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Synthesis and characterization of Pareto-optimal solutions for the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem[1]

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Abstract

The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem, nevertheless a continued research on it, does not have, up to now, an exact solution. This paper develops a methodology for multiobjective problem solution characterization, employing group properties of the Pareto-set. A multiobjective genetic algorithm is built on the basis of these properties. The solutions that are found for the $\mathcal{H}_2/\mathcal{H}_\infty$ control problem are both consistent and less conservative, when compared to other algorithms.

1 Introduction

The statement of a controller design problem as a multiobjective optimization problem is known to be more significant than as a single-objective one, under the viewpoint of real-life applications [6]. In abstract contexts, multiobjective design methods can be seen as a way for generating design alternatives that vary along some sets of the solution space that are known to have "good solutions". In fact, multiobjective controllers can present the nice property of producing large enhancements in some objectives for small debasements in other ones [2, 7].

This paper deals with one of the most traditional multiobjective control design problems: the $\mathcal{H}_2/\mathcal{H}_\infty$ design [4, 7], taken here in the contexts of state-feedback and static output feedback. Some group properties of the multiobjective solution sets (Pareto sets) are employed in order to aggregate the solutions of formerly proposed algorithms. These properties are also employed in the construction of a new "multiobjective genetic algorithm" that takes any set of solutions as an "initial population", in this way generating an enhanced consistent Pareto-set estimate.

2 General Problem Statement

Consider the following linear time-invariant dynamic system:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + \sum_{k=1}^{N} B_k w_k(t) \\ z_k(t) &= C_k x(t) + D_k u(t), \quad k = 1, \ldots, N \\ y(t) &= Cx(t) + \sum_{k=1}^{N} E_k w_k(t) \end{cases} \tag{1}$$

in which $x \in \mathbb{R}^n$ is the system state vector, $u \in \mathbb{R}^m$ is the control input vector and $w_k \in \mathbb{R}^p_k$, $k = 1, \ldots, N$ are the exogenous disturbance vector, $z_k \in \mathbb{R}^q_k$, are the controlled output and $y \in \mathbb{R}^r$ is the measurement output. This system configuration describes N channels from the disturbance input w_k to the controlled variable output z_k, and associated to each channel can be defined a performance index to be minimized or upper bounded.

For control purposes it is considered the static output control law: $u(t) = Ky(t)$. As a special case, in the standard setting of the static state feedback design problem, the state vector x is considered to be available for control law synthesis with $C = I$ and $u(t) = Kx(t)$.

Particularly, the closed-loop transfer functions from w_k to z_k are denoted[1] by

$$H_{z_k w_k}(s) = C_{cl}^{(k)}(sI - A_{cl})^{-1} B_{cl}^{(k)} \tag{2}$$

in which

$$\begin{aligned} A_{cl} &= A + BKC \\ B_{cl}^{(k)} &= B_k + BKE_k \\ C_{cl}^{(k)} &= C_k + D_k KC \end{aligned} \tag{3}$$

for static output feedback. For static state feedback just consider $C = I$ and $E_k = 0$, $k = 0, \ldots, N$.

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[1]The following assumption is made: $D_{cl} = D_k K E_k = 0$, $k = 1, \ldots, N$.

Several different performance criteria could be defined for the closed-loop transfer matrix $H_{z_k w_k}(s)$. In this paper the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms are used. One benefit of the strategy to be proposed here for the multiobjective control design is to deal with both static feedback problems above in a systematic way.

The main multiobjective problem to be addressed in this paper is stated in the sequel for $k = 2$.

Problem 1: The Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Controller Set Computation *Let γ_2 denote the value of the $\mathcal{H}_\infty$ norm of the closed-loop system with optimal $\mathcal{H}_2$ norm, and γ_∞ the optimal $\mathcal{H}_\infty$ norm. Let S denote the set of stabilizing static controllers with compatible dimensions. Determine the set $\mathcal{K}_{2\infty}$ such that:*

$$\mathcal{K}_{2\infty} \triangleq \left\{ K_{2\infty} \left| \left[\begin{array}{l} K_{2\infty} = \arg_K \inf_K \|H_{w_1 z_1}\|_2 \\ \\ \text{subject to} \left\{ \begin{array}{l} \|H_{w_2 z_2}\|_\infty \leq \gamma \\ K \in S \end{array} \right. \\ \\ \text{and} \quad \gamma_\infty \leq \gamma \leq \gamma_2 \end{array} \right] \right. \right\} \tag{4}$$

3 Multiobjective Approach

In a multiobjective setting, **Problem 1** receives another formulation that is mathematically equivalent. Let $\|H(K)\|_2$ be the $\mathcal{H}_2$ norm and $\|H(K)\|_\infty$ the $\mathcal{H}_\infty$ norm of the closed-loop system for controller K, considered in the appropriate channels. These norms define the control objectives, that are organized in the objective vector:

$$f(K) = [\ \|H(K)\|_2 \quad \|H(K)\|_\infty\ ]^T \tag{5}$$

A key concept in multiobjective optimization is the *efficient solution set*, or *Pareto-optimal set*, $\mathcal{P}$ defined by:

$$\mathcal{P} \triangleq \{K_p \mid \nexists K \text{ such that } f(K) \leq f(K_p) \text{ and } f(K) \neq f(K_p)\} \tag{6}$$

Note that the set $\mathcal{P}$ is a well-defined *object*, presenting some properties that are not related to each element of the set, but appear on the set as a whole. It can be shown [2] that $\mathcal{P} = \mathcal{K}_{2\infty}$. However, Equation (6) is more convenient for the purpose of analysing the "group properties" that emerge in set $\mathcal{P}$.

The research tradition in the problem of $\mathcal{H}_2/\mathcal{H}_\infty$ control synthesis has dealt with the question of finding *one* controller that is expected to belong to the set $\mathcal{K}_{2\infty}$, or at least approximates it [4, 7]. The approach that is proposed here, on the contrary, is based on a search for a representative set of solutions describing $\mathcal{P}$, in order to take advantage of the cross-validation possibilities of the solutions belonging to this set.

3.1 Multiobjective Analysis

For the purpose of discussing the Pareto-set properties that are relevant here, figure 1 presents typical structures that emerge after the execution of any computation in order to estimate some points belonging to $\mathcal{P}$.

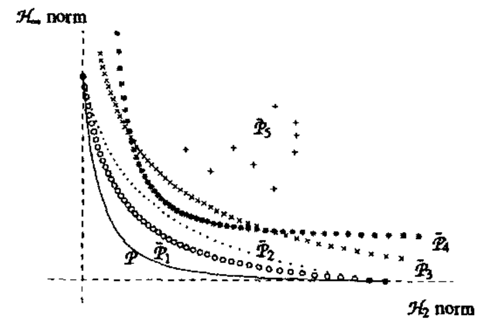

Figure 1: Typical computational estimates of the Pareto-set $\mathcal{P}$, in the space of objectives. In the figure: the exact Pareto-set $\mathcal{P}$ (continuous line), estimated set $\bar{\mathcal{P}}_1$ (o), estimated set $\bar{\mathcal{P}}_2$ (·), estimated set $\bar{\mathcal{P}}_3$ (x), estimated set $\bar{\mathcal{P}}_4$ (*), estimated set $\bar{\mathcal{P}}_5$ (+). The dashed lines denote the optimal values of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms.

The "exact" Pareto set $\mathcal{P}$ is the continuous curve in figure 1. This "exact" set, however, is in principle unavailable, in the sense that, even if one has a set of points that belong to it, there is no means, up to now, for proofing that fact. In order to characterize solutions that are "likely" to belong to $\mathcal{P}$ (the *Pareto candidates*), another relation is defined here, employing only points that are "available":

Definition 1 [Pareto Candidate Set]
Let $f(\cdot)$ be a vector of objectives and $\mathcal{K} \subset Dom(f(\cdot))$ be a set with a finite number of elements: $\mathcal{K} = \{K_1, \ldots, K_\nu\}$. The set $\Psi(\mathcal{K})$, defined by:

$$\begin{aligned} \Psi(\mathcal{K}) \quad \triangleq \quad & \{K_p \in \mathcal{K} \mid \nexists K \in \mathcal{K} \text{ such that } f(K) \leq f(K_p) \\ & \text{and } f(K) \neq f(K_p)\} \end{aligned} \tag{7}$$

is called the Pareto candidate set, *associated to the "sample set" $\mathcal{K}$* □

In fact, due to the unavailability of the set $\mathcal{P}$, the characterization of solution sets $\Psi(\cdot)$ as *Pareto candidates* is performed with *falsification* procedures, that can show that some sets are not candidates, but cannot ever show that any set is in fact a Pareto-set. This is the role of the *Pareto candidate set* concept.

Consistency: Given any set X, it can be considered as a *Pareto candidate* only if $\Psi(X) = X$. If this occurs, the set is said *auto-consistent*. Otherwise, the possibility of X being a subset of the Pareto-set $\mathcal{P}$ is *falsified*.

Ordering and Dominance: Given two sets, X_1 and X_2, ordering relations between these sets are defined:

$$\begin{aligned} & X_1 \prec X_2 \Leftrightarrow \{\Psi(X_1 \cup X_2) \supset X_1 \text{ and } \Psi(X_1 \cup X_2) \not\supset X_2\} \\ & X_1 \preceq X2 \Leftrightarrow \{\Psi(X_1 \cup X_2) \supset X_1\} \end{aligned} \tag{8}$$

In the case of $X_1 \prec X_2$, the possibility of set X_2 being a subset of the Pareto-set $\mathcal{P}$ is *falsified*, while the set X_1 remains being a *Pareto candidate*. In this case, X_1 is said to *dominate* X_2. There are two possibilities of *non-dominance*: if both relations $X_1 \preceq X_2$ and $X_2 \preceq X_1$ do not hold, then both sets become *falsified* as *Pareto candidates*; and if both relations $X_1 \preceq X_2$ and $X_2 \preceq X_1$ hold, both sets keep being *Pareto candidates*.

With these concepts, figure 1 is analysed. The set of estimates $\bar{\mathcal{P}}_5$ is not *auto-consistent*, and therefore is *falsified* as a Pareto candidate. The sets $\bar{\mathcal{P}}_1$ to $\bar{\mathcal{P}}_4$ are each one auto-consistent and could be considered, therefore, as *Pareto-candidates* if only one of them were available. There is an ordering relation among these Pareto-set estimates:

$$\mathcal{P} \prec \bar{\mathcal{P}}_1 \prec \bar{\mathcal{P}}_2 \prec \{\bar{\mathcal{P}}_3, \bar{\mathcal{P}}_4\} \prec \bar{\mathcal{P}}_5 \tag{9}$$

This ordering corresponds to a *dominance* ordering. If all these sets were available, the only *Pareto candidate* would be $\bar{\mathcal{P}}_1$, since:

$$\bar{\mathcal{P}}_1 = \Psi(\bar{\mathcal{P}}_1 \cup \bar{\mathcal{P}}_2 \cup \bar{\mathcal{P}}_3 \cup \bar{\mathcal{P}}_4 \cup \bar{\mathcal{P}}_5) \tag{10}$$

The only sets that are not "ordered", in figure 1 are $\bar{\mathcal{P}}_3$ and $\bar{\mathcal{P}}_4$. If they were both available, they would be both *falsified* as *Pareto candidates*, without need for any additional information.

Extension: Another kind of analysis that is useful for evaluation of a Pareto-set estimate is determining to what extent it covers the Pareto surface. For this purpose, some additional definitions are necessary.

Consider the space $\mathcal{Y}$ of the objective vectors. Let $\varepsilon > 0$ be a fixed real number, and $h \in \mathcal{Y}$ a solution point. The set $\delta(\cdot,\cdot)$ is defined by:

$$\delta(\varepsilon, h) = \{g \in \mathcal{Y} \text{ such that } |g - h| \leq \varepsilon\} \tag{11}$$

Take the set $X = \{x_1, \dots, x_\nu\}$, $X \subset \mathcal{Y}$.

Definition 2 **[ε-Extension]** *The set* $\Theta(\varepsilon, X)$ *defined by*

$$\Theta(\varepsilon, X) = \bigcup_{i=1}^{\nu} \delta(\varepsilon, x_i) \tag{12}$$

is the ε-extension *of the set* X. □

For any set X, $\Theta(\varepsilon, X) \supset X$ trivially. Consider now two sets X_1 and X_2 such that $X_1 \preceq X_2$ and $X_2 \preceq X_1$, i.e., both sets are *Pareto candidates*, and let be given an $\varepsilon > 0$. The following relations become defined:

$$\begin{aligned} \Theta(\varepsilon, X_1) \supset X_2 &\Leftrightarrow X_1 \overset{\varepsilon}{\supset} X_2 \\ \Theta(\varepsilon, X_1) \not\supset X_2 &\Leftrightarrow X_1 \overset{\varepsilon}{\not\supset} X_2 \end{aligned} \tag{13}$$

The following situations can occur:

$X_1 \overset{\varepsilon}{\supset} X_2$ **and** $X_2 \overset{\varepsilon}{\supset} X_1$**:** In this case, the sets X_1 and X_2 are said *extent-equivalent*.

$X_1 \overset{\varepsilon}{\supset} X_2$ **and** $X_2 \overset{\varepsilon}{\not\supset} X_1$**:** In this case, the set X_2 is said to be an *extent-subset* of set X_1.

$X_1 \overset{\varepsilon}{\not\supset} X_2$ **and** $X_2 \overset{\varepsilon}{\not\supset} X_1$**:** In this case, the sets are said to be *extent-incommensurable*.

If some set is an *extent-subset* or if it is *extent-incommensurable* when compared with another set, then it becomes *falsified* as a *Pareto candidate*.

Extremal Data: Specifically for the $\mathcal{H}_2/\mathcal{H}_\infty$ problem in the full state-feedback case, there are other consistency data that are *a priori* known: there are analytical tools for calculating the individual optima of both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ objectives. This means that the extremal points of the Pareto set are known. In figure 1, if this were the case, the sets $\bar{\mathcal{P}}_3$ and $\bar{\mathcal{P}}_4$ would become both *falsified* as *Pareto candidates*. The sets $\bar{\mathcal{P}}_1$ and $\bar{\mathcal{P}}_2$, taken individually, would remain as *candidates*.

4 Current Approaches

The following constrained mono-objective optimization problem is the usual formulation for the generation of solutions for the multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ problem:

Problem 2: The Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Single-Run Design. *Let the disturbance attenuation level* $\gamma > 0$ *be assigned with a fixed value. Let* S *denote the set of stabilizing static controllers of compatible dimensions. Find* $K_{2\infty} \in S$ *such that:*

$$\begin{cases} K_{2\infty} \text{ minimizes } \|H_{w_1 z_1}(K)\|_2 \\ \text{subject to } \|H_{w_2 z_2}(K)\|_\infty \leq \gamma \end{cases} \tag{14}$$

This is a constrained non-linear, non-smooth and non-convex mono-objective optimization problem with a possibly non-convex, non-compact and unbounded feasible set. The set of solutions of the multiobjective problem is obtained by varying the constraint paramenter γ. Note that, due to these characteristics, the set of exact solutions for different γ's cannot be affirmativelly characterized — this implies the need for a falsification procedure for solution characterization.

Let be given any optimization algorithm to solve **Problem 2** several times, with different γ's, in order to generate an estimate of the Pareto set. The estimated set is likely to be not only a *Pareto sub-optimum*, but even *non auto-consistent*, due to the facts: a single solution is found in each optimization algorithm run; the solutions are not taken as a set with set properties; the mono-objective optimization algorithms that are employed are likely to find only local minima of **Problem 2**; these minima are not necessarily related, from one run to another one. An exception occurs for the earlier LMI (Linear Matrix Inequalities) formulation of the mixed objective problem, in terms of conservative convex algorithms. It is based on sufficient but not necessary conditions, what means that **Problem 2** is modified in LMI formulation, being only approximativelly solved. Therefore, these algorithms lead to solutions that do not belong to the Pareto set and, in fact, can be significantly far from it. However, since the LMI formulation becomes convex, any single run of the optimization problem leads to its global solution. Due to this, the LMI algorithm does furnish points that are *auto-consistent*. However, it is an easy task finding other solutions that lie below the curve $\bar{\mathcal{P}}_{LMI}$ that is found with the LMI algorithm.

Different algorithms have been employed as optimization engine instances for solving **Problem 2**. Recently, an iterative non-convex algorithm that solves a sequence of LMI (Linear Matrix Inequalities) problems that approximate the exact

BMI (Bilinear Matrix Inequalities) form of **Problem 2** has been proposed in order to furnish less conservative solutions to $\mathcal{H}_2/\mathcal{H}_\infty$ problems [5]. Other heuristic solutions have been proposed for these problems, sometimes employing Genetic Algorithms [3], or other non-convex optimization schemes [7], with the aim of approaching solutions belonging to the set $\mathcal{P}$. All these algorithms can furnish solutions that are not *auto-consistent*.

The old, popular and conservative, LMI formulation and its succedaneum, the BMI formulation, are studied here as reference solutions that will initialize the multiobjective algorithm. Any other solutions could be employed for the same purpose.

4.1 Matrix Inequalities Formulations

In a matrix inequality setting the exact mixed control problem, as formulated above, is the direct combination of the actual $\mathcal{H}_2$ norm computation with the Bounded Real Lemma. Namely, assuming that the closed-loop system is asymptotically stable, the optimal $\mathcal{H}_2$ norm computation is performed by

$$\|H_{w_1z_1}\|_2^2 = \inf_{X_2,J}\{\mathrm{Tr}(J)\} \tag{15}$$

$$\text{s.t.} \quad \begin{bmatrix} J & * \\ (C_{cl}^{(1)})' & X_2 \end{bmatrix} > 0 \tag{16}$$

$$\begin{bmatrix} A_{cl}'X_2 + X_2A_{cl} & * \\ (B_{cl}^{(1)})'X_2 & -I \end{bmatrix} < 0 \tag{17}$$

Where for a symmetric block matrix, the symbol $*$ denotes the sub-matrices that lie above the main block-diagonal.

On the other hand the Bounded Real Lemma is stated in the following way: Let $\gamma > 0$ be given, A_{cl} is asymptotically stable and $\|H_{z_2w_2}\|_\infty < \gamma$ if and only if there exists a symmetric definite positive matrix X_∞ such that

$$\begin{bmatrix} A_{cl}'X_\infty + X_\infty A_{cl} & * & * \\ (B_{cl}^{(2)})'X_\infty & -I & * \\ C_{cl}^{(2)} & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{18}$$

Thus the exact mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem can be completely restated as in the following, with the simple substitution of the closed-loop matrices given in (3) into (16), (17), (18):

Problem 3: *Determine a stabilizing static feedback control K that achieves*

$$\Gamma = \min_{X_2,X_\infty,J,K}\{\mathrm{Tr}(J)\}$$

$$\text{s.t.} \quad X_\infty > 0, \quad \begin{bmatrix} J & * \\ (C_1 + D_1KC)' & X_2 \end{bmatrix} > 0$$

$$\begin{bmatrix} (A+BKC)'X_2 + X_2(A+BKC) & * \\ (B_1 + BKE_1)'X_2 & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} (A+BKC)'X_\infty + X_\infty(A+BKC) & * & * \\ (B_2+BKE_2)'X_\infty & -I & * \\ C_2 + D_2KC & 0 & -\gamma^2 I \end{bmatrix} < 0$$

4.1.1 Standard LMI Formulation: For the state feedback case, the conventional strategy adopted in the literature is based on the simple change of variables of type $K = ZW^{-1}$, with the imposition $W = X_2^{-1} = X_\infty^{-1}$, $C = I$ and $E_k = 0$, $k = 1,2$ in **Problem 3**. This allows to obtain the following optimization LMI control synthesis description:

Problem 4:

$$\Upsilon = \min_{Z,W,J} Tr\{J\}$$

$$\text{s.t.} \quad \begin{bmatrix} J & * \\ (C_1W + D_1Z)' & W \end{bmatrix} > 0$$

$$\begin{bmatrix} AW + WA' + Z'B' + BZ & * \\ B_1' & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} AW + WA' + Z'B' + BZ & * & * \\ B_2' & -I & * \\ C_2W + D_2Z & 0 & -\gamma^2 I \end{bmatrix} < 0$$

where $\|H_{z_1w_1}\|_2^2 \leq \Upsilon$, $\|H_{z_2w_2}\|_\infty < \gamma$ and the static state feedback gain is given by $K = ZW^{-1}$.

4.1.2 BMI Formulation: This formulation is derived from the recent paper [5] (for details one can see that paper). The key idea is to handle the non-affine characteristics introduced by non-positive quadratic terms when one substitutes (3) in (16)-(18) by means of matrix upper bounds. **Problem3** becomes, after these operations ([5]) :

Problem 5:

$$\Omega = \min_{X_2,X_\infty,J,K}\{\mathrm{Tr}(J)\}$$

$$\text{s.t.} \quad X_\infty > 0, \quad \begin{bmatrix} J & * \\ (C_1 + D_1KC)' & X_2 \end{bmatrix} > 0$$

$$\begin{bmatrix} \Phi_{21} & * & * & * \\ B_1'X_2 & \Phi_{22} & * & * \\ B'X_2 + KC & 0 & -I & * \\ B'X_2 & KE_1 & 0 & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} \Phi_{\infty 1} & * & * & * & * \\ B_2'X_\infty & \Phi_{\infty 2} & * & * & * \\ B'X_\infty + KC & 0 & -I & * & * \\ B'X_\infty & KE_2 & 0 & -I & * \\ C_2 + D_2KC & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix} < 0$$

where

$$\begin{aligned}
\Phi_{21} &= A'X_2 + X_2A - 2X_2BL_1 - 2L_1'B'X_2 + 2L_1'L_1 \\
&\quad -C'K'M - M'KC + M'M \\
\Phi_{\infty 1} &= A'X_\infty + X_\infty A - 2X_\infty BL_2 - 2L_2'B'X_\infty + 2L_2'L_2 \\
&\quad -C'K'M - M'KC + M'M \\
\Phi_{22} &= -I - E_1'K'N_1 - N_1'KE_1 + N_1'N_1 \\
\Phi_{\infty 2} &= -I - E_2'K'N_2 - N_2'KE_2 + N_2'N_2
\end{aligned}$$

$M = KC,\ L_1 = B'X_2,\ L_2 = B'X_\infty,\ N_1 = KE_1,\ N_2 = KE_2$

Note that **Problem 5** can be easily restated as a static state feedback mixed problem with $C = I$ and $E_k = 0,\ k = 1,2$. Adopting the above formulation, the following iterative algorithm is proposed in [5].

Iterative Algorithm

STEP 1 - Set $K^{(0)} = K$, where K is the optimal solution of Problem 4 as well as $(X_2^{(0)})^{-1} = (X_\infty^{(0)})^{-1} = W$ and $\Gamma^{(0)} = \Omega^{(0)} = \Upsilon$. Set $i = 1$.

STEP 2A - In Problem 3 set $X_2 = X_2^{(i-1)}$, $X_\infty = X_\infty^{(i-1)}$ and $\Gamma = \Gamma^{(i-1)}$.

STEP 2B - With $K = K^{(i-1)}$ fixed, solve Problem 3 with respect to $X_2^{(i-1)} > 0$, $X_\infty^{(i-1)} > 0$ and $\Gamma^{(i-1)}$.

STEP 2C - In Problem 5, set $M = K^{(i-1)}C$, $L_1 = B'X_2^{(i-1)}$, $L_2 = B'X_\infty^{(i-1)}$, $N_1 = K^{(i-1)}E_1$ and $N_2 = K^{(i-1)}E_2$ (for the particular case of state feedback $E_k = 0$, $k = 1,2$ and $C = I$).

STEP 3 - Solve Problem 5 for $X_2^{(i)} > 0$, $X_\infty^{(i)} > 0$, $K^{(i)}$ and $\Omega^{(i)}$.

STEP 4 - If $\|\Omega^{(i-1)} - \Omega^{(i)}\| < \varepsilon$ for a sufficiently small positive scalar ε, then stop. Else, set $i = i + 1$ and return to step 2a.

For the static output feedback problem, the algorithm can be started with any feasible controller K that assures a disturbance attenuation level γ, i.e. $\|H_{z_2w_2}\|_\infty < \gamma$ and with finite $\|H_{z_1w_1}\|_2^2$. In this case one can use, for example, the approach proposed in [1].

5 Multiobjective Genetic Algorithm

The problem of optimization of arbitrary functionals has been, since the early development of the optimization theory, a main goal. However, each different method that was developed was built with several assumptions on the structure of the functional to be optimized: linearity, convexity, differentiability, etc. The class of methods that has attained the best approximation to the problem of "arbitrary functional optimization" is the family of "stochastic optimization methods". A group of methods that has attained large applicability, from this class, is the family of "Genetic Algorithms".

Due to the "global optimization" properties of the Genetic Algorithms, they have become a natural tool for problems like the $\mathcal{H}_2/\mathcal{H}_\infty$ design [3]. Another potential reason for this suitability is pointed out here: since the genetic algorithms work with populations of candidate solutions, instead of a single candidate solution like other optimization methods, they are able to incorporate operators that exploit the group properties of the Pareto-set estimates.

5.1 Multiobjective Genetic Algorithm Construction

The multiobjective genetic algorithm can be built throught the modification of any mono-objective genetic algorithm. A "Genetic Algorithm" can be defined as the successive application of the following operations to a set of tentative solutions of the problem (called "population"):

crossover: The population is divided in pairs, and each pair of solutions is replaced by a new pair, that is generated employing information retained from the original pair;

mutation: Some solutions ("individuals") are randomly chosen to receive a perturbation in its parameters;

selection: The population that arises after the crossover and mutation operations is modified, with the exclusion of some "individuals" and the replication of other ones, being maintained the total size of the population. The probability of being replicated is greater for the greater optimization functional values (for maximization problems);

elitism: Some individuals (the "best" ones) are deterministically maintained in the population.

After some applications of these operations, the "population" converges to solutions that, in some sense, are "good approximations" of the global solutions of the problem.

Any mono-objective genetic algorithm can be adapted through the following guidelines, in order to make a multiobjective genetic algorithm:

- Select, from the initial population Q_0, the group of individuals that form the maximal consistent subset $\bar{\mathcal{P}}_0$. This operation is defined by: $\bar{\mathcal{P}}_0 = \Psi(Q_0)$.
- At each iteration, a new population Q_i is generated by the application of the genetic operators. Re-calculate the estimate $\bar{\mathcal{P}}_i = \Psi(Q_i)$, eventually excluding some individuals and including other ones. The set $\bar{\mathcal{P}}_i$ is employed as the "elit set" in the "elitism" operation.
- A "niche" technique should be employed, in order to avoid the inclusion of points that are much close one to the other in the set $\bar{\mathcal{P}}$. In this way, the solution $\bar{\mathcal{P}}_{mga}$ set suffers a pressure for covering the whole set $\mathcal{P}$.
- The functional that guides the selection operation should be composed with the individual functionals that compose the objective vector. In the specific implementation employed here, they are scaled and then aggregated with the operator *max*.

In this way, instead of searching for *single* solutions, the whole set $\mathcal{P}$ is searched, as an object with intrinsic properties. The design procedure starts with any *non-consistent* or *conservative* algorithm, that furnishes an initial solution set $\bar{\mathcal{P}}_a$ that is further refined by a Multiobjective Genetic Algorithm (MGA). Denote by $\bar{\mathcal{P}}_i$ the Pareto-set candidate produced by MGA at i-th iteration. The multiobjective genetic operators have been tailored such that:

niche+selection: Produces a pressure that leads the Pareto estimate $\bar{\mathcal{P}}_{mga}$ to increase its "extension".

elitism: Deterministically guarantees that: (i) $\bar{\mathcal{P}}_{i+1} \preceq \bar{\mathcal{P}}_i$ and (ii) $\bar{\mathcal{P}}_{i+1} \overset{\varepsilon}{\supset} \bar{\mathcal{P}}_i$.

selection: Produces the enhancement pressure that allows that eventually $\bar{\mathcal{P}}_{i+1} \prec \bar{\mathcal{P}}_i$ and $\bar{\mathcal{P}}_i \overset{\varepsilon}{\not\supset} \bar{\mathcal{P}}_{i+1}$.

The multiobjective genetic algorithm, therefore, extends the initial algorithm solutions in two senses: (i) Pareto set estimates that are under the former estimates are usually found by the MGA; (ii) Some "failures" in the estimate of $\mathcal{P}$ are "repaired" by the MGA. Some "holes" in the Pareto set estimates can be filled by the MGA, in this way enhancing the Pareto estimate.

6 Example

In this section, for the sake of space, only one example of algorithm combination with the multiobjective genetic algorithm is presented. A single-channel case is employed, for simplicity.

6.1 Full state feedback

A very simple system is presented, in order to allow the visualization of the objective space (of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ closed-loop norms). The matrices of the system are:

$$A = \begin{bmatrix} -0.3868 & 0.0751 \\ 0 & -0.0352 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.0591 & 0 \\ 0 & 1.7971 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.6965 \\ 1.6961 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0.0346 & 0.0535 \\ 0 & 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 \\ 0.5297 \end{bmatrix}$$

This system is controlled with a state-feedback controller:

$$u = \begin{bmatrix} K_1 & K_2 \end{bmatrix} x$$

The controller design problem is solved through: (i) the standard (conservative) LMI formulation defined in **Problem 4**; (ii) the "less conservative" BMI formulation defined in **Problem 5**; and (iii) the multiobjective genetic algorithm, starting both from the solution set of **Problem 4** and of **Problem 5**. The closed-loop norms obtained are ploted in figure 2.

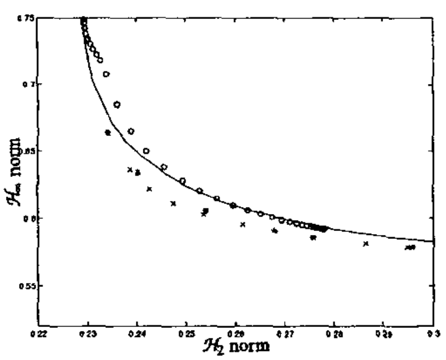

Figure 2: Pareto-set estimates, in the space of objectives, obtained from: (continuous line)- LMI standard formulation; (o)- BMI "less conservative" formulation; (x)- multiobjective genetic algorithm starting from the LMI solutions; (*)- multiobjective genetic algorithm starting from the BMI solutions.

Figure 2 shows that, in this case, the BMI formulation has generated some solutions that are even more conservative than the ones generated with the LMI formulation. Both initial condition sets for the multiobjective genetic algorithm have lead to the same solution set, that is less conservative and consistent with the characteristics of a Pareto-set. The MGA solution sets cover all the extension between the two individual optima, while the BMI solution set leaves some spaces unfilled.

7 Conclusions

By construction, it becomes a tautology that the output $\bar{\mathcal{P}}_{mga}$ of the proposed multiobjective genetic algorithm is the best estimate that is available for the Pareto-set in mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control design problems. The main consequences of this fact are: (i) Any algorithm that claims to find "the least conservative" solutions for this problem should have its output set $\bar{\mathcal{P}}_{lc}$ such that $\bar{\mathcal{P}}_{lc} \preceq \bar{\mathcal{P}}_{mga}$ and $\bar{\mathcal{P}}_{lc} \overset{\varepsilon}{\supset} \bar{\mathcal{P}}_{mga}$. The proposed scheme can be seen, therefore, as a strong validation procedure for any mixed criteria controller design algorithm; (ii) Otherwise, any algorithm should be coupled to MGA, in order to be able to generate the best approximation to the Pareto-set $\mathcal{P}$. Any algorithm that does not intend to completely solve the mixed problem, but only find some tentative solutions can be aggregated in this way.

Up to the present knowledge on the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, the best design procedure is always finished by the application of MGA, for finding better solutions, or for corroborating the conjecture (that cannot be proven) that some solution is already the best possible one.

The presented methodology, although being presented in the context of the $\mathcal{H}_2/\mathcal{H}_\infty$ control problem, is not specific for this domain. Any design problem with multiple objectives could be analysed under the proposed tools, with minor adaptations.

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