APPENDIX C

DERIVATION OF ANALYTICAL FORMULATION FOR EXPECTED LIFETIME SEISMIC DAMAGE COST

This appendix provides detailed derivation of the analytical formulation used in Chapter 7 to calculate the expected seismic damage cost over the lifetime. Only a single seismic hazard case is discussed; cases dealing with multiple hazards (earthquake, wind, etc.) can be found in Kang and Wen (2000). The following material is based on Wen and Kang (2001a,b).

C.1 General analytical formulation

Evaluation of the expected lifetime seismic failure cost necessitates consideration of the random number of occurrences and random occurrence times of the seismic hazard as well as the discounting of cost over time. Over a time period *T*, which may be the design life of a new structure or the remaining life of a retrofitted structure, the expected lifetime seismic damage cost can be expressed as a function of *T* and the design variable vector ***X***:

$$E\left[C_{seismic}(T,\mathbf{X})\right]=E\left[\sum_{i=1}^{N(T)}\sum_{j=1}^{K}C_j e^{-\lambda t_i}P_{ij}(\mathbf{X},t_i)\right] \tag{C.1}$$

where E[.] = expectation operator; ***X*** = design variable vector; *i* = number of severe seismic loading occurrence; t_i = loading occurrence time, a random variable; *N(T)* = number of severe seismic loading occurrences in *T*, a random variable; C_j = cost in present monetary value of the *j*-th damage state being reached at the time of loading occurrence; $e^{-\lambda t}$ = a discount factor of over time *t* that converts future damage-induced monetary cost into the present monetary value, in which λ = constant annual monetary discount rate; P_{ij} = probability of the *j*-th damage state being reached <u>given</u> the *i*-th hazard occurrence; *K* = total number of damage states considered. It is noted that this formula implicitly assumes that the structural system will be immediately retrofitted to its original intact condition after each damage-inducing hazard occurrence.

Assumption of time-invariance for structural resistance as well as for damage state monetary cost simplifies Equation C.1 as

$$E\left[C_{seismic}\right]=\left(\sum_{j=1}^{K}C_jP_j\right)E\left[\sum_{i=1}^{N(T)}e^{-\lambda t_i}\right] \tag{C.2}$$

where P_j is the <u>mean</u> value of the j-th damage state probability given seismic occurrence.

Under the assumption that the number of damage states of concern is small and the demands that cause these damage states are due to severe seismic hazards that occur infrequently, a simple Poisson process model of seismic hazard occurrences with an annual occurrence rate of ν would be appropriate. Conditioning the random number of occurrences N on n in the time interval (0, T), the occurrence times t_j (j = 1,2,…, n) of the seismic hazard are independent and uniformly distributed in (0, T) (Ang and Tang 1975). Taking the expectation with respect to the random hazard occurrence times t_j, one obtains

$$E\left[\sum_{i=1}^{n}e^{-\lambda t_i}\right]=\sum_{i=1}^{n}E\left[e^{-\lambda t_i}\right]=n\int_0^T\frac{e^{-\lambda\tau}d\tau}{T}=\frac{n}{\lambda T}\left(1-e^{-\lambda T}\right) \tag{C.2}$$

Application of the total probability theorem (Ang and Tang 1975) on the random seismic occurrence number leads to the following unconditional expected value:

$$\begin{aligned}E\left[\sum_{i=1}^{N(T)}e^{-\lambda t_i}\right]&=\sum_{n=0}^{\infty}\left\{E\left[\sum_{i=1}^{n}e^{-\lambda t_j}\right]P(N=n)\right\}=\sum_{n=0}^{\infty}\frac{n}{\lambda T}\left(1-e^{-\lambda T}\right)\frac{\left(\nu T\right)^n}{n!}e^{-\nu T}\\&=\frac{1}{\lambda T}\left(1-e^{-\lambda T}\right)\sum_{n=0}^{\infty}n\frac{\left(\nu T\right)^n}{n!}e^{-\nu T}=\frac{1}{\lambda T}\left(1-e^{-\lambda T}\right)\nu T\\&=\frac{\nu}{\lambda}\left(1-e^{-\lambda T}\right)\end{aligned} \tag{C.3}$$

Substitution of the above result into Equation C.1 leads to the final closed-form formula:

$$E[C_{seismic}(T,\mathbf{X})] = \frac{\nu}{\lambda}(1-e^{-\lambda T})\sum_{j=1}^{K} C_j P_j \tag{C.4}$$

which is also Equation 7.1.

C.2 Evaluation of damage state probabilities given seismic occurrence

Seismic performance indices (e.g., maximum interstory drift ratio) of a structure subject to each of the designated seismic hazard levels with *t*-year exceedance probability are calculated; then a lognormal curve is fitted for pairs of maximum drift ratio and the associated *t*-year exceedance probability (which could possibly be corrected to incorporate randomness/uncertainty effects). At this stage, the exceedance probability for drift ratio limits that define all damage states can be readily read from the fitted curve. This *t*-year exceedance probability of the drift ratio limit has to be converted to exceedance probabilities given seismic occurrence as needed in Equation C.4.

Because seismic occurrences are modeled as a Poisson process with an annual occurrence rate ν, the annual probability of exceeding the drift ratio limit Δ_j is then $\nu P(\Delta > \Delta_j)$, where $P(\Delta > \Delta_j)$ is the exceedance probability given seismic occurrence yet to be determined. The exceedance probability over a time period (0, *t*) is given by

$$P_t\left(\Delta > \Delta_j\right) = 1 - \left[\nu P(\Delta > \Delta_j)\,t\right]^0 e^{-\nu P(\Delta > \Delta_j)t} \Big/ 0! = 1 - e^{-\nu P(\Delta > \Delta_j)t} \tag{C.5}$$

from which one solves

$$P\left(\Delta > \Delta_j\right) = -\frac{1}{\nu t}\left\{\ln\left[1 - P_t\left(\Delta > \Delta_j\right)\right]\right\} \tag{C.6}$$

The damage state probability given seismic occurrence is then

$$P_j = \begin{cases} 1 - P\left(\Delta > \Delta_1\right), & \text{for } j = 1 \\ P\left(\Delta > \Delta_{j-1}\right) - P\left(\Delta > \Delta_j\right), & \text{for } j = 2, \ldots, K-1 \\ P\left(\Delta > \Delta_{K-1}\right), & \text{for } j = K \end{cases} \tag{C.7}$$

Substitution of Equations C.7 into C.6 shows that the annual earthquake occurrence rate ν is not needed if $C_1 = 0$.