Many-objective Problems: Challenges and Methods

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Abstract

This chapter presents a short review of the state-of-the-art of the efforts for understanding and solving problems with a large number of objectives (usually known as Many-objective optimization problems, MOPs). The first part of the chapter presents the current studies aimed at discovering the sources that make a Multiobjective Optimization Problem (MOP) harder when more objectives are added, degrading in this way, the performance of a Multiobjective Evolutionary Algorithm (MOEA). Next, some of the most relevant techniques designed to deal with Many-objective Optimization Problems (MOPs) are presented and categorized.

1 Introduction

Since the first implementation of a Multiobjective Evolutionary Algorithm (MOEA) in the mid 1980s [65], a wide variety of new MOEAs have been proposed, gradually improving in both their effectiveness and efficiency to solve MOPs [11]. However, most of these algorithms have been evaluated and applied to problems with only two or three objectives, in spite of the fact that many real-world problems have more than three objectives [32, 39, 72, 70].

Recent experimental [37, 76, 59] and analytical [74, 46] studies have shown that MOEAs based on Pareto optimality [58] scale poorly in MOPs with a high number of objectives (4 or more). These MOPs are usually known in the community as Many-objective Optimization Problems (MOPs). Although those scalability issues seems to affect mainly to Pareto-based MOEAs, as we will see later in this chapter, optimization problems with a large number of objectives introduce some difficulties common to any other multi-objective optimizer.

The goal of this chapter is presenting a general view of the difficulties posed by many-objective problems for Pareto-based MOEAs. Specifically, we present a review of the potential sources of difficulty currently found in the specialized literature. Likewise, we present a brief review of the current proposals to deal with these sources of difficulty. These proposals are classified into five classes. Among the most common approaches to deal with MOPs we can find the use of preference relations to further rank nondominated solutions, the removal of redundant objectives during or after the search, and the incorporation of preference information. Finally, at the end of the chapter some future research paths are outlined.

2 Basic Concepts and Notation

In this section, we will introduce the concepts and notation that will be used throughout the rest of the paper. Since some of these proposals are based on conflict information among the objectives, some definitions of conflict are provided also.

2.1 Multi-objective Optimization Problems

Definition 1 (Multi-objective optimization problem)**.** *A Multiobjective Optimization Problem (MOP) is defined as:*

$$\begin{aligned} &\textit{Minimize} \quad \mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})]^T, \\ &\textit{subject to} \quad \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{1}$$

The vector $\mathbf{x} \in \mathbb{R}^n$ is formed by n *decision variables* representing the quantities for which values are to be chosen in the optimization problem. The *feasible set* $\mathcal{X} \subseteq \mathbb{R}^n$ is implicitly determined by a set of equality and inequality constraints. The vector function $\mathbf{f} : \mathcal{X} \to \mathbb{R}^k$ is composed by $k \geq 2$ scalar *objective functions* $f_i : \mathcal{X} \to \mathbb{R}$ $(i = 1, \ldots, k)$. In multi-objective optimization, the sets $\mathbb{R}^n$ and $\mathbb{R}^k$ are known as *decision variable space* and *objective function space*, respectively. The image of $\mathcal{X}$ under the function $\mathbf{f}$ is a subset of the objective function space denoted by $\mathcal{Z} = \mathbf{f}(\mathcal{X})$ and referred to as the *feasible set in the objective function space*.

In order to define precisely the multi-objective optimization problem stated in Definition 1 we have to establish the meaning of minimization in $\mathbb{R}^k$. That is to say, we need to define how vectors $\mathbf{z} = \mathbf{f}(\mathbf{x}) \in \mathbb{R}^k$ have to be compared for different solutions $\mathbf{x} \in \mathbb{R}^n$. In single-objective optimization the relation "less than or equal" $(\leq)$ is used to compare the scalar objective values. By using this relation there may be many different optimal solutions $\mathbf{x} \in \mathcal{X}$, but only one optimal value $f^{\min} = \min\{f(\mathbf{x}) \,|\, x \in \mathcal{X}\}$ since the relation $\leq$ induces a total order in $\mathbb{R}$ (i.e., every pair of solutions is comparable, and thus, we can sort solutions from the best to the worst one). In contrast, in multi-objective optimization problems, there is no canonical order on $\mathbb{R}^k$, and thus, we need weaker definitions of order to compare vectors in $\mathbb{R}^k$.

In multi-objective optimization, the *Pareto dominance relation* is usually adopted. This relation was originally proposed by Fracis Ysidro Edgeworth in 1881 [23], but generalized by the French-Italian economist Vilfredo Pareto in 1896 [58].

Definition 2 (Pareto dominance relation)**.** *We say that a vector* $\mathbf{z}^1$ *dominates vector* $\mathbf{z}^2$*, denoted by* $\mathbf{z}^1 \prec \mathbf{z}^2$*, if and only if:*

$$\forall i \in \{1, \ldots, k\} : z_i^1 \leq z_i^2 \tag{2}$$

and

$$\exists i \in \{1, \ldots, k\} : z_i^1 < z_i^2. \tag{3}$$

If $\mathbf{z}^1 = \mathbf{z}^2$ or $z_i^1 > z_i^2$ for some i, then we say that $\mathbf{z}^1$ does not dominate $\mathbf{z}^2$ (denoted by $\mathbf{z^1} \nprec \mathbf{z^2}$). Thus, to solve a MOP we have to find those solutions $\mathbf{x} \in \mathcal{X}$ whose images, $\mathbf{z} = \mathbf{f}(\mathbf{x})$, are not dominated by any other vector in the feasible space. It is said that two vectors, $\mathbf{z^1}$ and $\mathbf{z^2}$, are *mutually nondominated vectors* if $\mathbf{z^1} \nprec \mathbf{z^2}$ and $\mathbf{z^2} \nprec \mathbf{z^1}$.

Definition 3 (Pareto optimality). *A solution* $\mathbf{x}^* \in \mathcal{X}$ *is Pareto optimal if there does not exist another solution* $\mathbf{x} \in \mathcal{X}$ *such that* $\mathbf{f}(\mathbf{x}) \prec \mathbf{f}(\mathbf{x}^*)$.

Definition 4 (Pareto optimal set). *The Pareto optimal set,* P_{opt}*, is defined as:*

$$P_{\text{opt}} = \{\mathbf{x} \in \mathcal{X} \mid \nexists\, \mathbf{y} \in \mathcal{X} \,:\, \mathbf{f}(\mathbf{y}) \prec \mathbf{f}(\mathbf{x})\}. \tag{4}$$

Definition 5 (Pareto front). *For a Pareto optimal set,* P_{opt}*, the Pareto front,* PF_{opt}*, is defined as:*

$$PF_{\text{opt}} = \{\mathbf{z} = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})) \mid \mathbf{x} \in P_{\text{opt}}\}. \tag{5}$$

In decision variable space, these vectors are referred to as decision vectors of the Pareto optimal set, while in objective space, they are called objective vectors of the Pareto optimal set. In practice, the goal of MOEAs is finding the "best" approximation set of the Pareto optimal front. An approximation set is a finite subset of $\mathcal{Z}$ composed of mutually nondominated vectors and is denoted by PF_{approx}. Currently, it is well accepted that the best approximation set is determined by the closeness to the Pareto optimal front, and the spread over the entire Pareto optimal front [20, 83, 11].

A common approach to deal with multi-objective optimization problems is formulating it as a single optimization problem by means of a kind of function called scalarizing function.

Definition 6 (Scalarizing function). *A scalarizing function is a parameterized function* $s : \mathbb{R}^k \to \mathbb{R}$. *Thus, the multi-objective problem is transformed into the following scalar problem:*

$$\begin{aligned} &\textit{Minimize} && s(\mathbf{z}), \\ &\textit{subject to} && \mathbf{z} \in \mathcal{Z}. \end{aligned} \tag{6}$$

A common scalarizing function is based on the Chebyshev distance (L_∞ metric) (see e.g., [55, 24]).

Definition 7 (Weighted Chebyshev scalarizing function). *The weighted Chebyshev scalarizing function (or Chebyshev function for short) is defined by*

$$s_\infty(\mathbf{z}, \mathbf{z}^{\text{ref}}) = \max_{i=1,\ldots,k} \{\lambda_i (z_i - z_i^{\text{ref}})\}, \tag{7}$$

where $\mathbf{z}^{\text{ref}}$ *is a reference point,* $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_k]$ *is a vector of weights such that* $\forall i\ \lambda_i \geq 0$ *and, for at least one* i, $\lambda_i > 0$.

2.2 Notions of Conflict Among Objectives

One important condition of a multi-objective problem is the conflict among their objectives. If the objectives have no conflict among them, then we could solve the problem optimizing each objective function independently. Nonetheless, it has been found that in some problems, although a conflict exists elsewhere, some objectives behave in a non-conflicting manner. Although different authors have proposed definitions for conflict (non-conflict) among objectives (see, e.g. [9, 60, 73, 7]), in this chapter we only present conflict (non-conflict) definitions relevant to this document.

Definition 8. *Let $S_{\mathcal{X}}$ be a subset of $\mathcal{X}$, then, according to Carlsson and Fullér, two objectives can be related in the following ways (assuming minimization):*

1. *f_i is in conflict with f_j on $S_{\mathcal{X}}$ if $f_i(\mathbf{x}^1) \leq f_i(\mathbf{x}^2)$ implies $f_j(\mathbf{x}^1) \geq f_j(\mathbf{x}^2)$ for all $\mathbf{x}^1, \mathbf{x}^2 \in S_{\mathcal{X}}$.*
2. *f_i supports f_j on $S_{\mathcal{X}}$ if $f_i(\mathbf{x}^1) \geq f_i(\mathbf{x}^2)$ implies $f_j(\mathbf{x}^1) \geq f_j(\mathbf{x}^2)$ for all $\mathbf{x}^1, \mathbf{x}^2 \in S_{\mathcal{X}}$.*
3. *f_i and f_j are independent on $S_{\mathcal{X}}$, otherwise.*

In the cases 2 and 3, those objectives are also called non-conflicting objectives. When $S_{\mathcal{X}} = \mathcal{X}$, it is said that f_i is in conflict with (or supports) f_j globally. However, in many MOPs the relation among the objectives changes when comparing different subsets of $\mathcal{X}$. Figure 1 shows an example in which two functions are in conflict in some subsets of $\mathcal{X}$, while in others, they support each other.

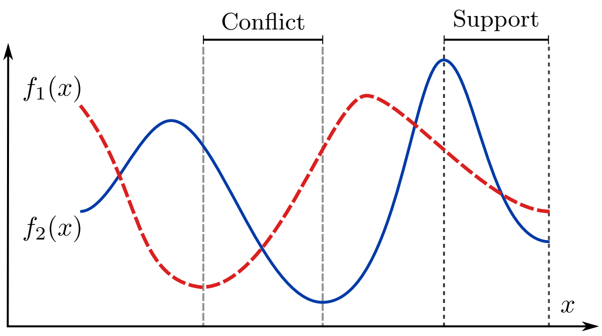

Figure 1: Two objective functions can be in conflict in some subsets of the feasible space, and can be supportive in other subsets.

Non-conflicting objectives are also known as nonessential or redundant objectives because, as pointed out by Gal and Hanne [33], when a non-conflicting objective is removed from the original set of objectives, the resulting Pareto front does not change. Based on the notion of nonessential objectives, Brockhoff and Zitzler [7] proposed a conflict definition that verifies whether the Pareto dominance relation changes when some objectives are removed, or not. The Pareto dominance relation induced by a given set of objectives, $F \subseteq \{f_1, f_2, \ldots, f_k\}$, is defined as $\preceq_F = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{X} \text{ and } \forall f_i \in F : f_i(\mathbf{x}) \leq f_i(\mathbf{y})\}$.

Definition 9. *Let $F_1, F_2 \subseteq \Phi$ be two subsets of objectives, where Φ is the entire set of objectives $\Phi = \{f_1, f_2, \ldots, f_k\}$. Then, we call F_1 non-conflicting with F_2 iff $(\preceq_{F_1} \subseteq \preceq_{F_2}) \wedge (\preceq_{F_2} \subseteq \preceq_{F_1})$.*

In other words, F_1 and F_2 are called non-conflicting if and only if the corresponding relations $\preceq_{F_1}$ and $\preceq_{F_2}$ are identical, but not necessarily $F_1 = F_2$. The non-conflicting definition is useful since if F and $F' \subset F$ are non-conflicting, then we can replace F with F' and obtain the same Pareto optimal front. The objectives in F' are then called essential objectives, whereas the objectives in $F \setminus F'$ are known as nonessential or redundant objectives.

In practice, however, it is useful to allow a certain extent of change on the Pareto front when an objective is omitted in order to define degrees of non-conflict among objectives. In this direction, Brockhoff and Zitzler proposed to use the additive ϵ-dominance indicator to measure the change between two dominance relations. The ϵ-dominance relation induced by a set F is defined by $\preceq_F^\epsilon = \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x}, \mathbf{y} \in \mathcal{X} \text{ and } \forall f_i \in F : f_i(\mathbf{x}) - \epsilon \leq f_i(\mathbf{y})\}$.

Definition 10. *Let $F_1, F_2 \subseteq F$ be two subsets of objectives, where F is the entire set of objectives. Then, we call F_1 δ-non-conflicting with F_2 iff $(\preceq_{F_1} \subseteq \preceq_{F_2}^\delta) \wedge (\preceq_{F_2} \subseteq \preceq_{F_1}^\delta)$.*

In this case, if an objective subset $F' \subset F$ is δ-non-conflicting with F, then we can omit all objectives in $F \setminus F'$ without causing a larger error than δ in the omitted objectives.

3 Sources of Difficulty to Solve Many-Objective Optimization Problems

3.1 Deterioration of the Search Ability

A widespread explanation for this problem is based on the fact that the proportion of nondominated solutions (i.e., equally good solutions according to Pareto dominance) in a population increases rapidly with the number of objectives [26, 78]. In order to illustrate this condition, Figure 2 shows the nondominated regions with respect to a given solution $\mathbf{z}$.

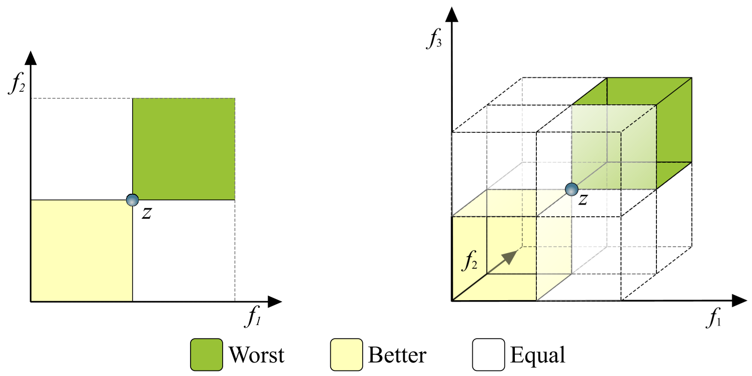

Figure 2: Example of the increasing proportion of nondominated solutions: for 2 objectives 1/2 of the search space is composed of nondominated regions, whereas for 3 objectives 3/4 of the search space consists of nondominated regions. In general, for k objectives, $(2^k - 2)/2^k$ of the objective space comprises nondominated regions.

In general, as presented by Farina and Amato [26], the expression to compute the proportion, e, of mutually nondominated regions and the whole search space is given by $e = (2^k - 2)/2^k$, where k is the number of objectives. This proportion goes to infinity when the number of objectives approaches infinity.

Therefore, since in MOPs with a high number of objectives almost all solutions are equivalent, many researchers have suggested [26, 61, 19, 46, 47, 42] that in such problems, the selection of the appropriate individuals for steering the population towards the Pareto optimal set gets more difficult. As a result, a MOP gets harder to solve as more objectives are added.

However, as pointed out by Schütze et al. [66], the increase of the number of nondominated individuals is not a sufficient condition for increase of the hardness of a problem. Specifically, they conclude that in a class of unimodal problems, their difficulty is marginally increased when more objectives are added despite the exponential growth of the proportion of nondominated solutions with k. Nonetheless, they suggest that the hardness increase observed in experimental studies might be the result of the addition of local optima to the problem as more objectives are aggregated.

Therefore, although the rise of the proportion of incomparable solutions does not significantly determine the difficulty of a MOP *per se*, it seems that the addition of objectives aggravates some particular difficulties observed in the context of 2 or 3 objectives. This is the case of the so called Dominance Resistant Solutions (DRSs) or outliers [40, 35, 20, 36]. DRSs are solutions with a poor value in at least one of the objectives, but with near optimal values in the others. In other words, those are nondominated solutions, but far from the Pareto optimal front. Figure 3 shows an example of DRSs in the well-known test problem DTLZ2 [20]. These kind of solutions represent potential difficulty since, as many researchers have pointed out [40, 35, 20, 36], the number of DRSs grows as the number of objectives is increased.

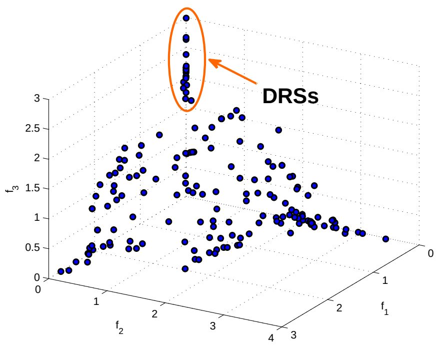

Figure 3: Illustration of some Dominance Resistant Solutions (DRSs) in problem DTLZ2: although solutions marked as DRSs seem to be dominated by some solution in the lower part of the circled solutions, they achieve marginal improvements in objectives f_1 or f_2, and therefore, they are nondominated solutions, but having poor values in objective f_3, though.

3.2 Effectiveness of Crossover Operators

In a combinatorial class of MOPs, Sato et al. [62] performed a series of experiments that revealed that solutions in the variable space become more distant[1] from each other as more objectives are added to the problem. In this scenario, the recombination of two parents close to the Pareto front might generate an offspring far from the Pareto front since a conventional crossover operator might be too disruptive.

3.3 Dimensionality of the Pareto front

Due to the 'curse of dimensionality', the number of points required to represent accurately a Pareto front increases exponentially with the number of objectives. Formally, the number of points necessary to represent a Pareto front with k objectives and resolution r is bounded by $O(kr^{k-1})$ (e.g., see [69]). This expression is derived assuming that each solution is contained inside a hypercube to preserve an even distribution. As can be seen in Figure 4, the number of hypercubes determines the resolution of the Pareto front, i.e., r is the number of hypercubes per dimension. An example of the shortest connected and non-degenerated 2-objective Pareto front (a straight line) is shown on the left side of Figure 4. The figure also shows a bound for the largest Pareto front for 2 and 3 objectives. In general, the bounding Pareto front is formed by k hyperplanes containing r^{k-1} hypercubes each (see, for example, the 3-objective case shown on the right side of Figure 4). This way, the maximum number of points of a 2-objective Pareto front with resolution $r = 6$ is $2 \cdot 6^{2-1} = 12$, whereas for 3 objectives and $r = 5$ is $3 \cdot 5^{3-1} = 75$. Table 1 shows the maximum number of points required to represent a Pareto front for different numbers of objectives using a resolution of $r = 25$, which is a conservative number considering that a resolution of $r = 50$ is usually used in several studies to obtain 100 solutions in 2-objective problems. Notwithstanding, for 5 objectives, we would require approximately 2 million points to represent a Pareto front with resolution $r = 25$. There are other formulations leading to a similar exponential expression with respect to k. For example, using the concept of ϵ-dominance, Laumanns et al. [49] and Schütze et al. [68] give a similar exponential bound for the size of an approximation of a Pareto front.

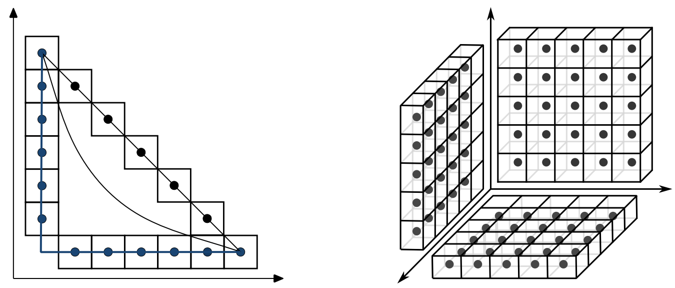

Figure 4: Number of points required to represent a Pareto front with a resolution r, i.e., the number of hypercubes per dimension.

[1]In terms of Hamming distance between binary encoded solutions.

k	Points
2	50
4	62 500
5	1 953 125
7	1 708 984 375

Table 1: Bound for the number of points required to represent a Pareto front with resolution $r = 25$.

This poses some difficulties to solve MOPs. The most important one is the number of function evaluations required to deal with a large number of solutions. This is a serious issue since plenty of real-world problems (e.g., [1, 6, 10, 44, 48, 71, 79]), due to time constraint reasons, have a small budget of function evaluations. In fact, there is an important research effort towards designing MOEAs that generate good approximations of the Pareto front using less than 1000 function evaluations (e.g., [25, 45, 34, 80]). Other challenges are related to the design of both data structures to efficiently manage that number of points, and density estimators to achieve an even distribution of the solutions along the Pareto front. Unfortunately, even if we could efficiently obtain an accurate approximation of the Pareto front, the selection of one solution among such a huge number of solutions would be a very difficult task for a decision maker (DM).

3.4 Visualization of the Pareto front

Clearly, with more than three objectives it is not possible to plot the Pareto front as usual. This is a serious problem since visualization plays a key role for a proper decision making process. Parallel coordinates [77] and self-organizing maps [57] are some of the methods proposed to ease decision making in high dimensional problems. The reader is referred to Chapters 8 and 9 of [5] for a good review of various visualization techniques. Nevertheless, more research in the many-objective optimization context is still required.

4 Current Approaches to Deal with Many-Objective Problems

Besides studies about the scalability of Pareto-based MOEAs, in the current literature we can find several proposals to overcome those scalability issues. The most common approaches can be categorized as follows:

1. Adopt or propose a preference relation that yields a finer solution ordering than the one yielded by Pareto optimality. In other words, these relations are able to further rank nondominated solutions. In addition, most of these preference relations share the property that their optimal set of solutions is a subset of the Pareto optimal set. Therefore, these techniques can also be used as a remedy to cope with the dimensionality of Pareto fronts in Many-objective Optimization Problems (MOPs).

2. Reduce the number of objectives of the problem during the search process or, a posteriori, once an approximation of the Pareto front has been

found [19, 7, 52]. The main goal of this kind of reduction techniques is to identify the non-conflicting objectives (at least to a certain extent) in order to discard them.

3. Scalarizing decomposition of a MOP. As described in the previous section, the degradation observed on MOEAs when dealing with many-objective problems is mainly attributed to the inefficiency of the Pareto relation in high-dimensional spaces. Therefore, methods that do not rely on Pareto dominance, like scalarizing decomposition methods, have been suggested as an alternative to deal with many-objective problems. The underlying idea of this type of methods is performing a number of single-objective searches along different search vectors evenly distributed over the objective space. Each single-objective search is formulated by means of a scalarizing function. This way, the approximation of the Pareto front is composed by the optima found by every single-objective search.

4. Incorporation of preference information interactively throughout the course of the optimization process. By incorporating preferences we can cope with MOPs in two aspects. First, the search can be focused on the decision maker's region of interest, avoiding this way, the evaluation of a huge number of solutions. Second, the preference relations usually used in interactive methods help to deal with a large number of objectives since they are able to rank incomparable nondominated solutions.

5. Use of specialized recombination operators or strategies to control the mating among parents. The first approach tries to diminish the disruptive effect of recombination operators by regulating the proportion in which the traits of each parent contribute to create the offspring. The second approach restricts which individuals can be paired for recombination, for instance using the similarity as mating criteria or the location in the objective space.

In the remainder of this section some of the most relevant approaches to deal with many-objective problems are presented.

4.1 Preference Relations to Deal with Many-Objective Problems

Bentley and Wakefield [3] proposed the Average Ranking (AR) and the Maximum Ranking (MR) preference relations. The AR relation computes, for each solution, a different rank considering each objective independently. The final rank is obtained by summing up the ranks on each objective. In turn, the MR relation takes the best rank as the global rank. Clearly, this method favors extreme solutions, i.e., solutions with high performance in some of the objectives, although with poor overall performance. Although it is less evident, the average ranking relation also favors extreme solutions.

In the *favour relation*, proposed by Drechsler et al. [22], a vector $\mathbf{z}^1$ is preferred to vector $\mathbf{z}^2$ with respect to the favour relation ($\mathbf{z}^1 \prec_{\text{favour}} \mathbf{z}^2$), if and only if:

$$\sharp\{i : z_i^1 < z_i^2, 1 \leq i \leq k\} > \sharp\{j : z_j^1 > z_j^2, 1 \leq j \leq k\}.$$

In other words, the favoured vector is that which outperforms the other one in more objectives. Unfortunately, this relation emphasizes extreme solutions.

The Preference Order Relation (POR), developed by di Pierro [21], is based on the concept of *efficiency of order* proposed by Das [13], which states that: A vector $\mathbf{z}^*$ is efficient of order q if it is not dominated by any other vector in all the $\binom{k}{q}$ objective subsets of size q.

Based on that definition, it is said that vector $\mathbf{z}^1$ is preferred to vector $\mathbf{z}^2$ ($\mathbf{z}^1 \prec_{\mathrm{POR}} \mathbf{z}^2$), if and only if, for some integer q and $\forall I \subseteq \{1, 2, \ldots, k\}$ such that $|I| = q$:

$$z_i^1 \leq z_i^2 \ \forall i \in I, \text{ and } \exists i \in I : \ z_i^1 < z_i^2.$$

In other words, if $\mathbf{z}^1$ and $\mathbf{z}^2$ do not dominate each other, then the solutions are compared in a lower-dimensional space in order to break the tie.

Sato, Aguirre and Tanaka [63] proposed a preference relation to control the dominance area of solutions. This method controls the degree of expansion or contraction of the dominance area by modifying each objective vector $\mathbf{z}$ with the expression:

$$z_i' = \frac{r \cdot \sin(\omega_i + s_i \cdot \pi)}{\sin(s_i \cdot \pi)} \quad \forall\, i = 1, 2, \ldots, k,$$

where $\mathbf{s} \in \mathbb{R}^k$ is a user-defined vector, $r = ||\mathbf{z}||$, and ω_i is the declination angle between $\mathbf{z}$ and the axis of f_i.

If the user adopts values $s_i < 0.5$ ($\forall\, i = 1, 2, \ldots, k$), the dominance area is expanded and produces a more fine-grained ranking of solutions which would strengthen the selection process. Thus, we can say that vector $\mathbf{z}$ is preferred to vector $\mathbf{y}$ with respect to the *expansion relation* ($\mathbf{z} \prec_{\text{expansion}} \mathbf{y}$), if and only if $\mathbf{z}' \prec \mathbf{y}'$.

Farina and Amato [27] proposed an alternative relation which takes into account the number of improved objectives between two solutions. This relation employs three quantities, $n_b(\mathbf{x}_1, \mathbf{x}_2), n_e(\mathbf{x}_1, \mathbf{x}_2)$ and $n_w(\mathbf{x}_1, \mathbf{x}_2)$, which denote the objectives where $\mathbf{x}_1$ is better, equal or worse than $\mathbf{x}_2$, respectively. Using these quantities the concepts of $(1-k)$-*Dominance* and k-*Optimality* are defined. A solution $\mathbf{x}_1$ $(1-k)$-dominates $\mathbf{x}_2$ if and only if

$$\begin{cases} n_e(\mathbf{x}_1, \mathbf{x}_2) < M \\ n_b(\mathbf{x}_1, \mathbf{x}_2) \geq \frac{M - n_e}{k+1} \end{cases}$$

In a similar way to Pareto optimality, a solution $\mathbf{x}^*$ is a k-*optimum* if and only if there is no $\mathbf{x}$ in the decision variable space such that $\mathbf{x}$ k-dominates $\mathbf{x}^*$.

An important remark that we have to take in mind with respect to a new preference relation is that in spite of the fact that some preference relations contribute to converge faster to the Pareto front than the Pareto dominance relation, they also stress the generation of solutions far from the knee region (usually the middle region of the Pareto front). This condition limits the applicability of these relations since, in the general case, it is commonly assumed that the DM prefers solutions from the knee region [12, 54, 4, 67].

4.2 Objective Reduction Approaches

Deb and Saxena [19] proposed a method for reducing the number of objectives based on principal component analysis. The main assumption is that if two objectives are negatively correlated (taking the generated Pareto front as the data set), then these objectives are in conflict with each other. To determine the most conflicting objectives (i.e., the most essential), the authors analyze in turn the eigenvectors (i.e., the principal components) of the correlation matrix. That is, by picking the most-negative and the most-positive elements from the first eigenvector, we can identify the two most important conflicting objectives. To aggregate more objectives to the set of essential objectives the remainder of the eigenvectors are analyzed in a similar way until the cumulative contribution of the eigenvalues exceeds a threshold cut (TC). This method is incorporated into an iterative scheme which uses a multi-objective optimizer (the actual implementation uses the Nondominated Sorting Genetic Algorithm II (NSGA-II) [15]) to obtain a reduced objective set containing only the non-redundant objectives according to the analysis of the eigenvectors. In this scheme, the evolutionary multi-objective optimizer is first run and then, the correlation analysis is carried out to obtain a reduced set of objectives. This process is repeated using the new reduced set of objectives. The process stops when the current subset is equal to the subset generated in the previous iteration.

Brockhoff and Zitzler [7] defined two kinds of objective reduction problems and two corresponding algorithms to solve them. The problems proposed are the following:

1. **The δ-MOSS problem**. Given a MOP, the δ-minimum objective subset problem is defined as follows.
   - **Input**: A Pareto front approximation of the MOP and a $\delta \in \mathbb{R}$.
   - **Task**: Compute the minimum objective subset $F' \subseteq F$ such that F' is δ-non-conflicting with F.
2. **The $\mathcal{K}$-EMOSS problem**. Given a MOP, the problem of finding the minimum objective subset of size $\mathcal{K}$ with minimum error is defined as follows.
   - **Input**: A Pareto front approximation of the MOP and a $\mathcal{K} \in \mathbb{N}$.
   - **Task**: Compute an objective subset $F' \subseteq F$ with size $|F'| \leq \mathcal{K}$, such that F' is δ-non-conflicting with F with the minimum possible δ.

Since both problems are $\mathcal{NP}$-hard, the authors proposed both an exact and a greedy algorithm for each of them. The exact algorithms for both problems have time complexity $O(m^2k \cdot 2^k)$, where m is the size of the given nondominated set and k is the number of objectives. On the other hand, the greedy algorithm for the δ-MOSS problem has time complexity $O(\min\{m^2k^3, m^4k^2\})$, while the greedy algorithm for the $\mathcal{K}$-EMOSS problem has time complexity $O(m^2k^3)$.

A similar approach was proposed by López et al. [52]. They proposed two different objective reduction algorithms:

1. An algorithm that finds a minimum subset of non-redundant objectives with the minimum error possible.
2. An algorithm that finds a $\mathcal{K}$-size subset of non-redundant objectives, yielding the minimum error possible.

Both algorithms are based on an unsupervised feature selection technique proposed by Mitra et al. [56], in which the correlation coefficient is used to estimate the conflict among objectives. Specifically, a negative correlation between a pair of objectives means that one objective increases while the other decreases and vice versa (see for example the functions in Fig. 1). On the other hand, if the correlation is positive, then both objectives increase or decrease at the same time. This way, we could interpret that the more negative the correlation between two objectives, the more the conflict between them.

These two algorithms were designed to be used after an approximation of the Pareto front has been found. From a general point of view, the removal of the non-conflicting objectives can help to the problem designer or the decision maker to gain knowledge about the relation and importance of the objectives according to the conflict. With regard to the decision making process, the removal of the non-conflicting objectives eases the visualization of the approximation of the Pareto front. In cases with a moderate number of objectives (i.e., 4 to 7), the reduced objective set might be visualized using traditional 3D plots.

However, an objective reduction technique can be also used in the course of the search. In [53], for instance, the authors proposed the incorporation of an objective reduction technique into a Pareto-based MOEA in order to cope with many-objective problems during the search. One possible approach is gradually reducing the number of objectives throughout different stages of the search until a target objective subset size has been reached. In each reduction stage, an objective reduction method is applied on the current Pareto front approximation. Towards the end of the search, the original objective set is used again to approximate the entire Pareto front. This kind of approach can be advantageous for solving real-world problems with expensive objective functions since only a small subset of the objective functions is evaluated. Additionally, the use of a small set of objectives throughout the course of the search makes possible the adoption of expensive ranking schemes (e.g., those based on the hypervolume indicator) in problems with a high number of objectives (see e.g., [8]).

A further approach, presented in [50] consists in partitioning the objective set into several subsets so that a different portion of the population focuses the search on a different subspace. The partitioning of the set of objectives is based on the analysis of the conflict information obtained from the current Pareto front approximation.

4.3 Preference Incorporation Approaches

Like the alternative preference relations reviewed in Section 4.1, the integration of DM's preferences provides a finer rank of the solutions. However, unlike preference relation approaches, in an interactive approach the region of interest can be changed during the search according to the requirements of the decision maker.

Among the earliest attempts to incorporate preferences in a MOEA, we can find Fonseca and Fleming's proposal [29, 31]. This proposal consisted of extending the ranking mechanism of Multiobjective Genetic Algorithm (MOGA) [30] using the so-called *preferability relation*. This relation accommodates goal information (equivalent to a reference point in other methods) and priorities in a single preference relation. The DM should define goal values and group objectives according to its priority. Using the preferability relation two solutions are first compared in terms of the group of objectives with the highest priority. If the objectives of both solutions meet all their goal values or, contrarily, violate some or all of their goal values in a similar way, the next priority objective group is considered. This process continues until reaching the lowest priority group, where solutions are compared using the Pareto dominance relation. By setting particular goals and priorities the authors derived the following special cases: the usual Pareto relation, lexicographic relation, constrained optimization, and goal programming. One disadvantage of this relation is that it is affected by the feasibility of the goal provided by the decision maker. If the given goal is far away from the feasible region, then the solutions will be mainly compared in terms of the objective priorities, reducing the relation to the lexicographic relation. In addition, if two solutions either do or do not meet their goals, the relation does not take into account the degree of under- or over-attainment.

Deb [14] proposed a technique to transform goal programming problems into multi-objective optimization problems which are then solved using a MOEA. In goal programming the DM has to assign goals that wishes to achieve for each objective, and these values are incorporated into the problem as additional constraints. The objective function then attempts to minimize the absolute deviations from the goals to the objectives. Unfortunately, as the previous method, this approach is sensitive to the feasibility of the goal values. If the goal is contained in the feasible space, it could prevent the generation of a better solution. On the other hand, if the goal is located far away from the feasible space, the effect of the method is practically nonexistent.

More recently, Deb and Sundar [19] incorporated a reference point approach into the NSGA-II [18]. They introduced a modification in the crowding distance operator in order to select from the last nondominated front the solutions that would take part of the new population. They used the Euclidean distance to sort and rank the population accordingly (the solution closest to the reference point receives the best rank). This method was designed to take into account a set of reference points. The drawback of this scheme is that it only guarantees weak Pareto optimality. That is to say, besides Pareto optimal solutions, the method might generate some weakly Pareto optimal solutions, particularly in MOPs with disconnected Pareto fronts. A similar approach was also proposed by Deb and Kumar [17], in which the light beam search procedure [43] was incorporated into the NSGA-II. Similar to the previous approach, they modified the crowding operator to incorporate DM's preferences. They used a weighted achievement function to assign a crowding distance to each solution in each front. Thus, the solution with the least distance will have the best crowding rank. Like in the previous approach, this algorithm finds a subset of solutions around the optimum of the achievement function adopting the usual outranking relation. A vector $\mathbf{z}^1$ outranks vector $\mathbf{z}^2$ if $\mathbf{z}^1$ is considered to be at least as good as $\mathbf{z}^2$. In [43] three kinds of thresholds are defined to determine if one solution outranks another one, namely, indifference, preference, and veto threshold. However, in [17] the veto threshold is the only one used. This relation depends on the crowding comparison operator. In contrast, the new preference relation presented in this work does not depend on external methods, and, therefore, it can be used in every Pareto-based MOEA.

Recently, Thiele et al. [75] proposed a variant of the Indicator-Based Evolutionary Algorithm (IBEA) [82], in which preference information is incorporated by means of an achievement scalarization function. The basic idea is to divide the original indicator value (which is to be maximized) by the achievement value (which is to be minimized). Thus, solutions with a smaller achievement value will be preferred since the modified indicator value is larger. In a further paper, the new IBEA of Thiele et al. was used in [28] in order to approximate the entire Pareto front by defining several reference points.

A recent interactive optimization method was proposed by López et al. [51] to deal with MOPs. This method is based on a Chebyshev achievement function. The basic idea of the Chebyshev preference relation is to combine the Pareto dominance relation and the achievement function to compare solutions in objective function space. The Chebyshev preference relation is defined as follows.

Definition 11. *A solution* $\mathbf{z}^1$ *is preferred to solution* $\mathbf{z}^2$ *with respect to the Chebyshev relation* ($\mathbf{z}^1 \prec_{\text{cheby}} \mathbf{z}^2$)*, if and only if:*

1. $s_\infty(\mathbf{z}^1, \mathbf{z}^{\text{ref}}) < s_\infty(\mathbf{z}^2, \mathbf{z}^{\text{ref}}) \ \wedge \{\mathbf{z}^1 \notin R(\mathbf{z}^{\text{ref}}, \delta) \vee \mathbf{z}^2 \notin R(\mathbf{z}^{\text{ref}}, \delta)\}$*, or,*

2. $\mathbf{z}^1 \prec \mathbf{z}^2 \wedge \{\mathbf{z}^1, \mathbf{z}^2 \in R(\mathbf{z}^{\text{ref}}, \delta)\}$,

where $R(\mathbf{z}^{\text{ref}}, \delta) = \{\mathbf{z} \,|\, s_\infty(\mathbf{z}, \mathbf{z}^{\text{ref}}) \leq s^{\min} + \delta\}$ *is the Region of Interest with respect to the vector of aspiration levels* $\mathbf{z}^{\text{ref}}$*.*

As an illustration of the preference relation, consider solutions $\mathbf{z}^1$ and $\mathbf{z}^2$ presented in Figure 5. Since $\mathbf{z}^2 \notin R(\mathbf{z}^{\text{ref}}, \delta)$ and $s_\infty(\mathbf{z}^1, \mathbf{z}^{\text{ref}}) < s_\infty(\mathbf{z}^2, \mathbf{z}^{\text{ref}})$, then $\mathbf{z}^1 \prec_{\text{cheby}} \mathbf{z}^2$.

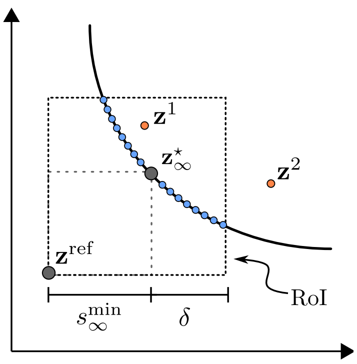

Figure 5: Nondominated solutions with respect to the Chebyshev relation.

5 Recombination Operators and Mating Restrictions

The idea of restricted mating is not new in the field of evolutionary optimization. For instance, in 1989 Deb and Goldberg [16] suggested the use of restrictive mating with respect to the phenotypic[2] distance using some metric. A different

[2] i.e., using the decoded values of the variables

approach consisted in distributing solutions on a logical topology. For example, Baita et al. [2] placed solutions on a grid and restricted the area within which each solution could mate. For more examples of restricted mating the reader is referred to [11].

Recently, specific mating techniques to deal with many-objective problems have been proposed. Sato et al. [64] describe a local recombination scheme that recombines individuals if they have similar search directions in the objective space. The search direction is defined by the polar coordinates of each solution, i.e., its norm and declination angles to the axis associated with the first $k-1$ objectives.

In order to control the disruptive effect of recombination, in [62] a crossover operator for binary representation was proposed, namely the Controlling Crossed Genes (CCG) operator. This technique was applied into the two point and uniform crossover operators. In two-point crossover, from the three binary segments in which two parents are divided, the middle segment is exchanged between the parents to produce two children. Thus, in the CCG operator for two-point crossover, the length of the middle segment is regulated by a user parameter. This way, as the middle segment gets shorter, the generated children become more similar to each parent.

Regarding uniform crossover, the number of exchanged bits between parents is regulated with the probability of writing a 1 or a 0 in the bit mask string that determines which parent bit will be copied into the produced offspring.

6 Scalarization Methods

Most of the scalarization methods have in common the following mechanisms (although they differ in the way in which they are implemented):

- A class of scalarizing function to evaluate solutions.
- A mechanism to generate a uniform distribution of search direction vectors.
- A mechanism to obtain an overall ranking of the solutions derived from the evaluation of each scalarizing function.

Hughes [38] proposed a method in which the weighted Chebyshev function and the vector angle distance scaling are used as scalarizing functions. The method to generate the search direction is formulated as the problem of maximizing the angle between each pair of neighboring search vectors. The fitness of each solution in the current population is based on the best result obtained over all the scalarizing function, i.e., the search direction in which the solution performs better.

Another algorithm that has been recently tested in many-objective problems is the Multiobjective Evolutionary Algorithm Based on Decomposition (MOEA/D) [81]. In [41], the performance of MOEA/D using either a weighted sum function or a Chebyshev function was studied using several instances of a knapsack problem. The results showed that the weighted sum function provided better results than the Chebyshev function, while in nonconvex problems, the Chebyshev function helped to achieve a better performance of MOEA/D.

7 Conclusions and Research Paths

This chapter presented a short review of the current advances to cope with optimization problems with a high number of objectives (MOPs) using MOEA. We covered results aimed at discovering and studying the causes that make a MOP more difficult as more objectives are aggregated. We also described and classified some of the current techniques to deal with MOPs.

Regarding the sources of difficulty of many-objective optimization problems we can realize that most of the initial works are based on experimental analysis, and only a few studies are focused on investigating the nature of the problem using theoretical considerations. When the interest on many-objective optimization problems begun, some hypotheses about the causes of the poor performance of MOEA on MOPs were suggested. Although some of them were considered highly probable and may turn out to be true, further investigation is still needed to confirm or refute these hypotheses. This was the case of the proportion of nondominated solutions, which was often taken as a sufficient condition to increase the difficulty of a MOP. However, recent studies have shown that there exists some problems, in which this proportion rises exponentially, while the hardness of the problem only increases marginally. In this sense, future research paths must be channeled to investigate other sources of difficulty. Some promising areas of future research are, for example, the following:

- Since Dominance Resistant Solutions are not present in every MOP, a characterization of the problems that promote the creation of DRSs is required.
- Investigate if recombination operators in continuous spaces also represent an issue as observed in discrete spaces.

Regarding the methods to solve MOPs, many proposals have been designed to improve the search ability of MOEAs in high dimensional scenarios. However, a few efforts are perceived for developing visualization methods specialized for MOPs. Similarly, more proposals for coping with the dimensionality of the Pareto front are needed. For instance, diversity mechanisms that are effective in large spaces or data structures to efficiently manage a large number of solutions. With respect to the assessment of a new MOEA in many-objective scenarios, our recommendation is adopting a diverse set of MOPs, taking instances from different families of test suites.

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