

- present two measures that can detect dominance between approximation sets and also show why the use of this type of measure is restricted, and
- classify and discuss existing unary measures.

Note that we focus on comparisons of approximation sets, i.e., assume that for each multiobjective EA only one run is performed. If we consider instead multiple runs, statistical methods are required (Grunert da Fonseca et al. 2001); this important issue will not be discussed in the present paper.

2 Scenario

Suppose an arbitrary optimization problem involving n objectives and the following preference orders on the set of objective vectors.

Definition 1 (Dominance relations) Let Z be the n -dimensional objective space and $\mathbf{z}^1 = (z_1^1, \dots, z_n^1), \mathbf{z}^2 = (z_1^2, \dots, z_n^2) \in Z$ two arbitrary objective vectors. We define the following relations on Z :

- $\mathbf{z}^1 \succ \mathbf{z}^2$ (\mathbf{z}^1 dominates \mathbf{z}^2) if \mathbf{z}^1 is not worse than \mathbf{z}^2 in any objective and is better in at least one objective,
- $\mathbf{z}^1 \succ\prec \mathbf{z}^2$ (\mathbf{z}^1 strictly dominates \mathbf{z}^2) if \mathbf{z}^1 is better than \mathbf{z}^2 in all objectives,
- $\mathbf{z}^1 \succeq \mathbf{z}^2$ (\mathbf{z}^1 weakly dominates \mathbf{z}^2) if \mathbf{z}^1 is not worse than \mathbf{z}^2 in any objective,
- $\mathbf{z}^1 \succeq_\epsilon \mathbf{z}^2$ (\mathbf{z}^1 ϵ -dominates \mathbf{z}^2) if \mathbf{z}^1 is not worse than \mathbf{z}^2 by a factor of ϵ in any objective for a fixed $\epsilon > 0$,
- $\mathbf{z}^1 \parallel \mathbf{z}^2$ (\mathbf{z}^1 and \mathbf{z}^2 are incomparable to each other) if neither \mathbf{z}^1 weakly dominates \mathbf{z}^2 nor \mathbf{z}^2 weakly dominates \mathbf{z}^1 .

The relations $\prec, \prec\prec, \preceq,$ and \preceq_ϵ are defined accordingly, i.e., $\mathbf{z}^1 \prec \mathbf{z}^2$ is equivalent to $\mathbf{z}^2 \succ \mathbf{z}^1$, etc.

The dominance relation reflects the weakest assumption about the preference structure of the decision maker: a solution is preferable to another solution if the former dominates the latter in objective space. Accordingly, those objective vectors that are not dominated by any other objective vector are denoted as *Pareto optimal* and the entirety of all of these objective vectors as *Pareto optimal front*.¹ Unfortunately, generating the Pareto-optimal set is often infeasible, and we can only hope to find a good approximation of it. By approximation, we usually mean the set of nondominated solutions found in one optimization run. In the following, the term approximation set is used in order to formally describe what we consider as the outcome of a multiobjective EA (Hansen and Jaszkiewicz 1998):

¹For a detailed discussion of these concepts, the interested reader is referred to (Deb 2001).

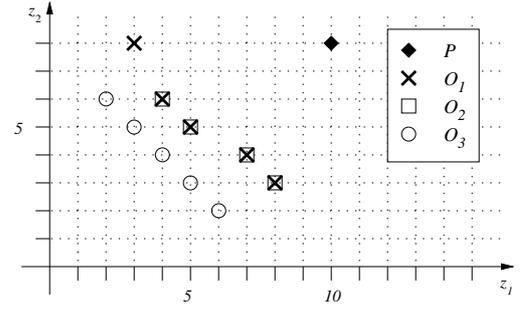


Figure 1: Outcomes of three hypothetical algorithms denoted as $O_1, O_2,$ and O_3 with respect to a two-dimensional maximization problem. The Pareto-optimal front P consist of a single objective vector.

Definition 2 (Approximation set) Let $A \subseteq Z$ be a set of objective vectors. A is called an approximation set if any two members of A do not dominate each other: $\forall \mathbf{z}^1, \mathbf{z}^2 \in A : \mathbf{z}^1 = \mathbf{z}^2 \vee \mathbf{z}^1 \parallel \mathbf{z}^2$. The set of all approximation sets is denoted as Ω .

Now, consider the outcomes of three hypothetical algorithms as depicted in Figure 1. Can we say that any of these approximation sets is better than another? To answer this question, we will extend the dominance relations from above to approximation sets.

Definition 3 (Dominance relations on approximation sets)

Let $A_1, A_2 \in \Omega$ be two approximations sets. We write $A_1 \succ A_2$ (A_1 dominates A_2) if every member in A_2 is dominated by at least one member in A_1 ; the relations $\succ, \succ\prec, \succeq, \succeq_\epsilon, \prec, \prec\prec, \preceq,$ and \preceq_ϵ are defined accordingly. Furthermore, we say A_1 is better than A_2 ($A_1 \triangleright A_2$ resp. $A_2 \triangleleft A_1$) if $A_1 \succeq A_2$ and $A_1 \neq A_2$; A_1 and A_2 are incomparable to each other ($A_1 \parallel A_2$) if neither $A_1 \succeq A_2$ nor $A_2 \succeq A_1$.

According to this definition, we consider an approximation set to be better than another ($A_1 \triangleright A_2$), if any solution in the latter is weakly dominated by the former and if the former contains at least one solution not weakly dominated by the latter. In the above example, O_1 is better than O_2 and strongly dominates O_3 ; O_2 dominates O_3 .

The statements we can make using the \triangleright relation is whether the outcome of one approximation algorithm is better than the outcome of another method or not. However, we would like to be able to make more precise statements:

- If one algorithm is better than another, can we express how much better it is?
- If no algorithm can be said to be better than the other, are there certain aspects in which respect we can say the former is better than the latter?

For this reason, quantitative quality measures have been introduced. As mentioned in the introduction, they usually

assign real numbers to approximation sets, and then a common metric can be used to quantify the quality difference of two approximation sets. In the following, we will use the term quality indicator (as quality measure is often used with different meanings): it is a function that assigns each tuple of approximation sets a number that somehow reflects aspects of the quality or quality differences.

Definition 4 (Quality indicator) An m -ary quality indicator I is a function $I : \Omega^m \rightarrow \mathbb{R}$, which assigns each vector (A_1, A_2, \dots, A_m) of m approximation sets a real value $I(A_1, \dots, A_m)$.

The goal is that we can draw conclusions about the relation between approximation sets by comparing their indicator values. Ideally, a greater (or smaller) indicator value would imply that one set is better than the other. On the other hand, we also would like to ensure that whenever A is better than B also the indicator value of A is greater (or smaller) than that of B . Thus, there are always two directions we have to consider: conclusions we can draw from the indicator values with respect to the dominance relations, and the implications of any dominance relation on the indicator values. In Section 4, we will introduce the terms compatibility and completeness for these purposes.

The remainder of this paper focuses on unary indicators as they are most commonly used in the literature; what makes them attractive is their capability of assigning quality values to an approximation set independent of other sets under consideration. We will classify and discuss existing unary indicator with regard to compatibility and completeness in Sections 4 and 5; first, however, we will investigate what we must not expect from them.

3 Limitations

Naturally, many studies have attempted to capture the multiobjective nature of approximation sets by deriving distinct indicators for the distance to the Pareto-optimal front and the diversity within the approximated front. Therefore, the question arises whether we can define a minimal combination of unary indicators $I = (I_1, I_2, \dots, I_k)$ such that better quality goes hand in hand with greater indicator values, i.e.,

$$(\forall 1 \leq i \leq k : I_i(A) > I_i(B)) \Leftrightarrow A \triangleright B$$

for any approximation sets A, B . Such a combination of indicators, applicable to any type problem, would be ideal, because then any approximation set could be characterized by, e.g., two real numbers that reflect the different aspects of the overall quality. The variety among the indicators proposed, however, suggests that this goal is, at least, difficult to achieve. The following theorem shows that in general it cannot be achieved: a fixed number of indicators is not sufficient for problems of arbitrary dimensionality. The

statement behind is that in order to detect weak dominance among objective vectors as many indicators as objectives are necessary.

Theorem 1 Let $Z = \mathbb{R}^n$ with $n \geq 2$ and $I = (I_1, I_2, \dots, I_k)$ be a vector of unary quality indicators such that for any $z^1, z^2 \in Z$:

$$(\forall 1 \leq i \leq k : I_i(\{z^1\}) \geq I_i(\{z^2\})) \Leftrightarrow z^1 \succeq z^2$$

Then it holds that $k \geq n$.

Proof. We will exploit the fact that in \mathbb{R} the number of disjoint open intervals $(a, b) = \{z \in \mathbb{R} ; a < z < b\}$ with $a < b$ is countable (Hrbacek and Jech 1999); in general, this means that \mathbb{R}^k contains only countably many disjoint open hyperrectangles $(a_1, b_1) \times (a_2, b_2) \times \dots \times (a_k, b_k) = \{(z_1, z_2, \dots, z_k) \in \mathbb{R}^k ; a_i < z_i < b_i, 1 \leq i \leq k\}$ with $a_i < b_i$. The basic idea is that whenever fewer indicators than objectives are available, uncountably many disjoint open hypercubes arise—a contradiction. Furthermore, we will show a slightly modified statement, which is more general: if Z contains an open hypercube $(u, v)^n$ with $u < v$ such that for any $z^1, z^2 \in (u, v)^n$:

$$(\forall 1 \leq i \leq k : I_i(\{z^1\}) \geq I_i(\{z^2\})) \Leftrightarrow z^1 \succeq z^2$$

then $k \geq n$.

Without loss of generality assume a maximization problem in the following. We will argue by induction.

$n = 2$: Let $a, b \in (u, v)$ with $a < b$ and consider the incomparable objective vectors (a, b) and (b, a) . If $k = 1$, then either $I_1(\{(a, b)\}) \geq I_1(\{(b, a)\})$ or vice versa; this leads to a contradiction to $(a, b) \not\succeq (b, a)$ and $(b, a) \not\succeq (a, b)$.

$n - 1 \rightarrow n$: Suppose $n > 2, k < n$ and that the statement holds for $n - 1$. Choose $a, b \in (u, v)$ with $a < b$, and consider the $n - 1$ dimensional open hypercube $S_c = \{(z_1, z_2, \dots, z_{n-1}, c) \in (u, v)^n ; a < z_i < b, 1 \leq i \leq n - 1\}$ for an arbitrary $c \in (u, v)$.

First, we will show that $I_i(\{(a, \dots, a, c)\}) < I_i(\{(b, \dots, b, c)\})$ for all $1 \leq i \leq k$. Assume $I_i(\{(a, \dots, a, c)\}) \geq I_i(\{(b, \dots, b, c)\})$ for any i . If $I_i(\{(a, \dots, a, c)\}) > I_i(\{(b, \dots, b, c)\})$, then $(b, \dots, b, c) \not\succeq (a, \dots, a, c)$, which yields a contradiction. If $I_i(\{(a, \dots, a, c)\}) = I_i(\{(b, \dots, b, c)\})$, then $I_i(\{z\}) = I_i(\{(a, \dots, a, c)\})$ for all $z \in S_c$, because $(b, \dots, b, c) \succeq z$ if $z \in S_c$. Then for any $z^1, z^2 \in S_c$ it holds

$$\forall 1 \leq j \leq k, j \neq i : I_j(\{z^1\}) \geq I_j(\{z^2\}) \Leftrightarrow z^1 \succeq z^2$$

which contradicts the assumption that for any $n - 1$ dimensional open hypercube in \mathbb{R}^{n-1} at least $n - 1$ indicators are necessary. Therefore, $I_i(\{(a, \dots, a, c)\}) < I_i(\{(b, \dots, b, c)\})$.

Now, we consider the image of S_c in indicator space. The vectors $\mathbf{I}(\{(a, \dots, a, c)\})$ and $\mathbf{I}(\{(b, \dots, b, c)\})$ determine an open hyperrectangle $H_c = \{(y_1, y_2, \dots, y_k) \in \mathbb{R}^k; I_i(\{(a, \dots, a, c)\}) < y_i < I_i(\{(b, \dots, b, c)\}), 1 \leq i \leq k\}$ where $\mathbf{I}(\mathbf{z}) = (I_1(\mathbf{z}), I_2(\mathbf{z}), \dots, I_k(\mathbf{z}))$. H_c has the following properties:

1. H_c is open in all k dimensions as for all $1 \leq i \leq k$: $\inf\{y_i; (y_1, y_2, \dots, y_k) \in H_c\} = I_i(\{(a, \dots, a, c)\}) < I_i(\{(b, \dots, b, c)\}) = \sup\{y_i; (y_1, y_2, \dots, y_k) \in H_c\}$.
2. H_c contains an infinite number of elements.
3. $H_c \cap H_d = \emptyset$ for any $d \in (u, v), d > c$: assume $\mathbf{y} \in H_c \cap H_d$; then $\mathbf{I}(\{(b, \dots, b, c)\}) \geq \mathbf{y} \geq \mathbf{I}(\{(a, \dots, a, d)\})$, which yields a contradiction as $(b, \dots, b, c) \not\geq (a, \dots, a, d)$.

Since c was arbitrarily chosen within (u, v) , there are uncountably many disjoint open hypercubes of dimensionality k in the k dimensional indicator space. This contradiction implies that $k \geq n$. \square

This theorem is a formalization of what is intuitively clear: we cannot reduce the dimensionality of the objective space without losing information. Unfortunately, the situation gets even worse when we consider approximation sets instead of single objective vectors. Theorem 2 states that there is no way of representing any dominance relation from Definition 3 by a finite combination of unary quality indicators— independent of the dimensionality of the objective space. This means the number of criteria, that determine what a good approximation set is, is infinite; or in another words: the aforementioned goal to define two (or more) indicators, one for distance and one for diversity, that uniquely characterize the quality of an approximation set, cannot be attained.

Theorem 2 *If $Z = \mathbb{R}^n$ with $n \geq 2$, then there is no vector of unary quality indicators $\mathbf{I} = (I_1, I_2, \dots, I_k)$ and a relation \triangleright such that for any approximation sets $A_1, A_2 \in \Omega$:*

$$\mathbf{I}(A_1) \triangleright \mathbf{I}(A_2) \Leftrightarrow A_1 \triangleright A_2 \quad (1)$$

where $\mathbf{I}(A) = (I_1(A), I_2(A), \dots, I_k(A))$ for $A \in \Omega$.

Note that \triangleright can be any relation ($<$, $>$, etc.), i.e., independently of what relation \triangleright we choose, there is no indicator vector such that Statement 1 holds.

To prove this theorem, we need the following fundamental results from set theory (Hrbacek and Jech 1999):

- \mathbb{R}, \mathbb{R}^k , and any open interval (a, b) in \mathbb{R} resp. hypercube $(a, b)^k$ in \mathbb{R}^k have the same cardinality, denoted as 2^{\aleph_0} , i.e., there is a bijection from any of these sets to any other;

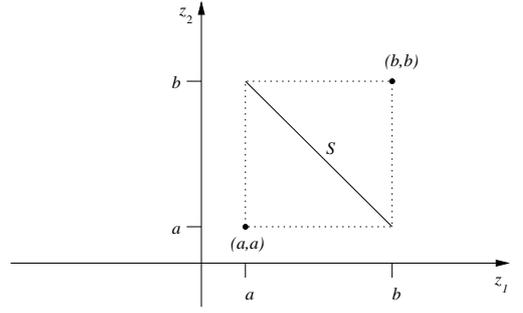


Figure 2: Illustration of the construction used in Theorem 2 for a two dimensional maximization problem. We consider an open hypercube $(a, b)^n$ and define a $n - 1$ dimensional hypercube S within. For S holds that any two objective vectors contained are incomparable to each other, and therefore any subset $A \subseteq S$ is an approximation set.

- If a set S has cardinality 2^{\aleph_0} , then the cardinality of the power set $\mathcal{P}(S)$ of S is $2^{2^{\aleph_0}}$, i.e., there is no injection from $\mathcal{P}(S)$ to any set of cardinality 2^{\aleph_0} .

The proof is based on the construction of a set S (cf. Figure 2) such that any two points contained are incomparable to each other. The power set of S is exactly the set of all approximation sets $A \subseteq S$, the cardinality of which is $2^{2^{\aleph_0}}$. As any two approximation sets must be mapped to a different indicator vector (shown in Lemma 1), an injection from a set of cardinality $2^{2^{\aleph_0}}$ to \mathbb{R}^k is required, which finally leads to a contradiction.

Lemma 1 *Let $Z = \{(z_1, z_2, \dots, z_n) \in \mathbb{R}^n; a < z_i < b, 1 \leq i \leq n\}$ be an open hypercube in \mathbb{R}^n with $n \geq 2$, $a, b \in \mathbb{R}$, and $a < b$. Furthermore, assume there exists a vector of unary quality indicators $\mathbf{I} = (I_1, I_2, \dots, I_k)$ and a relation \triangleright such that for any approximation sets $A_1, A_2 \in \Omega$:*

$$\mathbf{I}(A_1) \triangleright \mathbf{I}(A_2) \Leftrightarrow A_1 \triangleright A_2$$

where $\mathbf{I}(A) = (I_1(A), I_2(A), \dots, I_k(A))$ for $A \in \Omega$. Then, $\mathbf{I}(A_1) \neq \mathbf{I}(A_2)$ for all $A_1, A_2 \in \Omega$ with $A_1 \neq A_2$.

Proof. Let $A_1, A_2 \in \Omega$ be two arbitrary approximation sets with $A_1 \neq A_2$. If $A_1 \triangleright A_2$ or $A_2 \triangleright A_1$, then either $\mathbf{I}(A_1) \triangleright \mathbf{I}(A_2) \wedge \mathbf{I}(A_2) \not\triangleright \mathbf{I}(A_1)$ or vice versa. Thus, $\mathbf{I}(A_1) \neq \mathbf{I}(A_2)$. If $A_1 \parallel A_2$, there are two cases: (1) both A_1 and A_2 contain only a single objective vector, or (2) either set consists of more than one element.

Case 1: Choose $z \in Z$ with $A_1 \parallel \{z\}$ and $A_2 \parallel \{z\}$ (such an objective vector exists as Z is an open hypercube in \mathbb{R}^n). Then $A_1 \cup \{z\} \triangleright A_1$ and $A_1 \cup \{z\} \parallel A_2$. The former implies that $\mathbf{I}(A_1 \cup \{z\}) \triangleright \mathbf{I}(A_1) \wedge \mathbf{I}(A_1) \not\triangleright \mathbf{I}(A_1 \cup \{z\})$. Now suppose $\mathbf{I}(A_1) = \mathbf{I}(A_2)$; it follows that $\mathbf{I}(A_1 \cup \{z\}) \triangleright \mathbf{I}(A_2)$ and therefore

$A_1 \cup \{z\} \triangleright A_2$, which is a contradiction to $A_1 \cup \{z\} \parallel A_2$.

Case 2: Assume, without loss of generality, that A_1 contains more than one objective vector, and choose $z \in A_1$ with $\{z\} \parallel A_2$ (such an element must exist as $A_1 \parallel A_2$). Then, $A_1 \triangleright \{z\}$, which implies that $I(A_1) \blacktriangleright I(\{z\}) \wedge I(\{z\} \blacktriangleright I(A_1))$. Now suppose $I(A_1) = I(A_2)$; it follows that $I(A_2) \blacktriangleright I(\{z\})$ and therefore $A_2 \triangleright \{z\}$, which is a contradiction to $A_2 \parallel \{z\}$.

In summary, all cases ($A_1 \triangleright A_2$, $A_2 \triangleright A_1$, and $A_1 \parallel A_2$) imply that $I(A_1) \neq I(A_2)$. \square

Proof of Theorem 2. Let us suppose that such a quality indicator vector I in combination with a relation \blacktriangleright exists. Furthermore, assume, without loss of generality, that the first two objectives are to be maximized (otherwise the definition of the following set S has to be modified accordingly).

Choose $a, b \in \mathbb{R}$ with $a < b$, and consider $S = \{(z_1, z_2, \dots, z_n) \in Z ; a < z_i < b, 1 \leq i \leq n \wedge z_2 = b + a - z_1\}$; obviously, for any $z^1, z^2 \in Z$ either $z^1 = z^2$ or $z^1 \parallel z^2$, because $z_1^1 > z_1^2$ implies $z_2^1 < z_2^2$. Furthermore, let $\Omega_S \subseteq \Omega$ denote the set of approximation sets $A \in \Omega$ with $A \subseteq S$.

As $S \in \Omega$ and any subset of an approximation set is again an approximation set, Ω_S is identical to the power set $\mathcal{P}(S)$ of S . In addition, there is an injection f from the open interval (a, b) to S with $f(r) = (r, b + a - r, (b + a)/2, (b + a)/2, \dots, (b + a)/2)$, it follows that the cardinality of S is at least 2^{\aleph_0} . As a consequence, the cardinality of Ω_S is at least $2^{2^{\aleph_0}}$.

Now, we will use Lemma 1; it shows that for any $A_1, A_2 \in \Omega_S$ with $A_1 \neq A_2$ the quality indicator values differ, i.e., $I(A_1) \neq I(A_2)$. Therefore, there must be an injection from Ω_S to \mathbb{R}^k , the codomain of I . This means there is an injection from a set of cardinality $2^{2^{\aleph_0}}$ (or greater) to a set of cardinality 2^{\aleph_0} . From this absurdity, it follows that such a vector of unary quality indicators in combination with a relation \blacktriangleright cannot exist. \square

Note that Theorem 2 also holds (i) if we only assume that Z contains an open hypercube in \mathbb{R}^n for which I has the desired property, and (ii) if we consider any other dominance relation from Definition 3. However, if Z is finite, we can easily construct an appropriate unary indicator.

Corollary 1 *If Z is finite, there is a unary quality indicator I and a relation \blacktriangleright such that for any approximations sets $A_1, A_2 \in \Omega$:*

$$I(A_1) \blacktriangleright I(A_2) \Leftrightarrow A_1 \triangleright A_2$$

Proof. As Z is finite, also Ω is finite. Therefore, there exists an injection I from Ω to \mathbb{R} . Accordingly, the relation \blacktriangleright can be defined as $I(A_1) \blacktriangleright I(A_2) \Leftrightarrow I^{-1}(I(A_1)) \triangleright I^{-1}(I(A_2)) \Leftrightarrow A_1 \triangleright A_2$. \square

This result is rather of theoretical than of practical use, because we are mainly interested in indicators that are applicable to arbitrary problems. In general the power of unary indicators is restricted according to Theorems 1 and 2—so, what can we achieve using unary quality indicators?

4 Classification

There are two questions on the basis of which we will categorize quality indicators:

1. Which conclusions can be drawn from the indicator values with regard to the dominance relations?
2. Which portion of a specific dominance relation can be covered on the basis of the indicator values?

Let us go back to the example depicted in Figure 1 and consider the following unary indicator I_ϵ , which is inspired by concepts presented in (Laumanns et al. 2001).

Definition 5 (Unary ϵ -Indicator) *Without loss of generality assume a maximization problem and let $P \in \Omega$ be the Pareto-optimal front. The unary ϵ -indicator I_ϵ is defined as*

$$I_\epsilon(A) = \inf\{\epsilon \in \mathbb{R} ; A \succeq_\epsilon P\}$$

for $A \in \Omega$.

For the three algorithms we get $I_\epsilon(O_1) = 2$, $I_\epsilon(O_2) = 2$, and $I_\epsilon(O_3) = 2.5$. How does the order of the indicator values reflect the dominance relations?

In general, for any pair $(A, B) \in \Omega^2$ it holds

$$A \succ B \Rightarrow I_\epsilon(A) < I_\epsilon(B)$$

and (which follows from this)

$$I_\epsilon(A) < I_\epsilon(B) \Rightarrow A \not\prec B \Rightarrow A \not\triangleleft B$$

A smaller I_ϵ value tells us that an approximation is not worse than another; we say the pair $(I_\epsilon, <)$ is $\not\prec$ -compatible.² Furthermore, if an approximation set strongly dominates another, also its I_ϵ value is smaller; here, we say that $(I_\epsilon, <)$ is \succ -complete. Taken together that means: whenever A strongly dominates B , we will be able to infer that A is not worse than B . In our example, by looking at the I_ϵ values we can conclude that O_1 and O_2 are not worse than O_3 .

The terms compatibility and completeness address the two questions at the beginning of this section and will be used in the following to characterize and compare indicator-relation pairs.

²We use the same term as Hansen and Jaszkiewicz (1998) here, however, with a slightly different meaning.

| compatibility | | completeness | | | | | | |
|----------------------|----------------------|--------------|--------------|---------|------------------|-------------|----------------------|----------------------|
| | | none | $\succ\succ$ | \succ | \triangleright | $\not\prec$ | $\not\triangleright$ | $\not\triangleright$ |
| $\succ\succ$ | $\succ\succ$ | + | - | - | - | - | - | - |
| | \succ | + | ? | - | - | - | - | - |
| \triangleright | \triangleright | + | ? | ? | - | - | - | - |
| | $\not\prec$ | + | + | + | + | - | ? | ? |
| $\not\triangleright$ | $\not\prec$ | + | + | + | + | - | - | ? |
| | $\not\triangleright$ | + | + | + | + | - | - | - |

Table 1: Overview of possible compatibility/completeness combinations. A minus means there is no pair $(\mathbf{I}, \blacktriangleright)$ that is compatible regarding the row-relation and complete regarding the column-relation. A plus indicates that such a pair $(\mathbf{I}, \blacktriangleright)$ is known, while a question mark stands for a combination for which it is unclear whether a corresponding indicator-relation pair exists.

Definition 6 (Compatibility and completeness) Let \mathbf{I} be a vector of unary quality indicators $\mathbf{I} = (I_1, I_2, \dots, I_k)$ and \blacktriangleright a corresponding relation in indicator space. Furthermore, consider an arbitrary binary relation \gg on approximation sets. The pair $(\mathbf{I}, \blacktriangleright)$ is denoted as \gg -compatible if either for any $A, B \in \Omega$

$$\mathbf{I}(A) \blacktriangleright \mathbf{I}(B) \Rightarrow A \gg B$$

or for any $A, B \in \Omega$

$$\mathbf{I}(A) \blacktriangleright \mathbf{I}(B) \Rightarrow B \gg A$$

The pair $(\mathbf{I}, \blacktriangleright)$ is denoted as \gg -complete if either for any $A, B \in \Omega$

$$A \gg B \Rightarrow \mathbf{I}(A) \blacktriangleright \mathbf{I}(B)$$

or for any $A, B \in \Omega$

$$B \gg A \Rightarrow \mathbf{I}(A) \blacktriangleright \mathbf{I}(B)$$

We have seen that $(I_\epsilon, <)$ is $\not\prec$ -compatible and $\succ\succ$ -complete. However, it is neither \triangleright -compatible (as will be shown indirectly in Theorem 3 in Section 4.1) nor \triangleright -complete (as in the above example $O_1 \triangleright O_2$ but $I_\epsilon(O_1) = I_\epsilon(O_2)$).

Now, we can ask what combinations of compatibility and completeness are feasible. Theorem 2 proves that there does not exist any indicator-relation pair that is \triangleright -compatible and \triangleright -complete at the same time. This rules out also other combinations, Table 1 shows which. It reveals that the best we can achieve is either $\succ\succ$ -compatibility without any completeness, or $\not\prec$ -compatibility in combination with \triangleright -completeness.

In the following, we will classify and discuss existing unary indicators according to three categories: \triangleright -compatibility, $\not\prec$ -compatibility, and incompatibility, i.e., no compatibility with any dominance relation. Table 2 summarizes the results. In this context, we would also like to point out the relationships between the dominance relations, e.g., $\succ\succ$ -compatibility implies \triangleright -compatibility, $\not\prec$ -compatibility implies $\not\prec$ -compatibility, and \triangleright -completeness implies $\succ\succ$ -completeness.

4.1 \triangleright -Compatibility

In order to achieve \triangleright -compatibility, at least two indicators are needed as the following theorem shows.

Theorem 3 Consider $Z = \mathbb{R}^n$ with $n \geq 2$ and a unary quality indicator I . If for all $A_1, A_2 \in \Omega$

$$I(A_1) > I(A_2) \Rightarrow A_1 \triangleright A_2$$

then I is a constant function, i.e., $I(\Omega) = c$ with $c \in \mathbb{R}$.

Proof. Assume there are two approximation sets $A_1, A_2 \in \Omega$ with $I(A_1) > I(A_2)$; consequently, $A_1 \triangleright A_2$. Now consider $A_3 \in \Omega$ that is incomparable to both A_1 and A_2 ; as a consequence, $I(A_3) \leq I(A_2)$ and $I(A_1) \leq I(A_3)$. Therefore, $I(A_1) \leq I(A_3) \leq I(A_2)$ which contradicts the assumption. \square

However, even if we consider two or more indicators, the use of \triangleright -compatible indicator-relation pairs is restricted according to Theorem 1: in order to predict dominance between objective vectors at least as many indicators as objectives are required. Hence, it is not surprising that—to our best knowledge—no \triangleright -compatible indicators have been proposed in the literature; their design, though, is possible:

- Consider the line $L = \{(a, a, \dots, a) \in \mathbb{R}^n\}$ and let

$$\begin{aligned} I_1^L(A) &= \sup\{a \in \mathbb{R} ; \{(a, a, \dots, a)\} \triangleleft A\} \\ I_2^L(A) &= \inf\{b \in \mathbb{R} ; \{(b, b, \dots, b)\} \triangleright A\} \end{aligned}$$

We assume a maximization problem and that Z is bounded, i.e., $I_1^L(A)$ and $I_2^L(A)$ always exists. As illustrated in Figure 3, $I_1^L(A)$ determines the point (a, a, \dots, a) that is closest to and worse than A , and $I_2^L(A)$ gives the point (b, b, \dots, b) that is closest to and better than A . If we define the indicator $\mathbf{I}_L = (I_1^L, I_2^L)$ and the relation \blacktriangleright as $\mathbf{I}_L(A) \blacktriangleright \mathbf{I}_L(B) \Leftrightarrow I_1^L(A) > I_2^L(B)$, then the pair $(\mathbf{I}_L, \blacktriangleright)$ is \triangleright -compatible.

- Suppose a maximization problem and let

$$I_i^O(A) = \sup\{a \in \mathbb{R} ; \forall (z_1, \dots, z_n) \in A : z_i \geq a\}$$

for $1 \leq i \leq n$ and

$$I_{n+1}^O(A) = \begin{cases} 1 & \text{if } A \text{ contains two or more elements} \\ 0 & \text{else} \end{cases}$$

We see that I_1^O, \dots, I_n^O describe the closest objective vector that is weakly dominated by all points in A ; I_{n+1}^O serves to distinguish between single objective vectors and larger approximation sets. Let $\mathbf{I}_O = (I_1^O, \dots, I_{n+1}^O)$ and define the relation \blacktriangleright as $\mathbf{I}_O(A) \blacktriangleright \mathbf{I}_O(B)$ if and only if $I_i^O(A) > I_i^O(B)$ for

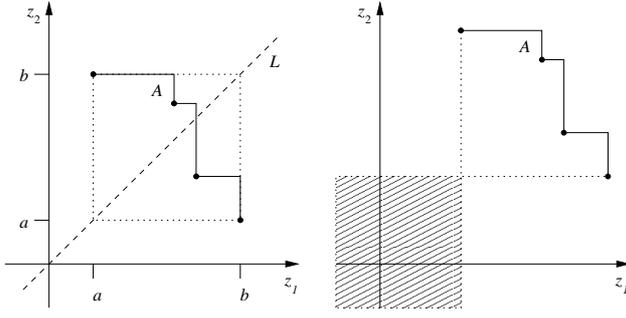


Figure 3: Two \triangleright -compatible indicators. On the left hand side, it is depicted how the I_L indicator defines a hypercube around an approximation set A , where $I_1^L(A) = a$ and $I_2^L(A) = b$. The right picture is related to the I_O indicator: for any objective vector in the shaded area we can detect that it is dominated by the approximation set A .

all $1 \leq i \leq n + 1$. Then, the pair $(I_O, \blacktriangleright)$ is \triangleright -compatible; it detects dominance between an approximation set and those objective vectors that are dominated by all members of this approximation set (see Figure 3).

Note that both indicators are even $\succ\succ$ -compatible, but neither is complete with regard to any dominance relation.

4.2 ∇ -Compatibility

As stated above, the unary ϵ -indicator is ∇ -compatible, and it is $\succ\succ$ -complete but neither \triangleright - nor \succ -complete. That is whenever $A \succ\succ B$, we will be able to state that A is not worse than B . On the other hand, there are cases $A \succ B$ for which this conclusion cannot be drawn, although A is actually not worse than B . The same holds for the two indicators proposed by Esbensen and Kuh (1996) and Czyzak and Jaskiewicz (1998). We will not discuss these in detail and only remark that the following example can be used to show that both indicators in combination with the $<$ relation are not \succ -complete (nor \triangleright -complete): let $A = \{(1, 3)\}$, $B = \{(1, 2)\}$, and the Pareto-optimal front be $P = \{(4, 4)\}$.

An indicator that is ∇ -compatible and \triangleright -complete is the hypervolume indicator I_H (Zitzler and Thiele 1998; Zitzler 1999). It gives the hypervolume of that portion of the objective space that is dominated by an approximation set A .³ We notice that from $A \triangleright B$ follows that $I_H(A) > I_H(B)$; the reason is that A must contain at least one objective vector that is not weakly dominated by B , thus, a certain portion of the objective space is dominated by A but not by B . This observation implies both ∇ -compatibility and \triangleright -completeness: by comparing the I_H values of two approx-

³Note that Z has to be bounded, i.e., there must exist a hypercube in \mathbb{R}^n that encloses Z . If this requirement is not fulfilled, it can be easily achieved by an appropriate transformation.

imation sets we are able to conclude that one is not worse than the other—provided either is actually better than the other.

Van Veldhuizen (1999) suggested an indicator, the error ratio I_{ER} , that is not ∇ -compatible but ∇ -compatible: the ratio of Pareto-optimal objective vectors in the approximation set. Obviously, if any approximation set A consist of only a single Pareto-optimal point, then $I_{ER}(A) \geq I_{ER}(B)$ for all $B \triangleright A$; if B contains not only Pareto-optimal points, then $I_{ER}(A) < I_{ER}(B)$. Therefore, $(I_{ER}, >)$ is not ∇ -compatible. However, if we consider just the total number (rather than the ratio) of Pareto-optimal points in the approximation set, we obtain ∇ -compatibility. Nevertheless, the power of these indicators is limited because neither is in combination with the $>$ relation complete with respect to any dominance relation.

4.3 Incompatibility

Section 3 has revealed the difficulties when trying to separate the overall quality of approximation sets into distinct goals. Nevertheless, it would be desirable if we could look at certain aspects such as diversity separately, and accordingly several authors suggested formalizations of specific aspects by means of unary indicators. However, we have to be aware that often these indicators are *generally* neither \triangleright -compatible nor \triangleright -compatible in combination with the $>$ and $<$ relations, which on the other hand does not mean that they may not be useful for specific applications. We only have to be careful what to infer from the indicator values.

One class of indicators that do not allow any conclusions to be drawn regarding the dominance relationship between approximation sets is represented by the various diversity indicators (Srinivas and Deb 1994; Schott 1995; Zitzler 1999; Deb 2001). If we consider a pair $(A, B) \in \Omega^2$ with $A \triangleright B$, the indicator value of A can be less or greater than or even equal to the value assigned to B (for the diversity indicators referenced above). Therefore, these indicators are neither compatible nor complete with respect to any dominance relation or complement of it.

The same holds for the three indicators proposed in (Van Veldhuizen 1999): overall nondominated vector generation, generational distance, and maximum Pareto front error. The first just gives the number of elements in the approximation set, and it is obvious that it does not provide compatibility and completeness. Why this also applies to the other two, both distance indicators, will only be sketched here. Assume a two-dimensional maximization problem for which the Pareto-optimal front P consists of the two objective vectors $(10, 0)$ and $(0, 10)$. Now, consider the three sets $A = \{(5, 5)\}$, $B = \{(4, 1), (1, 4)\}$, and $C = \{(0, 0)\}$. For both distance indicators holds $I(B) < I(A) < I(C)$, but $A \succ\succ B \succ\succ C$. Thus, we

| symbol | relation | name | reference | compatibility | completeness |
|--------------|-------------|--|------------------------------|---------------|------------------|
| I_L | $I_1 > I_2$ | reference line indicator | Section 4.1 | $\succ\succ$ | - |
| I_O | $>$ | objective vector indicator | Section 4.1 | $\succ\succ$ | - |
| I_H | $>$ | hypervolume | (Zitzler and Thiele 1998) | ∇ | \triangleright |
| I_W | $<$ | average best weight combination | (Esbensen and Kuh 1996) | ∇ | $\succ\succ$ |
| I_D | $<$ | distance from reference set | (Czyzak and Jaskiewicz 1998) | ∇ | $\succ\succ$ |
| I_ϵ | $<$ | unary ϵ -indicator | Definition 5 | ∇ | $\succ\succ$ |
| I_P | $>$ | number of Pareto points contained | Section 4.2 | ∇ | - |
| I_{ER} | $>$ | error ratio | (Van Veldhuizen 1999) | ∇ | - |
| I_{CD} | $<$ | chi-square like deviation | (Srinivas and Deb 1994) | - | - |
| I_S | $<$ | spacing | (Schott 1995) | - | - |
| I_{ONVG} | $>$ | overall nondominated vector generation | (Van Veldhuizen 1999) | - | - |
| I_{GD} | $<$ | generational distance | (Van Veldhuizen 1999) | - | - |
| I_{ME} | $<$ | maximum Pareto front error | (Van Veldhuizen 1999) | - | - |
| I_{MS} | $>$ | maximum spread | (Zitzler 1999) | - | - |
| I_{DU} | $<$ | deviation from uniform distribution | (Deb 2001) | - | - |

Table 2: Overview of unary indicators discussed in this paper. With respect to compatibility and completeness, not all relations are listed but only the strongest as, e.g., \succ -compatibility, implies \triangleright -compatibility (cf. Section 4).

cannot conclude whether one set is better or worse than another by just looking at the order of the indicator values.

Finally, one can ask whether it is possible to combine several non- ∇ -compatible indicators such that the resulting indicator vector is ∇ -compatible. Van Veldhuizen and Lamont (2000), for instance, used generational distance and overall nondominated vector generation in conjunction with the diversity indicator of Schott (1995), while Deb et al. (2000) applied a similar combination of diversity and distance indicators. As in both cases counterexamples can be constructed that show these combinations to be not ∇ -compatible, the above question remains open and is not investigated in more depth here.

5 Conclusions

In *general* the use of unary quality indicators is restricted: either the indicator values allow us to make strong statements (“an approximation set is better than another”) for a rather small number of approximation set pairs as, e.g., with the reference line indicator presented Section 4.1; or we cover a wider range of pairs, but at maximum can conclude that an approximation set is not worse than another as, e.g., with the hypervolume indicator (Zitzler and Thiele 1998). This does not mean that for a specific application unary indicators allow more powerful statements as above. However, binary indicators may be a promising alternative as they overcome the aforementioned problems, though are more difficult to handle. The investigation of this type of indicators and the statistical analysis of data from multiple optimization runs are the subject of ongoing research.

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