $\mathbf{z}^{\star}_{\infty}$. In our approach, the values of the weight vector, $\boldsymbol{\lambda}$, that appears in Eq. (5) are set according to Eq. (8). The vectors $\mathbf{z}^{\star\star}$ and $\mathbf{z}^{\text{nad}}$ are approximated using the current PF_{approx} achieved by the MOEA.

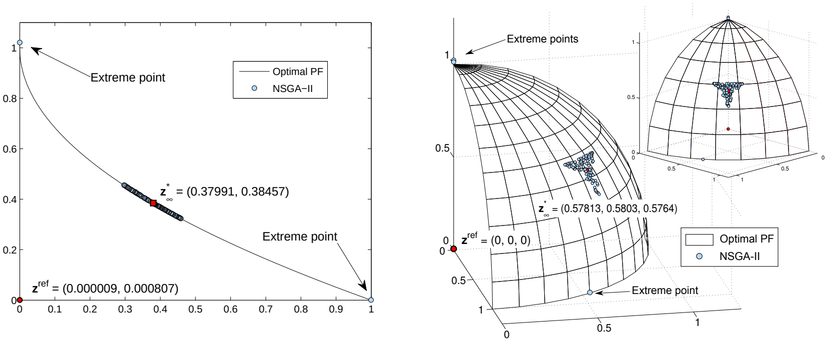
4.2. Central-guided Chebyshev Preference Relation

As previously mentioned, in many-objective problems the number of points needed to represent a Pareto front accurately grows exponentially with the number of objectives. Therefore, in many cases trying to approximate the whole Pareto front is not convenient. Additionally, in a many-objective context it might be very difficult for the DM to select a final solution.

When the DM does not have any knowledge about the MOP to be solved (e.g., trade-offs among the objectives, variation range of the objectives), a good idea might be to aim to converge to the ideal point, in which all the objectives are minimized simultaneously. In some cases, the solution that minimizes the distance to the ideal point is located in the central part of the Pareto front. If the Pareto front is symmetric, the closest solution to the ideal point is equivalent to the so-called knee of the front [10, 32, 3, 40]. In some cases the Pareto front is not symmetric, and therefore is not clear to define a central part of the front. Nonetheless, for the sake of brevity, we will use the term 'central part' to denote the region around the nearest solution to the ideal point.

In order to achieve the desired behavior we need to approximate the ideal point during the search process of the MOEA. To do so, we will use the lower bounds of the current approximation of the Pareto front. At each iteration we will determine one of the vectors that minimizes each objective separately. That is, we need to find the set of k vectors in PF_{approx}, $\Phi = \{\mathbf{z}^1, \ldots, \mathbf{z}^k \mid \mathbf{z}^i = \mathbf{f}(\mathbf{x}_i^\star), i = 1, \ldots, k\}$, where $\mathbf{x}_i^\star$ yields the minimum in objective $f_i(\mathbf{x})$.

There are some works, in which an evolutionary algorithm has been used to approximate the ideal point [40] or the nadir point [12]. Nonetheless, these approaches require a modification in a particular component of the MOEA (for instance, in the crowding operator or in the archive). In order to maintain the preference relation independent of an external module, e.g., an archive, we propose to modify the Chebyshev relation to implicitly maintain the extreme or boundary solutions Φ. To this end, besides emphasizing the points close to the



(a) ZDT1 (b) DTLZ2

Figure 5: Illustration of the central-guided Chebyshev preference relation incorporated into NSGA-II. In these plots $\mathbf{z}^{\text{ref}}$ is the approximation of the ideal point, and $\mathbf{z}^{\star}_{\infty}$ is the vector that we consider the central point of the Pareto front. In these examples we used $\tau = 0.1$.

central part of the Pareto front, the relation does not allow that extreme points are dominated.

Definition 11 (Central-guided Chebyshev preference relation). *A solution $\mathbf{z}^1$ is preferred to solution $\mathbf{z}^2$ with respect to the central-guided Chebyshev preference relation ($\mathbf{z}^1 \prec_{c\text{-}cheby} \mathbf{z}^2$) if and only if:*

$$\mathbf{z}^1 \prec_{cheby} \mathbf{z}^2, \;\; \textit{and} \;\; \mathbf{z}^2 \notin \Phi.$$

Figure 5 shows the Pareto front approximation obtained by NSGA-II using the central-guided Chebyshev preference relation with the approximated ideal point as a reference point. The figure shows the extreme points of problems ZDT1 and DTLZ2. It is worth noting that in both problems, the approximation of the ideal points is very accurate. Later in this section, we will quantitatively evaluate the accuracy of the approximation of the ideal point.

This variant of the proposed preference relation might be very useful in many-objective problems in which traditional visualization techniques, such as 2D or 3D plots, are no longer available. In this case, the DM can be assisted by the preference relation to find a set of solutions around the (usually) most interesting region of the Pareto front.

In order to assess the accuracy of the approximation of the ideal point yield by the central-guided Chebyshev relation we will compare the true ideal point against the obtained approximation. In these experiments, we use the Euclidean distance between the ideal point and its approximation as a measure of the error of the approximation. In these experiments we adopted the problems DTLZ2 and DTLZ7. The ideal point of the former is the origin of $\mathbb{R}^k$, while for the latter, $z_i^\star = 0$ for $i = 1, \ldots, k-1$ and $z_k^\star = \min f_k(\mathbf{x})$, where $x_i \in [0, 1]$, for $i = 1, \ldots, n$. The error was measured along the 200 generations of the search. Figure 6 shows the mean of the error over 30 runs using problem DTLZ2 with 4, 8, and 12 objectives. As we can see, after generation 20, the error is clearly below 0.005, which is a very good approximation.

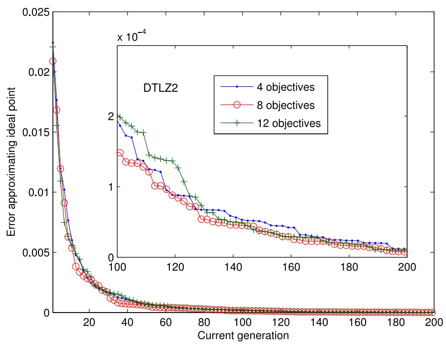


Figure 6: Distance between the ideal point and its approximation using the central-guided Chebyshev relation in problem DTLZ2.

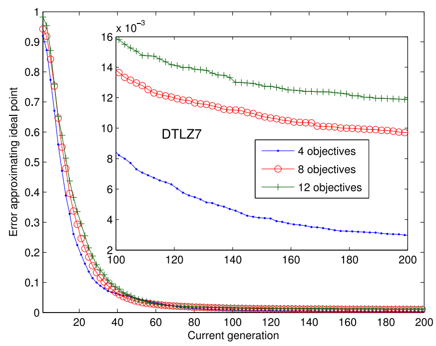


Figure 7: Distance between the ideal point and its approximation using the central-guided Chebyshev relation in problem DTLZ7.

The mean and the standard deviation of the error at generation 100 is shown at Table 2. From those results we can say that after generation 100, the relation uses a very good approximation of the ideal point.

# objs.	Mean	Std. Deviation
4	2.0133e-04	2.1932e-04
8	1.4873e-04	1.8505e-04
12	2.0693e-04	2.7470e-04

Table 2: Statistics at generation 100 of the distance between the ideal point and its approximation using the central-guided Chebyshev relation in problem DTLZ2.

# objs.	Mean	Std. Deviation
4	0.0086	0.0015
8	0.0139	0.0026
12	0.0160	0.0030

Table 3: Statistics at generation 100 of the distance between the ideal point and its approximation using the central-guided Chebyshev relation in problem DTLZ7.

The results of the error of the approximation of the ideal point for problem DTLZ7 are shown in Figure 7 and Table 3. From those results it is clear to see that the ideal point of DTLZ7 is harder to approximate than in the case of DTLZ2. However, after generation 40 the error is below 0.1, which is a useful approximation to guide the search towards the ideal point. We have to point out that, although the ideal point is approximated very well, even if the approximation is far from the ideal point, when τ reaches a value of 1, the whole Pareto front can be generated.

One of the advantages of the basic Chebyshev preference relation and the central-guided variant over other preference relations is their low time complexity. The evaluation of the achievement function for the entire population has complexity $O(km)$, where m is the size of the population and k is the number of objectives. Regarding the central-guided variant, the process of finding the extreme points has complexity $O(km)$. Therefore, the total process of the central-guided variant also has complexity $O(km)$. In order to illustrate the computational savings using the Chebyshev relation, let us take as an example, the ranking procedures of NSGA-II and Multiobjective Genetic Algorithm (MOGA) [24]. Both NSGA-II's nondominated sorting [44, 14] and MOGA's nondominated ranking [23] have complexity $O(km^2)$ using the Pareto dominance relation. Using any of the Chebyshev relations we need to compare a single real value instead of a k-dimensional vector for each pair of solutions. Therefore, using the Chebyshev relation, these ranking procedures have complexity $O(km + m^2)$. Figure 8 shows the complexities of the ranking procedures using the Pareto relation, and any of the Chebyshev relations, respectively. In this discussion we have assumed that the entire population is exclusively compared using the achievement function. In practice, however, the actual complexity depends on the proportion of solutions compared using the achievement function and the usual Pareto dominance relation. Nonetheless, for small values of τ and a set of points evenly distributed over the objective space, the resulting complexity is similar to the one defined above. For instance, if $\tau = 0.1$, then approximately 90% of the population is compared using the Chebyshev relation.

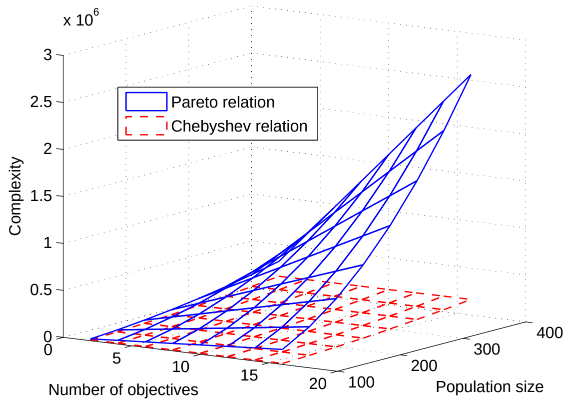


Figure 8: Plots of the complexity of NSGA-II and MOGA's ranking procedures using the Pareto dominance relation ($O(km^2)$), and the Chebyshev relation ($O(km + m^2)$).

4.3. An Interactive Method Using the Chebyshev Relation

The two variants of the Chebyshev preference relation can be used in an interactive way. When the DM does not have enough knowledge about the problem to provide a reference point, the central-guided Chebyshev relation can be used to obtain a first set of solutions. However, in real situations it is common that the DM counts with a previous best known solution of the given problem. In that case, the previous solution can serve as a good reference point. Then, the process can follow the usual steps of the interactive techniques. That is, at each iteration the DM must provide new aspiration levels in the form of a reference point. Additionally, the DM can change the value of the threshold τ that controls the size of the region of interest. Initially, a default value $\tau = 1$ can be used to obtain a first approximation of the entire Pareto front. Later, the user can, for example, set $\tau = 0.5$ in order to obtain about half of the Pareto front around the reference point. This helps the DM to know the trade-offs among the objectives. At subsequent iterations, the value of τ could be reduced to concentrate the search towards a region of interest chosen by the DM. In order to show the set of solutions of the region of interest, some visualization tool designed for problems with more than three objectives could be used, such as parallel coordinates plots, heatmap graphs, or scatter plots (see e.g., [4]). To ease the visualization of the solutions, a technique for truncating the approximation set can be used. For example, a clustering technique can be employed, such as the one used in SPEA2 [55], or a technique similar to that used in archiving methods. Therefore, the interactive process requires an additional parameter indicating the number of solutions to visualize. If the number of nondominated solutions found by the MOEA is lower than the number requested by the DM, then all solutions are visualized.

In a next step, the employed MOEA is again executed using the Chebyshev relation in order to find a new set of solutions that best satisfies the aspirations of the DM. This process continues until the DM is satisfied with a solution of the current set of solutions. Algorithm 1 shows the whole interactive process. The user also has to decide the parameter values for the MOEA employed, for instance the number of generations to stop each search of Step 2.

Algorithm 1 Interactive technique using the Chebyshev preference relation.

Step 1: Ask the DM to specify the threshold τ.
If the DM has some knowledge about the problem, he/she can provide a reference point. Otherwise, the central-guided preference relation can be used to converge towards the ideal point.

Step 2: **If** a reference point was provided, **then**
  Execute the MOEA using the Chebyshev relation with the reference point provided by the decision maker.
else
  Execute the MOEA using the central-guided Chebyshev relation.

Step 3: Ask the DM to define how many solutions of the current approximation should be shown.
Additionally, from the use of the central-guided relation the DM can be informed of the current ideal point in order to decide the new aspiration levels.

Step 4: **If** the DM is satisfied with some solution of the current set, **then**
  STOP.
else
  Go to **Step 1**.

5. Experiments using the interactive method

In order to illustrate the interactive method presented in the previous section we will use three variants of a multiobjective aerodynamic airfoil shape optimization problem adapted from [46]. An airfoil is the cross-section of a lifting surface such as an airplane's wing. The goal in this set of problems is to optimize the shape of a standard-class glider, aiming at obtaining optimum performance for a sailplane at different flight conditions. We experiment with problems with 2, 3, and 6 objectives.

5.1. Geometry parameterization

In all the variants of the aerodynamic airfoil shape optimization problems, we adopt a modified PARSEC airfoil representation [42]. Figure 9 illustrates the 12 basic parameters used for this representation: $r_{\mathrm{le_{up}}}$ / $r_{\mathrm{le_{lo}}}$ leading edge radius for upper/lower surfaces, $X_{\mathrm{up}}/X_{\mathrm{lo}}$ location of maximum thickness for upper/lower surfaces, $Z_{\mathrm{up}}/Z_{\mathrm{lo}}$ maximum thickness for upper/lower surfaces, $Z_{xx\mathrm{up}}/Z_{xx\mathrm{lo}}$ curvature for upper/lower surfaces, at maximum thickness locations, Z_{te} trailing edge coordinate, ΔZ_{te} trailing edge thickness, α_{te} trailing edge direction, and β_{te} trailing edge wedge angle. The PARSEC geometry representation adopted allows us to define independently the leading edge radius, both for upper and lower surfaces (the original representation uses the same value both for upper and lower surfaces). Thus, 12 variables are used in total. We employed two different instances of the problem A720 presented in [46] (for the 2- and 3-objectives problem), and one instance of problem NLF0416 described in [51] (for the 6-objectives problem). Their allowable ranges are defined in Table 4.

The PARSEC airfoil geometry representation uses a linear combination of shape functions for defining the upper and lower surfaces. These linear combinations are given by:

$$Z_{\mathrm{upper}} = \sum_{n=1}^{6} a_n x^{(n-1)/2}, \quad Z_{\mathrm{lower}} = \sum_{n=1}^{6} b_n x^{(n-1)/2}. \tag{9}$$

Table 4: Parameter ranges for the PARSEC airfoil representation for problems A720 (2 and 3 objs.) and NLF0416 (6 objs.).

	A720		**NLF0416**	
Variable	**Lower**	**Upper**	**Lower**	**Upper**
$r_{\mathrm{le_{up}}}$	0.0085	0.0126	0.0055	0.0215
$r_{\mathrm{le_{lo}}}$	0.0020	0.0040	0.0055	0.0215
α_{te}	7.0000	10.0000	-2.0000	21.0000
β_{te}	10.0000	14.0000	1.0000	15.0000
Z_{te}	-0.0060	-0.0030	-0.0200	0.0200
ΔZ_{te}	0.0025	0.0050	0.0000	0.0000
X_{up}	0.4100	0.4600	0.2875	0.5345
Z_{up}	0.1100	0.1300	0.0880	0.1195
$Z_{xx_{\mathrm{up}}}$	-0.9000	-0.7000	-1.0300	-0.4200
X_{lo}	0.2000	0.2600	0.3060	0.5075
Z_{lo}	-0.0230	-0.0150	-0.0650	-0.0500
$Z_{xx_{\mathrm{lo}}}$	0.0500	0.2000	-0.0490	0.8205

The coefficients a_n, and b_n are determined as function of the 12 geometric parameters by solving two systems of linear equations, one for each surface. It is important to note that the geometric parameters $r_{\mathrm{le_{up}}}/r_{\mathrm{le_{lo}}}$, $X_{\mathrm{up}}/X_{\mathrm{lo}}$, $Z_{\mathrm{up}}/Z_{\mathrm{lo}}$, $Z_{xx_{\mathrm{up}}}/Z_{xx_{\mathrm{lo}}}$, Z_{te}, ΔZ_{te}, α_{te}, and β_{te} are the actual design variables in the optimization process. In turn, coefficients a_n, b_n serve as intermediate variables for interpolating the airfoil's coordinates, which are used by the Computational Fluid Dynamics (CFD) solver (we used the Xfoil tool proposed in [17]) for its discretization process.

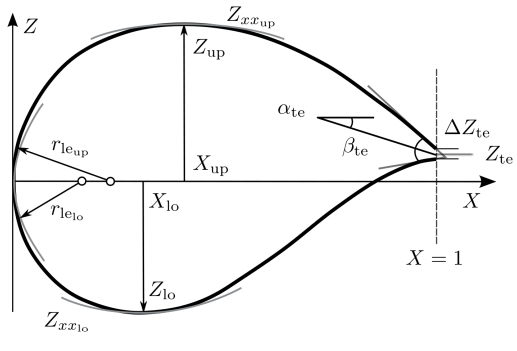


Figure 9: PARSEC airfoil parametrization.

5.2. Airfoil Shape Problem with 2 Objectives

The goal in the 2-objective problem is to optimize the shape of a standard-class glider, aiming at obtaining optimum performance for a sailplane.

5.2.1. Objective functions

Two conflicting objective functions are defined in terms of a sailplane average weight and operating conditions [46]:

1. Min $f_1 = C_D/C_L$, s.t. $C_L = 0.63, Re = 2.04 \times 10^6$, $M = 0.12$.
2. Min $f_2 = C_D/C_L^{3/2}$, s.t. $C_L = 1.05, Re = 1.29 \times 10^6, M = 0.08$.

Objective f_1 represents the inverse of the glider's gliding ratio, whereas f_2 represents the sink rate. Both objectives are important performance measures for this aerodynamic optimization problem. Each objective is evaluated at different prescribed flight conditions, given in terms of Mach (M) and Reynolds (Re) numbers, and the drag and lift coefficients, denoted by C_D and C_L respectively. The aim of solving this MOP is to find a better airfoil shape, which improves a reference design.

Next, we will show a simulation of the interactive process using NSGA-II with the Chebyshev relation using a reference point given by the DM. We adopted the following parameters for NSGA-II: a crossover probability of 0.9, a mutation probability of $1/n$ (n is the number of decision variables), and the distribution indices for crossover and mutation were set as 15 and 20, respectively. A population composed of 60 individuals was employed. It is worth mentioning that the evaluation of the objective functions is very expensive in terms of processing time. A run for the 6-objective problem with 60 individuals and 80 generations (4800 evaluations) required around 9 hours using a processor running at 2.67GHz.

In all the experiments included in this paper, we used $\rho = 10^{-5}$ for Eq. (7). In the first step of the process, we used $\tau = 0.8$ in order to get a global perspective of the entire Pareto front. As a reference point we employed the vector $\mathbf{z}^{\text{ref}} = [0.007610, 0.005236]$. This reference point corresponds to the evaluation of the reference airfoil shape A720 [46] in both objectives. Then, NSGA-II was executed for 15 generations. The resulting approximation set is shown in Figure 10 (denoted by triangles). As can be seen, the reference point was dominated by almost all solutions in the approximation set. This illustrates how the relation is able to correctly compare solutions better than the reference point provided. On the other hand, due to the nature of the objective space of the problem, only 25 solutions, from the total of 60, are nondominated. Therefore, in this case, the clustering technique to reduce the size of the approximation set was not needed.

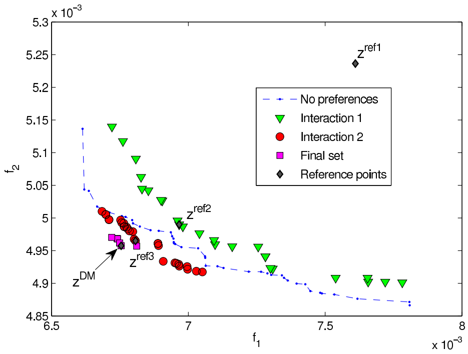


Figure 10: Approximation sets obtained during the simulation of the interactive method applied on the 2-objective problem A720.

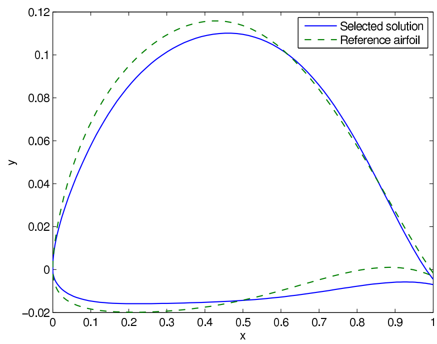


Figure 11: Most preferred airfoil from the simulation of the interactive method applied on the 2-objective problem A720.

Since the initial reference point was improved, we decided to choose one solution of the approximation set as the next reference point, namely, the nearest solution to the ideal point (diamond). For the next execution, the region of interest was reduced to $\tau = 0.2$. Similar to the previous DM interaction, the next reference point was the nearest solution of PF_{approx} to the ideal point. In order to obtain a final approximation to select the most preferred solution, the region of interest was reduced to a small region using $\tau = 0.05$. This time, NSGA-II was executed for 40 generations. At this stage only 8 solutions were obtained and the most preferred solution for the DM was the one with objective values $[0.006754, 0.004957]$. Figure 11 shows the airfoils corresponding to the initial reference point and to the most preferred solution. In this example, an improvement of approximately 11.24% and of 5.32% was attained for the first and second objective, respectively. From a practical point of view, these improvements are quite significant in increasing the aerodynamic efficiency of the sailplane.

Figure 10 also shows the PF_{approx} achieved by NSGA-II with no preferences during the same number of generations than that used in the interactive method. As one can expect, the final approximation set obtained articulating preferences is closer to the ideal point than the one generated with no preferences. This can be explained by the fact that the incorporation of preferences concentrates all the function evaluations to improve the region of interest. On the other hand, when the task is to approximate the entire Pareto front, some function evaluations are used to approximate regions outside the region of interest. These are clearly different tasks, and therefore, a fair performance comparison is not possible. Nonetheless, we want to emphasize the computational savings of using an interactive approach over an *a posteriori* approach, specially when the function evaluations are expensive in terms of CPU time.

5.3. Airfoil Shape Problem with 3 Objectives

In this section the interactive method is evaluated using the airfoil shape optimization problems with 3 objectives. Unlike the previous example, in this case we will simulate the DM using the Chebyshev achievement function. That is to say, at each interaction point, the new reference point will be the solution in the current PF_{approx} with the best achievement value (which is to be minimized). To solve this problem we used 4 interaction points with the DM during the search using a total of 100 generations. The parameters at each interaction point are shown in Table 5. The initial threshold for both problems was set to $\tau = 0.8$.

In order to evaluate the performance of the interactive method we carried out 30 runs of the interactive method. As for NSGA-II, we adopted the same parameter values used in the 2-objective problem. For each run, the best achievement value of the final PF_{approx} was measured. As a reference, we also computed the best achievement value obtained by NSGA-II with no preferences. The 3-

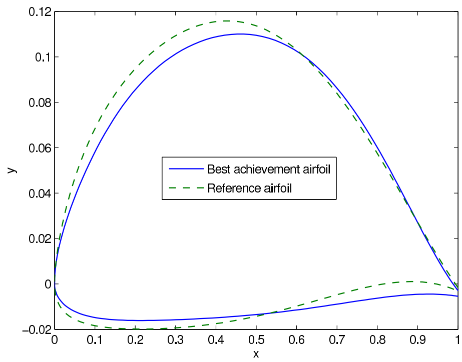


Figure 12: Airfoil with the best achievement value and the reference airfoil for the problem with 3 objectives.

objective problem is a variant of the problem A720 in which the first and third objectives are objectives f_1 and f_2 of the 2-objective problem of the previous section. The second objective is defined as

- Min $f_2 = C_D/C_L$, s.t. $C_L = 0.86, Re = 1.63 \times 10^6$, $M = 0.1$.

The bounds for the variables are the same described in Table 4. For this problem, we used the vector [0.007610, 0.005895, 0.005236] as our initial reference point. The results for the 3-objective problem are shown in Table 6. As can be seen, both approaches yield achievement values results less than zero, which means that the reference point was improved in all cases. In addition, as expected, the interactive approach obtained better results than the approach with no preferences articulated. The solution with the best achievement value was $[0.006772, 0.005244, 0.004960]$. Objectives were improved by 11.01%, 11.04% and 5.27%, respectively. The airfoil of this solution is presented in Figure 12, along with that of the reference point.

Table 5: Parameter values at each interaction point.

Problem		**Int. 1**	**Int. 2**	**Int. 3**	**Int. 4**
3-obj	Gen	15	35	55	80
	τ	0.5	0.2	0.1	0.025
6-obj	Gen	15	35	55	–
	τ	0.43	0.18	0.025	–

Table 6: Statistics of the achievement function values obtained with preferences and without them in the 3-objective problem.

	Best	**Median**	**Worst**	**Std. dev.**
Preferences	-0.2196	-0.2111	-0.1982	0.0047
No prefs.	-0.2183	-0.2020	-0.1816	0.0101

5.4. Airfoil Shape Problem with 6 Objectives

The 6-objective airfoil optimization problem presented in this section was taken from [51]. The goal of this problem is to optimize the airfoil shape of a low-speed unmanned aerial vehicle to cover a range of different flight condition (e.g., take-off and cruise). The 6 objectives to be minimized are described in Table 7, whereas the bounds for the variables are presented in Table 4.

First, we will present the simulation of the interactive process in order to see how the DM might guide the search in a problem with more than 3 objectives. In this experiment we used a population composed of 40 solutions. However, the DM can decide to visualize a lower number of solutions. As reference point we employed a representative airfoil of the NLF series [49], namely the NLF0416 [43], $\mathbf{z}^{\mathrm{ref}} = [0.00523, 0.00595, 0.01048, 0.33373, 0.90135, 2.93083]$. Given the low number of function evaluations used in this simulation (4000 using 40 solutions during 100 generations), we do not expect to improve an airfoil such as NLF0416, which was specially designed for real aerial missions. Nonetheless, the experiment is a good example to show the simplicity of the interactive method even in problems with a large number of objectives.

In Step 1, a value of $\tau = 1$ was adopted, i.e., an approximation of the entire Pareto front. After 45 generations, 36 solutions are presented to the DM. Those solutions were selected using a clustering technique. The set of solutions are presented in the parallel coordinate plot shown in Figure 13. In this plot, the objective values are normalized with respect to the minimum and maximum values obtained in each objective. The closest generated solution to the reference point is $[0.0057, 0.0054, 0.0236, 0.4230, 0.7306, 8.0836]$.

Since the third objective is far from being achieved, the DM decides to relax the aspiration level of that objective in the hope of improving the others. The new reference point is then [0.0055, 0.0059, 0.02, 0.333738, 0.901354, 2.9308]. In addition, the value of τ is reduced to 0.2. After 35 more generations, the set of solutions shown in Figure 14 are presented to the DM. This time, the DM decides that 20 solutions are enough. The closest generated solution to the reference point is [0.0057, 0.0053, 0.0243, 0.3756, 0.7144, 6.5218]. As can be noted, all the objectives, except for objective 3, were improved.

For the next and last optimization phase, NSGA-II is executed for 20 generations (a total of 100 generations). The same reference point is used. However, the value of τ is reduced to 0.05 to find solutions very close to the reference point in order to choose a final solution. The final set of solutions is presented in Figure 15. Only 8 solutions are presented to the DM. Finally, the DM selects the closest generated solution to the reference point, i.e., [0.0057, 0.0053, 0.0260, 0.3326, 0.6472, 2.7616]. The airfoil shape corresponding to the preferred solution is shown in Figure 16.

Besides the parameters of the MOEA, the parameters that have to be selected by the DM are the reference point, τ, and the number of solution to be visualized. We consider that selecting a new reference point is an easy task to the DM since its interpretation is intuitive. In turn, the parameter τ can be easily set since it is given in terms of the current range approximation of the Pareto front.

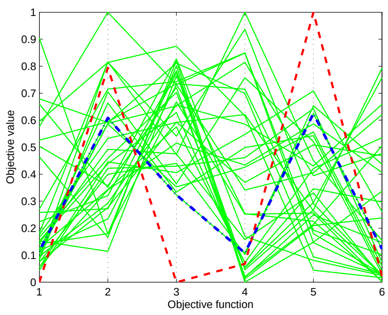


Figure 13: Set of solutions presented to the DM after 45 generations for the 6-objective problem. The dashed line is the reference point, and the dash-dot line is the closest solution to the reference point.

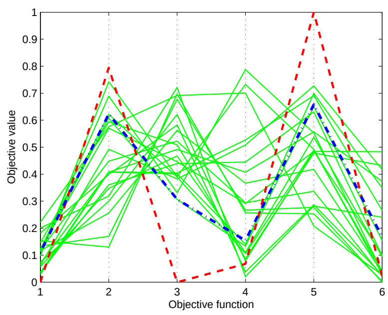


Figure 14: Set of solutions presented to the DM after 80 generations for the 6-objective problem. The dashed line is the reference point, and the dash-dot line is the closest solution to the reference point.

Table 7: Objectives of the airfoil design problem with 6 objectives (to be minimized).

Objective	**Comments**
$f_1 = C_D$	$C_L = 0.5, \mathrm{Re} = 4 \times 10^6, \mathrm{Ma} = 0.3$
$f_2 = C_D/C_L^{3/2}$	$\mathrm{Re} = 4 \times 10^6, \mathrm{Ma} = 0.3$
$f_3 = C_{m_0}^2$	$\mathrm{Re} = 4 \times 10^6, \mathrm{Ma} = 0.3$
$f_4 = 1/C_{\max}^2$	$\mathrm{Re} = 4 \times 10^6, \mathrm{Ma} = 0.3$
$f_5 = 1/C_L^2$	$\alpha = 5^\circ, \mathrm{Re} = 2 \times 10^6, \mathrm{Ma} = 0.15$
$f_6 = 1/x_{tr}$	$\alpha = 5^\circ, \mathrm{Re} = 2 \times 10^6, \mathrm{Ma} = 0.15$

In the remainder of this section, we use the 6-objective airfoil problem to evaluate the interactive method simulating the DM through the Chebyshev achievement function. For this problem we used 3 interaction points and a total of 80 generations. The value of threshold τ was modified as shown in Table 5.

The results presented in Table 8 show that for this problem the reference point was not improved by any of the two approaches. However, the interactive approach found better airfoils than the those of the approach without preferences. The solution corresponding with the best achievement value found by the interactive approach is the following: [0.004962, 0.007022, 0.007275, 0.346273,0.920056, 2.929393]. This solution improves objectives f_1, f_3 and f_6 by an amount of 5.12%, 30.58% and 0.04%, respectively. The airfoil of this solution is presented in Figure 17.

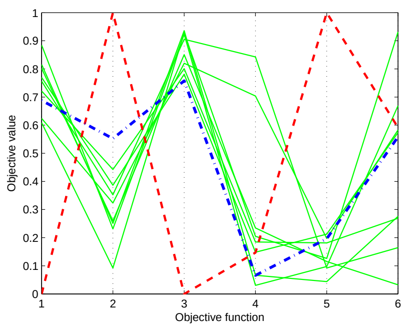


Figure 15: Set of solutions presented to the DM after 100 generations for the 6-objective problem. The dashed line is the reference point, and the dash-dot line is the closest solution to the reference point.

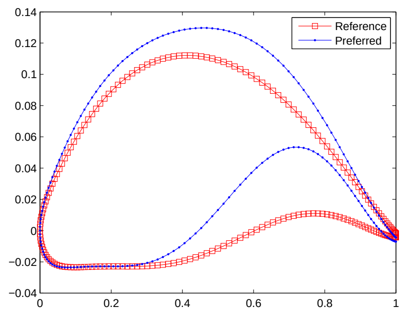


Figure 16: Airfoil corresponding to the solution selected by the decision maker at the end of the simulation of the interactive method applied on the 6-objective problem.

Table 8: Statistics of the achievement function values obtained with preferences and without them in the 6-objective problem.

	Best	**Median**	**Worst**	**Std. dev.**
Preferences	0.0047	0.0473	0.0914	0.0183
No prefs.	0.0157	0.2506	0.4787	0.1480

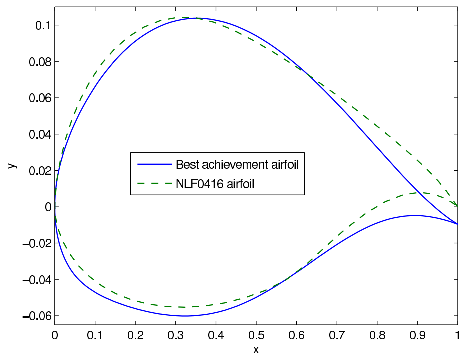


Figure 17: Airfoil with the best achievement value and the reference airfoil for the problem with 6 objectives.

6. Conclusions and Future Work

In this paper, we have proposed a new preference relation based on an achievement scalarizing function. The purpose of the new preference relation is to provide an easy approach to integrate decision maker's preferences into a MOEA without modifying the original structure of the MOEA. In addition, an interactive method including the new preference relation was presented.

The new preference relation divides the objective function space into two subspaces. The solutions of one of these subspaces are compared adopting the usual Pareto dominance relation, while the other is compared using the achievement function. Besides finding the optimal solution of the achievement function, the new preference relation allows us to find a set of solutions around such an optimal solution. Additionally, the size and range of that set can be easily regulated by the DM. In order to incorporate preferences into a MOEA, the user only needs to change the Pareto dominance checking functions by the new preference relation.

The interactive optimization method proposed was evaluated using 3 different instances of an airfoil shape design problem (2, 3 and 6 objectives). From solving those problems, we can see that setting the parameters is an intuitive task. Likewise, the experimental results show that the interactive method achieves better results than those obtained by its *a posteriori* counterpart.

Since our preference relation is able to induce a finer order on the objective space (in terms of number of dominance ranks) than one generated by the Pareto dominance relation, we believe that it can be used to overcome the scalability problems observed in Pareto-based MOEAs to deal with many-objective problems.

As previously indicated, the current implementation of our preference relation compares solutions using the achievement function and the Pareto dominance relation. However, it is possible to couple the achievement function with a different preference relation. Thus, in the future, we want to assess the performance of the achievement function coupled with a different preference relation. In particular, we want to investigate preference relations recently proposed that have shown promising results in many-objective problems, e.g., the average Hausdorff distance indicator proposed by Schütze et al. [39]. Finally, we also want to investigate the suitability of the proposed preference relation to approximate the entire Pareto front in many-objective problems.

[1] Bentley, J.L., Kung, H.T., Schkolnick, M., Thompson, C.D., 1978. On the average number of maxima in a set of vectors and applications. J. ACM 25, 536–543.

[2] Branke, J., 2008. Consideration of Partial User Preferences in Evolutionary Multiobjective Optimization, in: Branke, J., Deb, K., Miettinen, K., Slowinski, R. (Eds.), Multiobjective Optimization. Interactive and Evolutionary Approaches. Springer. Lecture Notes in Computer Science Vol. 5252, Berlin, Germany, pp. 157–178.

[3] Branke, J., Deb, K., Dierolf, H., Osswald, M., 2004. Finding Knees in Multi-Objective Optimization, in: Parallel Problem Solving from Nature - PPSN VIII, Springer-Verlag. Lecture Notes in Computer Science Vol. 3242, Birmingham, UK. pp. 722–731.

[4] Branke, J., Deb, K., Miettinen, K., Slowinski, R. (Eds.), 2008. Multiobjective Optimization. Interactive and Evolutionary Approaches. Springer. Lecture Notes in Computer Science Vol. 5252, Berlin, Germany.

[5] Branke, J., Kaußler, T., Schmeck, H., 2000. Guiding Multi Objective Evolutionary Algorithms Towards Interesting Regions. Technical Report 398. Institute für

Angewandte Informatik und Formale Beschreibungsverfahren, Universität Karlsruhe. Karlsruhe, Germany.

[6] Chankong, V., Haimes, Y.Y., 1983. Multiobjective Decision Making Theory and Methodology. Elsevier Science, New York.

[7] Coello Coello, C.A., Lamont, G.B., Van Veldhuizen, D.A., 2007. Evolutionary Algorithms for Solving Multi-Objective Problems. Springer, New York. second edition.

[8] Cvetković, D., Parmee, I.C., 1999. Use of Preferences for GA–based Multi–objective Optimisation, in: Banzhaf, W., Daida, J., Eiben, A.E., Garzon, M.H., Honavar, V., Jakiela, M., Smith, R.E. (Eds.), GECCO–99: Proceedings of the Genetic and Evolutionary Computation Conference, Morgan Kaufmann Publishers, Orlando, Florida, USA. pp. 1504–1509.

[9] Cvetković, D., Parmee, I.C., 2002. Preferences and their Application in Evolutionary Multiobjective Optimisation. IEEE Transactions on Evolutionary Computation 6, 42–57.

[10] Das, I., 1999. On characterizing the "knee" of the Pareto curve based on Normal-Boundary Intersection. Structural Optimization 18, 107–115.

[11] Deb, K., 1999. Solving Goal Programming Problems Using Multi-Objective Genetic Algorithms, in: 1999 Congress on Evolutionary Computation, IEEE Service Center, Washington, D.C.. pp. 77–84.

[12] Deb, K., Chaudhuri, S., Miettinen, K., 2006. Towards Estimating Nadir Objective Vector Using Evolutionary Approaches, in: Keijzer, M. (Ed.), 2006 Genetic and Evolutionary Computation Conference (GECCO'2006), ACM Press, Seattle, Washington, USA. pp. 643–650.

[13] Deb, K., Kumar, A., 2007. Light beam search based multi-objective optimization using evolutionary algorithms, in: IEEE Congress on Evolutionary Computation, pp. 2125–2132.

[14] Deb, K., Pratap, A., Agarwal, S., Meyarivan, T., 2002a. A Fast and Elitist Multiobjective Genetic Algorithm: NSGA–II. IEEE Transactions on Evolutionary Computation 6, 182–197.

[15] Deb, K., Sundar, J., 2006. Reference Point Based Multi-objective Optimization Using Evolutionary Algorithms, in: GECCO '06: Proceedings of the 8th annual conference on Genetic and evolutionary computation, ACM, New York, NY, USA. pp. 635–642.

[16] Deb, K., Thiele, L., Laumanns, M., Zitzler, E., 2002b. Scalable Multi-Objective Optimization Test Problems, in: Congress on Evolutionary Computation (CEC'2002), IEEE Service Center, Piscataway, New Jersey. pp. 825–830.

[17] Drela, M., 1989. XFOIL: An Analysis and Design System for Low Reynolds Number Aerodynamics, in: Conference on Low Reynolds Number Aerodynamics, University Of Notre Dame, IN. pp. 1–12.

[18] Edgeworth, F.Y., 1881. Mathematical Physics. P. Keagan.

[19] Ehrgott, M., 2005. Multicriteria Optimization. Springer, Berlin. second edition.

[20] Farina, M., Amato, P., 2002. On the Optimal Solution Definition for Many-criteria Optimization Problems, in: Proceedings of the NAFIPS-FLINT International Conference'2002, IEEE Service Center, Piscataway, New Jersey. pp. 233–238.

[21] Figueira, J.R., Liefooghe, A., Talbi, E.G., Wierzbicki, A.P., 2010. A parallel multiple reference point approach for multi-objective optimization. European Journal of Operational Research 205, 390–400.

[22] Fonseca, C.M., Fleming, P.J., 1993. Genetic Algorithms for Multiobjective Optimization: Formulation, Discussion and Generalization, in: Forrest, S. (Ed.), Proceedings of the Fifth International Conference on Genetic Algorithms, University of Illinois at Urbana-Champaign. Morgan Kauffman Publishers, San Mateo, California. pp. 416–423.

[23] Fonseca, C.M., Fleming, P.J., 1995. An Overview of Evolutionary Algorithms in Multiobjective Optimization. Evolutionary Computation 3, 1–16.

[24] Fonseca, C.M., Fleming, P.J., 1998. Multiobjective Optimization and Multiple Constraint Handling with Evolutionary Algorithms—Part I: A Unified Formulation. IEEE Transactions on Systems, Man, and Cybernetics, Part A: Systems and Humans 28, 26–37.

[25] Gong, M., Liu, F., Zhang, W., Jiao, L., Zhang, Q., 2011. Interactive moea/d for multi-objective decision making, in: Proceedings of the 13th Annual Conference on Genetic and Evolutionary Computation, ACM, New York, NY, USA. pp. 721–728.

[26] Hughes, E.J., 2005. Evolutionary Many-Objective Optimisation: Many Once or One Many?, in: 2005 IEEE Congress on Evolutionary Computation (CEC'2005), IEEE Service Center, Edinburgh, Scotland. pp. 222–227.

[27] Jaszkiewicz, A., Slowinski, R., 1999. The light beam search approach -an overview of methodology and applications. European Journal of Operational Research 113, 300–314.

[28] Khare, V., Yao, X., Deb, K., 2003. Performance Scaling of Multi-objective Evolutionary Algorithms, in: Fonseca, C.M., Fleming, P.J., Zitzler, E., Deb, K., Thiele, L. (Eds.), Evolutionary Multi-Criterion Optimization. Second International Conference, EMO 2003, Springer. Lecture Notes in Computer Science. Volume 2632, Faro, Portugal. pp. 376–390.

[29] Knowles, J., Corne, D., 2007. Quantifying the Effects of Objective Space Dimension in Evolutionary Multiobjective Optimization, in: Obayashi, S., Deb, K., Poloni, C., Hiroyasu, T., Murata, T. (Eds.), Evolutionary Multi-Criterion Optimization, 4th International Conference, EMO 2007, Springer. Lecture Notes in Computer Science Vol. 4403, Matshushima, Japan. pp. 757–771.

[30] Kukkonen, S., Lampinen, J., 2007. Ranking-Dominance and Many-Objective Optimization, in: 2007 IEEE Congress on Evolutionary Computation (CEC'2007), IEEE Press, Singapore. pp. 3983–3990.

[31] Luque, M., Miettinen, K., Eskelinen, P., Ruiz, F., 2009. Incorporating preference information in interactive reference point methods for multiobjective optimization. Omega 37, 450–462.

[32] Mattson, C., Mullur, A., Messac, A., 2004. Smart Pareto filter: Obtaining a minimal representation of multiobjective design space. Engineering Optimization 36, 721–740.

[33] Miettinen, K.M., 1999. Nonlinear Multiobjective Optimization. Kluwer Academic Publishers, Boston, Massachusetts, USA.

[34] Molina, J., Santana, L.V., Hernndez-Daz, A.G., Coello, C.A.C., Caballero, R., 2009. g-dominance: Reference point based dominance for multiobjective metaheuristics. European Journal of Operational Research 197, 685–692.

[35] Pareto, V., 1896. Cours D'Economie Politique. F. Rouge.

[36] Praditwong, K., Yao, X., 2007. How Well Do Multi-Objective Evolutionary Algorithms Scale to Large Problems, in: 2007 IEEE Congress on Evolutionary Computation (CEC'2007), IEEE Press, Singapore. pp. 3959–3966.

[37] Purshouse, R.C., Fleming, P.J., 2007. On the Evolutionary Optimization of Many Conflicting Objectives. IEEE Transactions on Evolutionary Algorithms 11, 770–784.

[38] Said, L.B., Bechikh, S., Ghedira, K., 2010. The r-dominance: A new dominance relation for interactive evolutionary multicriteria decision making. Evolutionary Computation, IEEE Transactions on 14, 801–818.

[39] Schütze, O., Esquivel, X., Lara, A., Coello Coello, C.A., 2012. Using the averaged hausdorff distance as a performance measure. IEEE Transactions on Evolutionary Computation 16, 504–522.

[40] Schütze, O., Laumanns, M., Coello Coello, C.A., 2008. Approximating the Knee of an MOP with Stochastic Search Algorithms, in: Rudolph, G., Jansen, T., Lucas, S., Poloni, C., Beume, N. (Eds.), Parallel Problem Solving from Nature–PPSN X. Springer. Lecture Notes in Computer Science Vol. 5199, Dortmund, Germany, pp. 795–804.

[41] Sindhya, K., Ruiz, A.B., Miettinen, K., 2011. A preference based interactive evolutionary algorithm for multi-objective optimization: Pie, in: Takahashi, R., Deb, K., Wanner, E., Greco, S. (Eds.), Evolutionary Multi-Criterion Optimization. Springer Berlin Heidelberg. volume 6576 of *Lecture Notes in Computer Science*, pp. 212–225.

[42] Sobieczky, H., 1998. Parametric Airfoils and Wings. Notes on Numerical Fluid Mechanics 68, 71–88.

[43] Somers, D., Aeronautics, U.S.N., Scientific, S.A., Branch, T.I., Center, L.R., 1981. Design and Experimental Results for a Natural-Laminar-Flow Airfoil for General Aviation Application. National Aeronautics and Space Administration, Scientific and Technical Information Branch.

[44] Srinivas, N., Deb, K., 1994. Multiobjective Optimization Using Nondominated Sorting in Genetic Algorithms. Evolutionary Computation 2, 221–248.

[45] Steuer, R.E., 1986. Multiple Criteria Optimization: Theory, Computation and Application. John Wiley, New York, 546 pp.

[46] Szöllös, A., Smíd, M., Hájek, J., 2009. Aerodynamic optimization via multi-objective micro-genetic algorithm with range adaptation, knowledge-based reinitialization, crowding and epsilon-dominance. Advances in Engineering Software 40, 419–430.

[47] Teytaud, O., 2007. On the hardness of offline multi-objective optimization. Evolutionary Computation 15, 475–491.

[48] Thiele, L., Miettinen, K., Korhonen, P.J., Molina, J., 2009. A preference-based evolutionary algorithm for multi-objective optimization. Evolutionary Computation 17, 411–436.

[49] University of Illinois at Urbana-Champaign, Applied Aerodynamics Group., 2011. UIUC Airfoil Coordinates Database. `http://www.ae.illinois.edu/m-selig/ads/coord_database.html`.

[50] Wagner, T., Beume, N., Naujoks, B., 2007. Pareto-, Aggregation-, and Indicator-Based Methods in Many-Objective Optimization, in: Obayashi, S., Deb, K., Poloni, C., Hiroyasu, T., Murata, T. (Eds.), Evolutionary Multi-Criterion Optimization, 4th International Conference, EMO 2007, Springer. Lecture Notes in Computer Science Vol. 4403, Matshushima, Japan. pp. 742–756.

[51] Wickramasinghe, U.K., Carrese, R., Li, X., 2010. Designing airfoils using a reference point based evolutionary many-objective particle swarm optimization algorithm, in: IEEE Congress on Evolutionary Computation, pp. 1–8.

[52] Wierzbicki, A., 1980a. A methodological guide to multiobjective optimization, in: Iracki, K., Malanowski, K., Walukiewicz, S. (Eds.), Optimization Techniques, Part 1, Springer, Berlin. pp. 99–123.

[53] Wierzbicki, A., 1980b. The use of reference objectives in multiobjective optimisation, in: G., F., T., G. (Eds.), Multiple Criteria Decision Making Theory and Application, Springer Verlag, Heilderberg, Berlin. pp. 468–486.

[54] Zitzler, E., Künzli, S., 2004. Indicator-based selection in multiobjective search, in: Conference on Parallel Problem Solving from Nature (PPSN VIII), Springer. pp. 832–842.

[55] Zitzler, E., Laumanns, M., Thiele, L., 2002. SPEA2: Improving the Strength Pareto Evolutionary Algorithm, in: Giannakoglou, K., Tsahalis, D., Periaux, J., Papailou, P., Fogarty, T. (Eds.), EUROGEN 2001. Evolutionary Methods for Design, Optimization and Control with Applications to Industrial Problems, Athens, Greece. pp. 95–100.

[56] Zitzler, E., Thiele, L., Laumanns, M., Fonseca, C.M., da Fonseca, V.G., 2003. Performance Assessment of Multiobjective Optimizers: An Analysis and Review. IEEE Transactions on Evolutionary Computation 7, 117–132.