Aggregated Partial Hypervolumes - An Overall Indicator for Performance Evaluation of Multimodal Multiobjective Optimization Methods

Ali Ahrari[1][0000−0001−7232−7967], Ruhul Sarker[1][0000−0002−1363−2774], and Carlos A. Coello Coello[2][0000−0002−8435−680X]

[1] School of Systems and Computing, University of New South Wales, Canberra, Australia a.ahrari@unsw.edu.au, r.sarker@adfa.edu.au

[2] CINVESTAV-IPN, Departamento de Computación, Mexico City, Mexico ccoello@cs.cinvestav.mx

Abstract. Multimodal multiobjective optimization (MMMOO) can be perceived as the combination of multiobjective optimization (MOO) and multimodal optimization (MMO). The performance of an MMMOO method should be thus assessed from both perspectives, leading to the prevalence of dual-metric indicators in the existing literature. This study first analyzes the ideal outcome of MMMOO for informed decision-making to determine the prerequisites of a theoretically and practically sound performance indicator. Then, it critically evaluates existing indicators, especially those that intend to measure success from the MMO perspective. Subsequently, it introduces Aggregated Partial Hypervolumes (APHVs) as a novel overall parametric performance indicator that not only addresses the drawbacks of existing ones but can also reflect the relative importance of MMO for the decision-maker. Finally, a few descriptive MMMOO examples are studied to verify that the optimal population according to APHVs matches our understanding of the ideal outcome of MMMOO, taking into account the relative importance of both the MMO and the MOO perspectives.

Keywords: Performance indicator · Multimodal multiobjective optimization · Hypervolume.

1 Introduction

Multiobjective optimization (MOO) aims to find a set of diverse non-dominated solutions that approximate the Pareto front (PF). These solutions reveal the trade-off between the potentially conflicting objectives of the problem. The decision-maker can then determine the best overall trade-off among the objectives to select a single solution for its implementation using an *a posteriori* decision-making approach [1].

Quite often, the availability of distinct solutions for the selected trade-off can be beneficial. Such distinct solutions provide alternatives to support a reliable decision-making process [2]. The importance of such distinct solutions has already been analyzed and highlighted for several real-world multiobjective problems, such as path planning [3], space mission design [4], distillation plant layout [5], functional brain imaging [6], and diesel engine design [7].

Multimodal multiobjective optimization (MMMOO) aims to provide such distinct solutions. The goal of MMMOO is to find the whole Pareto set (PS), even though a part of the PS can represent the whole PF. Even solutions that are slightly dominated can be of interest [2]. MMMOO can be perceived as the integration of multimodal optimization (MMO) [8] with MOO, two relatively well-studied fields that can help to advance the knowledge in the less-establish and more complex field of MMMOO [2]. Evaluation of an MMMOO method requires assessing it from both the MOO and the MMO perspectives, resulting in the prevalence of dual-metric indicators in the existing literature:

- a metric that measures the success from the MOO perspective. Most studies used either hypervolume (HV) [9,10,11][12], IGD [13,14,15,16,17,18,19], or both of them [20,21,22,23] [24,25] for this purpose.
- a metric that measures the success from the MMO perspective. Most existing studies used either IGDX ([21,16,10,17,25,18,19,12]), Pareto set proximity (PSP) [20,9,14,11] or both [22,13,23,15,26] for this purpose.

A significant drawback of dual-metric indicators arises when method A is better than method B according to one metric but worse according to the other one. In such cases, a dual-metric indicator cannot determine the superior method. Besides, existing metrics to measure the performance from the MMO perspective suffer from some theoretical shortcomings, which will be explained in Section 3.

To the best of our knowledge, IGDM [27] is the only overall indicator for MMMOO, which addresses some of the limitations of existing dual-metric indicators. Nevertheless, it requires tuning a sensitive parameter. Besides, like IGD, it depends on the procedure used to generate uniformly distributed reference points on the PF, which can cause some biases in the comparison [28]. Therefore, developing other overall performance indicators that can overcome these shortcomings has been encouraged [28].

Another limitation of existing indicators is disregarding the relative importance of MMO and MOO for the decision-maker, which is referred to as *MMO-MOO trade-off* in this study. It implies that improving the performance from the MMO perspective comes at the cost of deteriorating it from the MOO perspective. There are two reasons for this claim. First, it is difficult and sometimes conflicting to efficiently address both MMO and MOO challenges at the same time since each demands certain strategies that can negatively affect the other one. This explains why MMMOO methods are not as good as well-known MOO methods when only MOO is pursued [2]. Second, for a given problem, the theoretically ideal outcome of MMMOO may have a worse HV or IGD than the ideal outcome of MOO. This means that regardless of the efficiency of the employed method, improved diversity in solution space may necessitate some sacrifice in the diversity in the objective function space.

The MMO-MOO trade-off questions whether the added benefits of MMO can justify the decline in MOO capability. The answer to this question depends on how much the decision-maker is interested in the availability of diverse solutions, a feature of the problem that should be specified a priori. Accordingly, performance indicators should be able to reflect the relative importance of MMO for the decision-maker, a feature that is missing in existing ones.

The shortage of overall and pragmatic performance indicators is a major obstacle to the progress of this field since most developments in the field of evolutionary MMMOO rely on experimental evaluations and comparisons of heuristic and meta-heuristic methods and strategies. This study aims to mitigate this shortcoming by introducing a novel performance metric that can reliably measure the success of MMMOO methods. The contributions of this study are as follows:

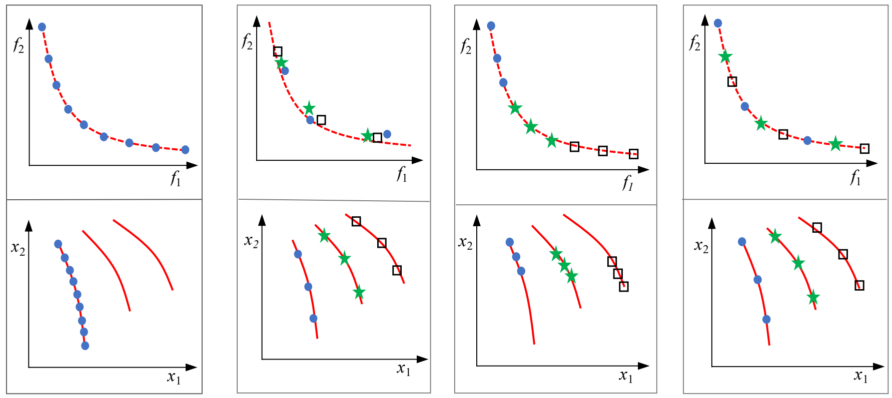
– It analyzes different potential outcomes of MMMOO to determine prerequisites of a theoretically and practically sound performance indicator.
– It scrutinizes existing and popular performance indicators for MMMOO.
– It introduces an overall parametric indicator, called Aggregated Partial Hypervolumes (APHVs) to address the shortcomings of the existing ones.
– It analyzes APHVs on some distinct examples to confirm that indications of APHVs match our understanding of the optimal outcome of MMMOO for informed decision-making.

The rest of this study is organized as follows. Section 2 analyzes some potential outcomes of MMMOO. Section 3 reviews and analyzes relevant performance metrics for MMMOO. Section 4 introduces APHVs. Section 5 designs descriptive examples to study APHVs. Finally, our conclusions are drawn in Section 6.

2 Qualitative Analysis of Potential MMMOO Outcomes

Fig. 1 illustrates a typical MMMOO problem, in which the PS consists of three regions. Each of these regions can represent the whole PF. Four cases for the final population are considered:

– The population in Case I (Fig. 1a) has the ideal outcome from the MOO perspective, but a poor one from the MMO perspective since two regions of the PS have not been detected at all.
– In Case II (Fig. 1b), the population could detect all three regions and for each region, it has provided the three most important solutions, i.e., those that maximize the HV or IGD of that subpopulation. Such an outcome has generally been used in the MMMOO literature to represent a simple MMMOO problem where solutions from different PS regions (PSRs) map to the same point on the PF (e.g., in [14,29,30,31,11,25]). It provides the best approximation of individual regions of the PS. Once a trade-off among



(a) case I (b) case II (c) case III (d) case IV

Fig. 1. Potential outcomes of MMMOO. The problem has three distinguishable PSRs.

the objectives has been selected, three distinct solutions are available for the decision-maker. However, we argue that this is not the ideal outcome when both the MOO and the MMO perspectives are important because all nine solutions almost map to three points on the PF, resulting in an inferior performance from the MOO perspective when compared to the population in Fig. 1a. Population in Case II is superior to that of Case I only if the MMO perspective was quite important for the decision-maker.

- In Case III (Fig. 1c), the population has nicely approximated the PF and detected all three regions of the efficient set. This outcome is indeed superior to the one in Case I; however, it is still not the ideal outcome since solutions from each part of the PF belong to one particular region of the PS. Once the decision-maker specifies the desired trade-off among the objectives, there is limited diversity in the available solutions.
- Case IV (Fig. 1d) shows the pragmatic ideal outcome of MMMOO when performance form both the MOO and the MMO perspectives is important. Convergence and diversity in the objective space are ideal. For each part of the PF, there are solutions from different regions of the PS, and by a slight deviation from the selected trade-off, three distinct solutions are available for the decision-maker.

3 Critical Assessment of Existing Indicators

Hypervolume (HV) [32] and Inverted Generational Distance (IGD) [33] are two of the most popular metrics for assessing performance of MMMOO methods form the MOO perspective. Given a set of solutions $\mathbb{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{|\mathbb{X}|}\}$ with normalized objective values $\mathbf{0}$ $\mathbb{F} = \left\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_{|\mathbb{X}|}\right\}$, HV is the size of the space that is dominated by solutions in $\mathbb{F}$ and dominates a reference point $\boldsymbol{r}$. The ideal and nadir points have normalized objective values of $\mathbf{0}$ and $\mathbf{1}$, respectively, and the reference point should be slightly dominated by the nadir point, e.g. $\boldsymbol{r} = \mathbf{1.1}$, as suggested in [34]. HV is the only known unary Pareto-compliant performance indicator.

IGD is another popular performance indicator for MOO:

$$IGD(\mathbf{F}^*, \mathbf{F}) = \frac{1}{|\mathbb{F}^*|} \sum_{i=1}^{|\mathbf{F}^*|} \min_{\boldsymbol{f}_j \in \mathbb{F}} d(\boldsymbol{f}_i^*, \boldsymbol{f}_j), \tag{1}$$

in which $d()$ calculates the Euclidean distance between two points, and $\mathbb{F}^* = \left\{\boldsymbol{f}_1^*, \boldsymbol{f}_2^*, \ldots, \boldsymbol{f}_{|\mathbb{F}^*|}^*\right\}$ is a set of uniformly distributed reference points on PF. The main drawback of IGD is that it is not Pareto-compliant [35]. Besides, the IGD value depends on the algorithm used to generate the reference points, which may be difficult to reproduce across studies. IGD$^+$ [36] is an enhanced version of IGD which is weakly Pareto compliant; however, it does not resolve the challenge of sampling reference points.

IGDX [21] measures the spread of solutions in the solution space to evaluate the success from the MMO perspective. Analogous to IGD, it samples a set of uniformly distributed solutions $\mathbf{X}^* = \left\{\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \ldots, \boldsymbol{x}_{|\mathbb{X}^*|}^*\right\}$ on PS. Then, IGDX is calculated as follow:

$$IGDX(\mathbf{X}^*, \mathbb{X}) = \frac{1}{|\mathbf{X}^*|} \sum_{i=1}^{|\mathbf{X}^*|} \min_{\boldsymbol{x}_j \in \mathbf{X}} d(\boldsymbol{x}_i^*, \boldsymbol{x}_j). \tag{2}$$

Like IGD, IGDX suffers from the dependency on the employed algorithm for generating the reference point. This dependency is more prominent for IGDX since the solution space has generally a higher dimensionality than the objective

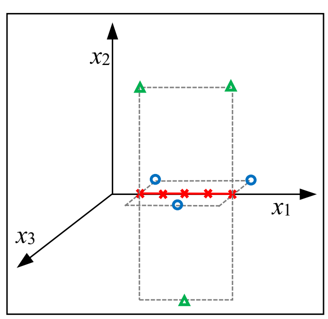
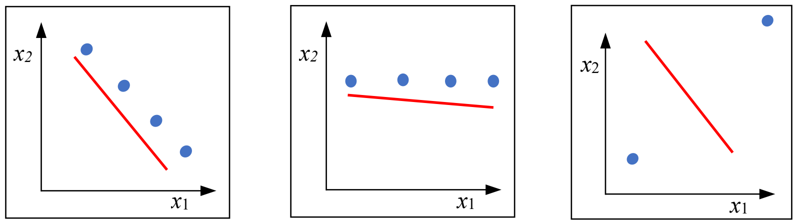


Fig. 2. Given Reference points (crosses), IGDX of population $\mathbf{P}_1$ (shown by circles) is much smaller than that of $\mathbf{P}_2$ (shown by triangles).



(a) (b) (c)

Fig. 3. Actual PS (solid line) and final populations (circles) in for three different cases for problems with two decision parameters.

space, which makes generating reference points that are uniformly distributed more challenging. Besides, directional sensitivity of the objective function(s) can result in misleading IGDX values. Fig 2 depicts such a situation in which at PS (solid line segment on x_1 axis), the fitness function is assumed to be more sensitive along x_3 than x_2. Two populations are considered: $\mathbf{P}_1$ (circles) has a slight deviation from PS along x_3 while $\mathbf{P}_2$ (triangles) has a large deviation along x_2. The fitness values of the solutions can be identical, but population $\mathbf{P}_1$ has a much smaller IGDX, and thus should be regarded as a better one according to the IGDX metric. However, $\mathbf{P}_2$ provides a greater diversity in the solution space; therefore, it should be a better choice from the MMO perspective if diversity of solutions is desired.

Like IGDX, PSP [9] has been frequently used to evaluate the success from the MMO perspective. It is the ratio of the Cover Rate (CR) and IGDX:

$$PSP = \frac{CR}{IGDX}, CR = \left(\prod_{D}^{k=1} \delta_k\right)^{1/(2D)}, \tag{3}$$

in which δ_k is the coverage of the PS for the k^{th} dimension. It is calculated as follows:

$$\delta_k = \begin{cases} 1 & \text{if } V_k^{\max} = V_k^{\min} \\ 0 & \text{if } v_k^{\min} \geq V_k^{\max} \text{ or } v_k^{\max} \leq V_k^{\min} \\ \frac{\min\{v_k^{\max}, V_k^{\max}\} - \max\{v_k^{\min}, V_k^{\min}\}}{V_k^{\max} - V_k^{\min}} & \text{otherwise} \end{cases}, \tag{4}$$

in which $v_k^{\min}$ and $v_k^{\max}$ are the minimum and maximum of the k^{th} element of the solutions in the population, and $V_k^{\min}$ and $V_k^{\max}$ are the minimum and maximum of the PS along the k^{th} coordinate.

CR is inspired by the maximum spread [37], a less popular performance indicator for MOO. CR takes into account only the hypercube that contains final populations, and compares it with a similarly defined hypercube for the actual

PS. Comparing such limited information can easily result in misleading conclusions. For instance, PSP is sensitive to the orientation of PS. Fig. 3a shows a situation in which the final population has approximated the PS quite successfully. The value of CR is about 0.9 in this case. Fig. 3b represents the same problems subject to a linear rotation of the search space, and it is assumed that the population has converged to exactly the previous solutions after undergoing the exact same rotation; therefore, the approximation quality has remained unchanged. It is expected that an indicator assessing the convergence to the PS has identical values for the cases depicted in Figs. 3a and 3b; however, we have *CR=0* for the latter. Fig. 3c reveals a more serious drawback of CR. The final population (two solutions here) could not approximate the PS properly, yet, we have $CR = 1$ in this case, which is the best possible outcome from a MMO perspective according to the CR indicator. Since PSP is proportional to CR, the drawbacks of CR are also present in PSP, even though combining CR with IGDX may mitigate these drawbacks to some extent.

So far, IGDM [27] is the only overall performance indicator for MMMOO [28]. Like IGD, IGDM requires a set of uniformly distributed reference points on the PF ($\boldsymbol{f}_i^*, i = 1, 2, \ldots, |\mathbf{F}^*|$). For each $\boldsymbol{f}_i^*$, it finds all the solutions in the PS that map to that reference point ($\boldsymbol{x}_{i,j}^*$s). Then, the population members are assigned to the closest reference solutions according to the minimum Euclidean distance criterion in the solution space. This means that for every $\boldsymbol{x}_{i,j}^*$, there is a subset of population members, denoted by $\mathbf{P}_{ij}$. Then, the distance between all $\boldsymbol{x}_{ij}^*$s and all the population members in $\mathbf{P}_{ij}$ in the objective space is calculated, and the smallest one is considered d_{ij}. IGDM is the mean of all these d_{ij} values.

IGDM addresses some of the drawbacks of dual-metric indicators. Most importantly, it is an overall indicator which facilitates comparison of MMMOO methods. However, it has the following drawbacks:

- It depends on the algorithm used for generating reference solutions on the PF.
- Some of $\mathbf{P}_{ij}$'s can be empty, for which IGDM sets $d_{ij} = d_{\max}$, which is a default value for reference solutions with no matching population member. This makes the relative values of IGDM sensitive to $d_{\max}$, particularly knowing that IGDM is the mean of d_{ij}s, and the mean is not a robust statistical measure.
- It cannot reflect the relative importance of MOO and MMO for the decision-maker. Based on our analysis, the ideal population with minimal IGDM resembles the one depicted in Case II (1b), which is not generally the best outcome when both MMO and MMO are important.

4 Aggregated Partial Hypervolumes

Our proposed performance measure is based on the summation of partial hypervolumes (PHVs). Let us assume that the PS, consists of K distinct regions, each of which represents the whole or a part of the PF, i.e., $\mathbf{PS} = \cup_{k=1}^{K} \mathbf{PSR}_k$, in which $\mathbf{PSR}_k$ is the k^{th} distinct region of $\mathbf{PS}$. Let $\mathbf{P}_k$ be a subset of population $\mathbf{P}$ for which the closest PSR is $\mathbf{PSR}_k$. It is assumed that $\mathbf{P}_k$ approximates $\mathbf{PSR}_k$. PHV_k is the partial hypervolume that corresponds to $\mathbf{P}_k$, and THV is the total hypervolume that corresponds to $\mathbf{P}$. Aggregated Partial Hypervolumes (APHVs) is defined as follows:

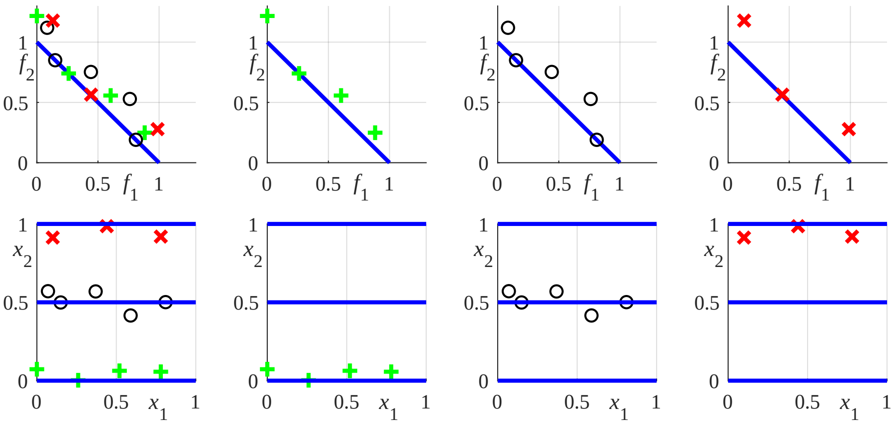
$$\begin{aligned} APHVs &= \alpha_t\, THV + (1-\alpha_t) MPHVs, \\ MPHVs &= \frac{1}{K}\sum_{k=1}^{K} PHV_k,\ 0 \leq \alpha_t \leq 1 \end{aligned} \tag{5}$$

APHVs is the weighted average of two terms. The first term, THV, measures the quality of $\mathbf{P}$ from the MOO perspective (convergence and spread), disregarding which PSRs have been approximated. In contrast, the second term ($MPHVs$), measures how good every PSR has been approximated on average, indicating the quality of $\mathbf{P}$ from the MMO perspective. Parameter α_t specifies the relative importance of MOO and MMO for the decision-maker, i.e., the MMO-MOO trade-off. $\alpha_t = 1$ means MMO has no value for the decision maker, whereas for $\alpha_t = 0$, the importance of finding every region of the PS is maximal.

Fig. 4 illustrates how APHVs is calculated for a simple bi-objective minimization problem. The PS consists of three distinct regions ($\mathbf{PSR}_1, \mathbf{PSR}_2, \mathbf{PSR}_3$), and each region may represent the whole PF. The population $\mathbf{P}$ has successfully approximated all these regions, and can be divided into $\mathbf{P}_1$ (pluses), $\mathbf{P}_2$ (circles), and $\mathbf{P}_3$ (crosses), based on the distance of its members to the each PSR. For this example, Figure 4a calculates $THV = 0.557$ given the reference point $[1.1, 1.1]^T$. Figures 4b, 4c, and 4d illustrate how PHVs are calculated for each subpopulation. For this example, $APHVs = 0.557\alpha_t + (0.460 + 0.476 + 0.384)(1 - \alpha_t)/3$.

When compared with existing indicators for MMMOO, APHVs has the following advantages:

- Like IGDM, APHVs is an overall indicator, which facilitates comparison of MMMOO methods.
- Although APHVs is the weighted average of two terms, these two terms have the same nature. They are all HVs calculated in the objective space with respect to one fixed reference point.
- APHVs uses a distance metric in the solution space only to group population members based on the PSR that they are approximating. Unlike IGDX, MPHVs (or APHVs when $\alpha_t = 0$) does not use any distance metric in the solution space to quantify the performance from the MMO perspective.
- MPHVs implicitly takes the quality of solutions into account. It can deal with potential differences in the sensitivity of the objective function at different parts of the PS or to certain directions, whereas IGDX ignores such information.
- Unlike IGDX, IGD, and IGDM, APHVs does not require uniformly distributed reference points on the PS or the PF. The reference point for the



(a) $THV = 0.557$ (b) $PHV_1 = 0.460$ (c) $PHV_2 = 0.476$ (d) $PHV_3 = 0.384$

Fig. 4. Calculation of different terms in APHVs for a typical problem with three distinct PSRs, represented by solid lines. The reference for the calculation of APHVs is $[1.1, 1.1]^T$.

calculation of APHVs can be easily set (e.g. **1.1** when the objectives are normalized [35]).

– Unlike IGDM, APHVs can easily reflect the relative importance of the MMO perspective for the decision maker by setting α_t at the problem level.
– Unlike IGDM, APHVs does not have any additional and sensitive parameter to account for regions of the PS that have not been detected. These subregions simply have a zero PHV.

Quite often, the PSRs are distinguishable given the mathematical description of the PS. This is the case with most existing mathematically defined benchmark problems for MMMOO, such as those proposed in the CEC 2019 special session on MMMOO [38], in which the PS consists of either end-joined or disjoint continuous PSRs. For a PS with a complex geometry, a mathematically meaningful procedures to divide the PS into PSRs is required at the problem level, which can be regarded as a limitation of APHVs.

5 Descriptive Examples

This section designs descriptive examples to highlight distinctive features of APHVs, especially its capability to reflect the relative importance of MMO and MOO for the decision-maker. The examples are simple but distinct so that the search-space can be exhaustively searched and indications of APHVs can be visually compared with our intuition of the ideal outcome of MMMOO for informed decision-making. For the same reason, only bi-objective problems with two decision parameters ($0 \leq x_1, x_2 \leq 1$) are considered. Both objectives should be minimized, and the reference point for the calculation of APHVs is $\boldsymbol{r} = \mathbf{1.1}$.

For each problem, a set of Pareto optimal solutions of size N_P is provided. The optimal population of size n_P is exhaustively sought such that APHVs is maximized for the predefined α_t. Population members are a subset of the N_P provided Pareto optimal solutions. For each α_t, we report THV and MPHVs and their relative ratios to the maximum possible values in parenthesis. It is possible that multiple of such optimal populations exist in the problem; therefore, for each value of α_t, N_{best} indicates the number of different populations with the maximum APHVs. In such cases, the first population with the maximal APHVs is illustrated. The outcomes are distinct for the selected values of α_t.

5.1 Example 1

This example analyzes APHVs in a relatively simple but insightful scenario. The problem objectives are:

$$\begin{cases} f_1(\boldsymbol{x}) = x_1 \exp(g), f_2(\boldsymbol{x}) = (1 - x_1) \exp(g) \\ g = \sin^2(2\pi x_2) \end{cases}. \tag{6}$$

The PS consists of three disjoint regions with $x_2 = 0, 0.5, 1$, respectively, and each can represent the whole front. For this problem, $N_P = 3 \times 17 = 51$ and $n_P = 6$. Fig. 5 illustrates the ideal population that maximizes APHVs for selected values of α_t. As observed:

- When $\alpha_t = 0$ (maximal importance of the MMO perspective), the optimal population resembles the one depicted in Fig. 1b, which is the ideal outcome from the MMO perspective. The HV is 22.1% less than the ideal case from the MOO perspective due to a lack of sufficient diversity of the whole population in the objective space. In fact, from the MOO perspective, the union of subpopulations is no better than each subpopulation alone.
- when $\alpha_t = 0.1$, subpopulations map to different points of the PF, even though they are still in two distinguishable regions of the PF. THV is now only 9.3% inferior to the ideal outcome form the MOO perspective. At the same time, MPHVs has reduced only 0.5%, which is practically negligible. There are 6 possibilities for the ideal population in this case, which are formed by swapping the relative solutions in each subpopulation.
- A greater α_t is used when the MMO perspective is less important for the decision-maker, e,g,, when the formulated optimization problem is a more accurate model of the actual problem. A better approximation of the whole PF thus becomes more important. Comparing the plots in Fig 5 demonstrates that APHVs nicely reflects this preference, where the optimal population, according to the APHVs indicator, should have a higher diversity in the objective space to maximize THV, in exchange for a weaker approximation of individual PSRs. The former becomes less important for a greater α_t.
- The ideal outcome for $0.3 \leq \alpha_t \leq 0.99$ resembles the outcome in Fig. 1d, which has a nice balance between the importance of MMO and MOO. This shows that the indications of APHVs are robust with respect to α_t.

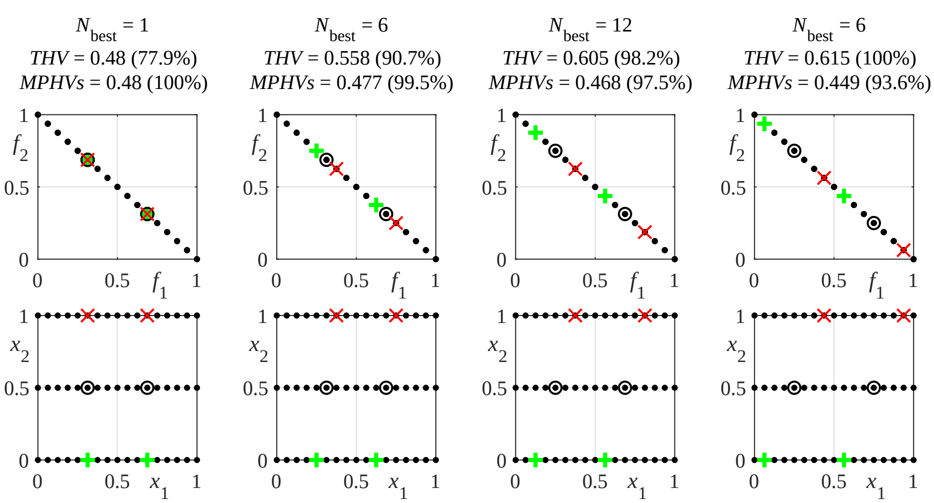
5.2 Example 2

In the second test problem, PS consisted of two regions, $\mathbf{PSR}_1$ and $\mathbf{PSR}_2$ with $x_2 = 0$ and $x_2 = 1$, respectively. Each PSR can represent the whole front, but $\mathbf{PSR}_2$ is four times larger:

$$\begin{cases} f_1(\boldsymbol{x}) = g\, x_1/h, f_2(\boldsymbol{x}) = g\, |1 - x_1/h| \\ g = 2 + \cos(2\pi x_2 + \pi), h = (1 + Ax_2)/(A+1), A = 3 \end{cases}. \tag{7}$$

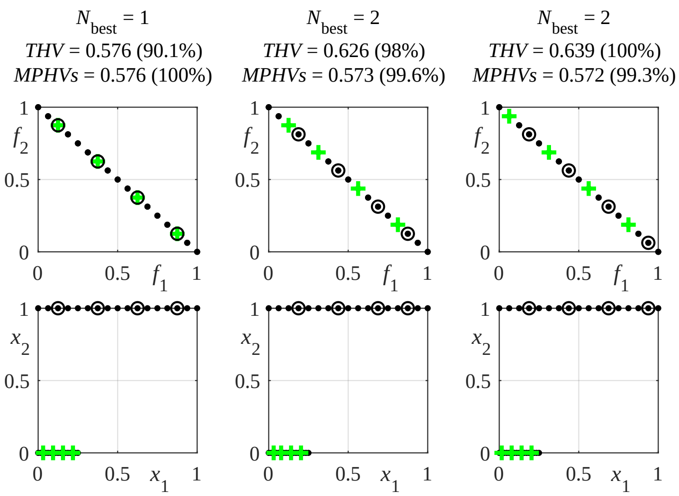
The set of available Pareto optimal solutions consists of 17 solutions on each PSR ($N_P = 2 \times 17 = 34$). For this problem $n_P = 8$. Fig. 6 illustrates the optimal outcome according to the APHVs indicator for different values of α_t. As observed:

- When $\alpha_t = 0$, the population forms two equally sized subpopulations $\mathbf{P}_1$ and $\mathbf{P}_2$, which are uniformly distributed on $\mathbf{PSR}_1$ and $\mathbf{PSR}_2$, respectively, even though $\mathbf{PSR}_2$ is four times larger. Both subpopulations map to exactly the same points in the objective space to maximize their MPHVs, providing the decision-maker with the most important trade-off solutions for each PSR. Once a trade-off is chosen, two distinct solutions are available.
- According to the IGDX indicator, the ideal outcome from the MMO perspective would have only one or two points on $\mathbf{PSR}_1$. The reason for this



(a) $\alpha_t = 0$ (b) $\alpha_t = 0.1$ (c) $\alpha_t = 0.3$ (d) $\alpha_t = 0.8$

Fig. 5. The optimal MMMOO outcome for the first example according to the APHVs indicator with different values of α_t. The dots represent N_P solutions on the PS. Pluses, circles, and crosses are used to divide the population according to which region of the PS they are approximating.



(a) $\alpha_t = 0$ (b) $\alpha_t = 0.1$ (c) $\alpha_t = 0.2$

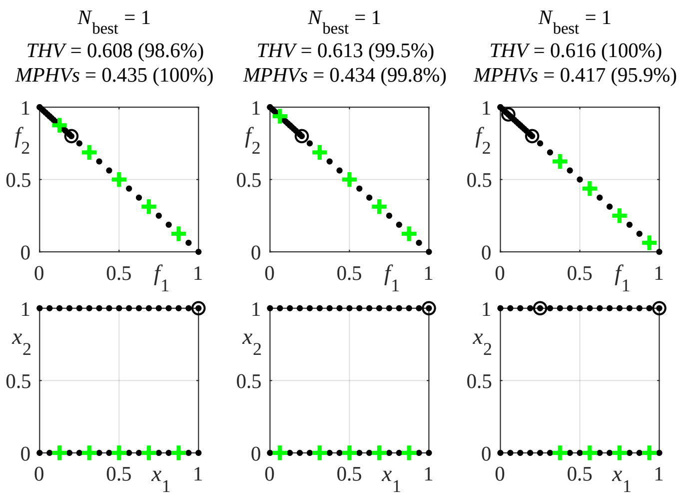
Fig. 6. The optimal MMMOO outcome for the second test problem according to the APHVs indicator with different values of α_t. The dots represent N_P solutions on the PS. Pluses and circles are used to divide the population according to the region of the PS that they are approximating.

is that the number of reference points on $\mathbf{PSR}_2$ would be four times larger than the number of reference points on $\mathbf{PSR}_1$ since reference points for the calculation of IGDX are uniformly distributed on the PS. This observation reveals an intrinsic difference between IGDX and MPHVs (or APHVs when $\alpha_t = 0$): MPHVs and IGDX specify two fundamentally different populations when the ideal outcome from the MMO perspective is desired. From the decision-making perspective, the one specified by IGDX is inferior because distinct solutions might not be available for the selected trade-off.
– When α_t increases, the diversity of the optimal population in the objective space improves while the quality of the approximation of individual PSRs declines. Nevertheless, for the large range of $0.2 \leq \alpha_t \leq 0.99$, the ideal outcome does not change, indicating the robustness of the APHVs metric to the choice of α_t.

5.3 Example 3

The third example investigates a scenario which can be regarded as the opposite of the one in Example 2: $\mathbf{PSR}_1$ at $x_2 = 0$ and $\mathbf{PSR}_2$ at $x_1 = 0$ have equal lengths but the former maps to a small region of the PF:

$$\begin{cases} f_1(\boldsymbol{x}) = g\,x_1/h, f_2(\boldsymbol{x}) = g\,(1 - x_1)/h \\ g = 2 + \cos(2\pi x_2 + \pi), h = (1 + Ax_2), \quad A = 4 \end{cases} . \tag{8}$$



(a) $\alpha_t = 0$ (b) $\alpha_t = 0.2$ (c) $\alpha_t = 0.99$

Fig. 7. The optimal MMMOO outcome for the third example according to the APHVs indicator with different values of α_t. The dots represent N_P solutions on the PS. Pluses and circles are used to divide the population according to the region of the PS that they are approximating.

2×17 uniformly distributed points are provided on the PS from which the optimal population of size $n_P = 6$, which consists of two subpopulations, $\mathbf{P}_1$ and $\mathbf{P}_2$, should be selected. Fig. 7 illustrates the optimal outcome for different values of α_t. As observed:

- For $\alpha_t = 0$, the two subpopulations are not equally sized. Proper approximation of $\mathbf{PSR}_1$ is more important because the contribution of $\mathbf{PSR}_2$ to MPHVs is limited. For decision-making, detection of $\mathbf{PSR}_2$ is useful only if the decision-maker is interested in a trade-off on the upper left part of the PF. This characteristic of the problem is well-captured by the APHVs indicator. The only member of $\mathbf{P}_2$ is on the right corner of $\mathbf{PSR}_2$ to present the most important trade-off that can be offered by a solution in $\mathbf{PSR}_2$. In contrast, IGDX recommends a completely different outcome in which both subpopulations are equally sized and uniformly distributed on PSRs, disregarding the fact that all solutions in $\mathbf{PSR}_2$ are useless unless the decision-maker is interested in a small region of the PF on the top left.
- As expected, a larger α_t emphasizes more on diversity of the whole population in the objective space. Surprisingly, when $\alpha_t = 0.99$, $\mathbf{PSR}_2$ has two members in the optimal population. An extra member on $\mathbf{PSR}_2$ has improved THV because the available points on $\mathbf{PSR}_2$ can provide a better approximation of the upper left part of PF.

6 Summary and Conclusions

This study introduced an overall indicator, called APHVs, which overcomes the discussed theoretical and practical shortcomings of existing indicators. APHVs is the weighted average of two terms: THV, which measures the success from the MOO perspective, and MPHVs, which quantifies the success from the MMO perspective. Both terms are inherently hypervolumes calculated on the objective space with respect to a fixed reference point. The weight parameter, $0 \leq \alpha_t \leq 1$, specifies the importance of the MMO perspective for the decision-maker, a feature that does not exist in the currently available indicators for MMMOO.

Our descriptive test problems presented evidence indicating that APHVs matches our understanding of the desirable outcome of MMMOO with an arbitrary MOO-MMO trade-off. Besides, the parameter α_t of APHVs can reliably reflect the relative importance of MOO and MMO perspectives when evaluating MMMOO methods. When $\alpha_t = 0$, the optimal population aims to make the best approximation of individual regions of the PS. By increasing α_t, the optimal population focuses more on providing a better approximation of the PF; nevertheless, the optimal population is not sensitive to the choice of α_t, and for a large range of α_t, the optimal population should have a good performance from both the MMO and the MOO perspectives.

When compared with existing dual-metric indicators, APHVs takes the relation between PS and PF into account when quantifying MMO success, resulting in a fundamentally different and practically more meaningful perception of the MMO success. For example, if a large region of PS maps to a small part of PF, APHVs gives less importance to that region of the PS. Besides, for APHVs, diverse solutions are those that belong to different regions of the PS, which might not necessarily be far from each other in the solution space.

The limitation of APHVs arises when PSRs cannot be easily determined given the PS; however, this case is scarce in existing benchmark problems for MMMOO. Formulating a mathematically sound procedure to divide the PS into PSRs can address this limitation, which can be the subject of future studies.

References

1. R. C. Purshouse, K. Deb, M. M. Mansor, S. Mostaghim, R. Wang, A review of hybrid evolutionary multiple criteria decision making methods, in: 2014 IEEE congress on evolutionary computation (CEC), IEEE, 2014, pp. 1147–1154.
2. R. Tanabe, H. Ishibuchi, A review of evolutionary multimodal multiobjective optimization, IEEE Transactions on Evolutionary Computation 24 (1) (2019) 193–200.
3. J. Liang, C. Yue, G. Li, B. Qu, P. Suganthan, K. Yu, Problem definitions and evaluation criteria for the cec 2021 on multimodal multiobjective path planning optimization (2020).
4. O. Schutze, M. Vasile, C. A. C. Coello, Computing the set of epsilon-efficient solutions in multiobjective space mission design, Journal of Aerospace Computing, Information, and Communication 8 (3) (2011) 53–70.

5. M. Preuss, C. Kausch, C. Bouvy, F. Henrich, Decision space diversity can be essential for solving multiobjective real-world problems, in: Multiple Criteria Decision Making for Sustainable Energy and Transportation Systems: Proceedings of the 19th International Conference on Multiple Criteria Decision Making, Auckland, New Zealand, 7th-12th January 2008, Springer, 2010, pp. 367–377.
6. M. Sebag, N. Tarrisson, O. Teytaud, J. Lefevre, S. Baillet, A multi-objective multimodal optimization approach for mining stable spatio-temporal patterns., in: IJCAI, 2005, pp. 859–864.
7. T. Hiroyasu, S. Nakayama, M. Miki, Comparison study of spea2+, spea2, and nsga-ii in diesel engine emissions and fuel economy problem, in: 2005 IEEE congress on evolutionary computation, Vol. 1, IEEE, 2005, pp. 236–242.
8. S. Das, S. Maity, B.-Y. Qu, P. N. Suganthan, Real-parameter evolutionary multimodal optimization—a survey of the state-of-the-art, Swarm and Evolutionary Computation 1 (2) (2011) 71–88.
9. C. Yue, B. Qu, J. Liang, A multiobjective particle swarm optimizer using ring topology for solving multimodal multiobjective problems, IEEE Transactions on Evolutionary Computation 22 (5) (2017) 805–817.
10. F. Ming, W. Gong, Y. Yang, Z. Liao, Constrained multimodal multi-objective optimization: Test problem construction and algorithm design, Swarm and Evolutionary Computation 76 (2023) 101209.
11. S. Agrawal, A. Tiwari, B. Yaduvanshi, P. Rajak, Differential evolution with nearest better clustering for multimodal multiobjective optimization, Applied Soft Computing 148 (2023) 110852.
12. Y. Wang, Z. Liu, G.-G. Wang, Improved differential evolution using two-stage mutation strategy for multimodal multi-objective optimization, Swarm and Evolutionary Computation 78 (2023) 101232.
13. Y. Sun, S. Zhang, A decomposition and dynamic niching distance-based dual elite subpopulation evolutionary algorithm for multimodal multiobjective optimization, Expert Systems with Applications (2023) 120738.
14. X. Zhang, H. Liu, L. Tu, A modified particle swarm optimization for multimodal multi-objective optimization, Engineering Applications of Artificial Intelligence 95 (2020) 103905.
15. W. Li, T. Zhang, R. Wang, H. Ishibuchi, Weighted indicator-based evolutionary algorithm for multimodal multiobjective optimization, IEEE Transactions on Evolutionary Computation 25 (6) (2021) 1064–1078.
16. W. Li, X. Yao, K. Li, R. Wang, T. Zhang, L. Wang, Coevolutionary framework for generalized multimodal multi-objective optimization, IEEE/CAA Journal of Automatica Sinica 10 (7) (2023) 1544–1556.
17. Z. Lv, S. Li, H. Sun, H. Zhang, A multimodal multi-objective evolutionary algorithm with two-stage dual-indicator selection strategy, Swarm and Evolutionary Computation (2023) 101319.
18. J. Ji, T. Wu, C. Yang, Multimodal multiobjective differential evolutionary optimization with species conservation, IEEE Transactions on Systems, Man, and Cybernetics: Systems (2023).
19. Z. Ding, L. Cao, L. Chen, D. Sun, X. Zhang, Z. Tao, Large-scale multimodal multiobjective evolutionary optimization based on hybrid hierarchical clustering, Knowledge-Based Systems 266 (2023) 110398.
20. W. Zhang, G. Li, W. Zhang, J. Liang, G. G. Yen, A cluster based pso with leader updating mechanism and ring-topology for multimodal multi-objective optimization, Swarm and Evolutionary Computation 50 (2019) 100569.

21. A. Zhou, Q. Zhang, Y. Jin, Approximating the set of pareto-optimal solutions in both the decision and objective spaces by an estimation of distribution algorithm, IEEE Transactions on evolutionary computation 13 (5) (2009) 1167–1189.
22. J. Liang, K. Qiao, C. Yue, K. Yu, B. Qu, R. Xu, Z. Li, Y. Hu, A clustering-based differential evolution algorithm for solving multimodal multi-objective optimization problems, Swarm and Evolutionary Computation 60 (2021) 100788.
23. C. Yue, B. Qu, K. Yu, J. Liang, X. Li, A novel scalable test problem suite for multimodal multiobjective optimization, Swarm and Evolutionary Computation 48 (2019) 62–71.
24. T. Zhou, X. Han, L. Wang, W. Gan, Y. Chu, M. Gao, A multiobjective differential evolution algorithm with subpopulation region solution selection for global and local pareto optimal sets, Swarm and Evolutionary Computation 83 (2023) 101423.
25. C. Yang, T. Wu, J. Ji, Two-stage species conservation for multimodal multi-objective optimization with local pareto sets, Information Sciences 639 (2023) 118990.
26. M. Xiong, W. Xiong, Z. Liu, Y. Liu, C. Han, A multi-modal multi-objective evolutionary algorithm based on dual decomposition and subset selection, Swarm and Evolutionary Computation (2023) 101431.
27. Y. Liu, G. G. Yen, D. Gong, A multimodal multiobjective evolutionary algorithm using two-archive and recombination strategies, IEEE Transactions on Evolutionary Computation 23 (4) (2018) 660–674.
28. Y. Liu, L. Xu, Y. Han, X. Zeng, G. G. Yen, H. Ishibuchi, Evolutionary multimodal multiobjective optimization for traveling salesman problems, IEEE Transactions on Evolutionary Computation (2023).
29. T. Zhou, Z. Hu, Q. Su, W. Xiong, A clustering differential evolution algorithm with neighborhood-based dual mutation operator for multimodal multiobjective optimization, Expert Systems with Applications 216 (2023) 119438.
30. J. Zou, Q. Deng, Y. Liu, X. Yang, S. Yang, J. Zheng, A dynamic-niching-based pareto domination for multimodal multiobjective optimization, IEEE Transactions on Evolutionary Computation (2023).
31. J. Zhang, J. Zou, S. Yang, J. Zheng, An evolutionary algorithm based on independently evolving sub-problems for multimodal multi-objective optimization, Information Sciences 619 (2023) 908–929.
32. E. Zitzler, L. Thiele, Multiobjective evolutionary algorithms: a comparative case study and the strength pareto approach, IEEE Transactions on Evolutionary Computation 3 (4) (1999) 257–271.
33. C. A. Coello Coello, M. Reyes Sierra, A study of the parallelization of a coevolutionary multi-objective evolutionary algorithm, in: MICAI 2004: Advances in Artificial Intelligence: Third Mexican International Conference on Artificial Intelligence, Mexico City, Mexico, April 26-30, 2004. Proceedings 3, Springer, 2004, pp. 688–697.
34. H. Ishibuchi, R. Imada, Y. Setoguchi, Y. Nojima, How to specify a reference point in hypervolume calculation for fair performance comparison, Evolutionary computation 26 (3) (2018) 411–440.
35. H. Ishibuchi, R. Imada, N. Masuyama, Y. Nojima, Comparison of hypervolume, igd and igd+ from the viewpoint of optimal distributions of solutions, in: Evolutionary Multi-Criterion Optimization: 10th International Conference, EMO 2019, East Lansing, MI, USA, March 10-13, 2019, Proceedings 10, Springer, 2019, pp. 332–345.

36. H. Ishibuchi, H. Masuda, Y. Tanigaki, Y. Nojima, Modified distance calculation in generational distance and inverted generational distance, in: Evolutionary Multi-Criterion Optimization: 8th International Conference, EMO 2015, Guimarães, Portugal, March 29–April 1, 2015. Proceedings, Part II 8, Springer, 2015, pp. 110–125.
37. C.-S. Tsou, H.-H. Fang, H.-H. Chang, C.-H. Kao, An improved particle swarm pareto optimizer with local search and clustering, in: Asia-Pacific Conference on Simulated Evolution and Learning, Springer, 2006, pp. 400–407.
38. J.-J. Liang, B. Qu, D. Gong, C. Yue, Problem definitions and evaluation criteria for the cec 2019 special session on multimodal multiobjective optimization, Computational Intelligence Laboratory, Zhengzhou University (2019) 353–370.