Improving the Integration of the IGD$^+$ Indicator into the Selection Mechanism of a Multi-objective Evolutionary Algorithm

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***Abstract*—In recent years, the design of new selection mechanisms based on quality indicators has become a popular trend in the development of Multi-Objective Evolutionary Algorithms (MOEAs). This trend has been motivated by the well-known limitations of Pareto-based MOEAs when dealing with many-objective optimization problems (i.e., problems having more than 3 objectives). In this paper, we propose a selection mechanism (called IGD$^+$-H) which is based on the combination of the Inverted Generational Distance$^+$ (IGD$^+$) indicator and Kuhn-Munkres' (Hungarian) algorithm to solve Linear Assignment Problems (LAPs). The proposed selection scheme is compared with respect to other selection mechanisms based on the IGD indicator and with respect to the use of the Δ_p indicator. Our proposed technique is incorporated into a MOEA and is validated using standard test functions. Our comparative study indicates that both Δ_p and IGD present some limitations when selecting solutions in degenerate multi-objective problems. Our results show that the transformation of the selection mechanism into a linear assignment problem speeds up the convergence of the MOEA and it is able to solve many-objective problems in an effective and efficient manner. We show that our proposed IGD$^+$-H-based selection mechanism is able to achieve a significant speed up (of up to 200x) with respect to the exclusive use of any of the indicators adopted in our study.**

I. Introduction

A large number of real-world problems that arise in academic and industrial areas, have several (often conflicting) objectives which need to be optimized at the same time. They are generically called multi-objective optimization problems (MOPs) and their solution involves finding the best possible trade-offs among all their objectives. These trade-offs, when defined in decision variable space, constitute the so-called *Pareto optimal set*. The image of the Pareto optimal set is called the Pareto front (PF). Multi-Objective Evolutionary Algorithms (MOEAs) have become an increasingly common approach for solving MOPs, particularly in the last 15 years, mainly because of their flexibility and ease of use.

For several years, MOEAs adopted selection mechanisms based on Pareto optimality. However, in recent studies, it has been found that Pareto-based MOEAs can not properly solve many-objective problems (problems with more than three objectives) [1]. This has motivated the development of indicator-based selection mechanisms, since this sort of mechanism seems to work properly in many-objective problems. The most popular performance indicator has been the hypervolume, mainly because of its nice mathematical properties (it's the only unary indicator which is known to be Pareto compliant [2], [3], [4]). However, the main drawback of hypervolume-based MOEAs is the high computational cost associated with the computation of the exact hypervolume contributions, which becomes unaffordable when trying to solve many-objective optimization problems. A possible way to deal with this limitation is to adopt a different indicator to select solutions in a MOEA such as the Inverted Generational Distance (IGD [5]), which is defined as the average distance from each reference point to its nearest solution. Although the IGD indicator has a low computational cost, it is Pareto non-complaint.

Recently, a variation of the well-known IGD indicator was introduced by Ishibuchi [6]. This indicator, which is called IGD$^+$, was shown to be weakly Pareto compliant and presents some evident advantages with respect to the original IGD (for more details about the weak Pareto compliance of IGD$^+$, see [6]). This indicator is also considered to be a good candidate for being adopted in the selection mechanism of a MOEA. In [7], a new mechanism based on transforming the selection mechanism of a MOEA into a linear assignment problem (LAP) was proposed. This mechanism was shown to be able to deal with many-objective problems as well, without being based on a performance (or quality) indicator.

In this paper, we propose to adopt the aforementioned mechanism that transforms the selection mechanism of a MOEA into a linear assignment problem, but in this case, we adopt the modified Euclidean distance (d+) as our cost function. In order to compute d+, we also incorporate the reference set adopted by the IGD$^+$ indicator.

Thus, we analyze here the impact on this selection mechanism (using a transformation to a linear assignment problem) when it interacts with different indicators based on reference sets.

The remainder of this paper is organized as follows. Section II provides some basic concepts related to multi-objective

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optimization. The most relevant previous related work is described in Section III. In Section IV, we explain how to transform the selection mechanism into a LAP using the IGD$^+$ indicator, and we also briefly describe the procedure adopted to solve the resulting LAP. Our experimental results are presented in Section V-D, including our methodology and a short discussion of our main findings. Finally, our conclusions and some possible paths for future research are provided in Section VI.

II. Basic Concepts

We are interested in solving problems of the type:

$$\text{minimize} \quad \vec{f}(\vec{x}) := [f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x})]^T \tag{1}$$

subject to:

$$g_i(\vec{x}) \leq 0, \quad i = 1, 2, \ldots, p \tag{2}$$

$$h_j(\vec{x}) = 0, \quad j = 1, 2, \ldots, q \tag{3}$$

where $\vec{x} = [x_1, x_2, \ldots, x_n]$ is the vector of decision variables, $f_k : \mathbb{R}^n \to \mathbb{R}$, $k = 1, \ldots, m$ are the objective functions and $g_i, h_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$ are the constraint functions of the problem. Next, we introduce some definitions that will be used in the paper.

Definition 1: Let $\vec{x}, \vec{y} \in \mathbb{R}^m$, we say that $\vec{x}$ "dominates" $\vec{y}$ (denoted by $\vec{x} \prec \vec{y}$), if and only if: i) $x_i \leq y_i$ for all $i \in \{1, \ldots, m\}$ and ii) $x_j < y_j$ for at least one $j \in \{1, \ldots, m\}$.

Definition 2: We say that a vector of decision variables $\vec{x} \in \mathcal{X} \subset \mathbb{R}^n$ is "nondominated" with respect to $\mathcal{X}$, if there does not exist another $\vec{x}' \in \mathcal{X}$ such that $\vec{f}(\vec{x}') \prec \vec{f}(\vec{x})$.

Definition 3: We say that a vector of decision variables $\vec{x} \in \mathcal{F} \subset \mathbb{R}^n$ (where $\mathcal{F}$ is the feasible region) is "Pareto-optimal" if it is nondominated with respect to $\mathcal{F}$.

Definition 4: The Pareto Optimal Set $\mathcal{P}^*$ is defined by:

$$\mathcal{P}^* = \{\vec{x} \in \mathcal{F} | \vec{x} \text{ is Pareto-optimal}\}$$

Definition 5: The "Pareto Front" $\mathcal{PF}^*$ is defined as follows:

$$\mathcal{PF}^* = \{\vec{f}(\vec{x}) \in \mathbb{R}^m | \vec{x} \in \mathcal{P}^*\}$$

As we have already explained before, the IGD$^+$ indicator evaluates the quality of an objective vector using a reference set. In oder to introduce IGD$^+$, we need to describe some indicators which are based on the use of a reference set. First, we have the Generational Distance indicator (GD [8]) which is the averaged semi-distance from the image of a candidate set $\mathcal{A}$ to our discretization of the true Pareto front. Next, we present its formal definition.

Given a candidate set $\mathcal{A} \subset \mathbb{R}^m$ and a reference set $\mathcal{Z} \subset \mathbb{R}^m$, then:

$$GD(\mathcal{A}, \mathcal{Z}) = \frac{1}{|\mathcal{A}|} \left(\sum_{j=1}^{|\mathcal{A}|} d_j(\vec{z}, \vec{a})^p \right)^{1/p} \tag{4}$$

where d_j denotes the nearest Euclidean distance from $a_i \in \mathcal{A}$ to $\mathcal{Z}$. Second, the formal definition of the IGD indicator is defined as [5]:

$$IGD(\mathcal{A}, \mathcal{Z}) = \frac{1}{|\mathcal{Z}|} \left(\sum_{j=1}^{|\mathcal{Z}|} d'_j(\vec{z}, \vec{a})^p \right)^{1/p} \tag{5}$$

where d'_j denotes the minimal Euclidean Distance from $z_j \in \mathcal{Z}$ to $\mathcal{A}$.

Another reference set-based indicator is Δ_p [9], which is considered as an "Averaged Hausdorff Distance" between the approximate set and the reference set. This indicator is based on both GD and IGD. It is defined as:

$$\Delta_p = \max(GD(\mathcal{A}, \mathcal{Z}), IGD(\mathcal{A}, \mathcal{Z})) \tag{6}$$

In spite of the fact that Δ_p is a pseudo-metric which simultaneously evaluates proximity to the Pareto front and spread of solutions along it, it is not Pareto compliant.

Finally, according to [6], the IGD$^+$ indicator can be viewed as follows:

$$IGD^+(\mathcal{A}, \mathcal{Z}) = \frac{1}{|\mathcal{Z}|} \left(\sum_{j=1}^{|\mathcal{Z}|} d' +_j (\vec{z}, \vec{a})^p \right)^{1/p} \tag{7}$$

where $d'+$ is the nearest modified Euclidean distance from $z_j \in \mathcal{Z}$ to $\mathcal{A}$ and the modified euclidian distance $d+(\vec{a}, \vec{z})$ is defined as:

$$d+(\vec{z}, \vec{a}) = \sqrt{(\max\{a_1 - z_1, 0\})^2 + \cdots + (\max\{a_m - z_m, 0\})^2}. \tag{8}$$

Therefore, a low IGD^+ value means that the set $\mathcal{A}$ constitutes a better approximation to the real $\mathcal{PF}$ if we consider the reference set as $\mathcal{PF}_{True}$.

The Pareto compliance property between two objective vectors (i.e., $\vec{a} \prec \vec{b} \implies I(\vec{a}, \vec{z}) < I(\vec{b}, \vec{z})$) does not always hold when the Euclidean distance is used. The authors of IGD$^+$ proved that it is weakly Pareto compliant. Additionally, they also showed that IGD and Δ_p have inconsistencies with respect to the Pareto dominance relation since if a reference point $\vec{z}$ and an objective vector $\vec{a}$ are non-dominated with each other, it is possible to obtain $I(\vec{a}, \vec{z}) > I(\vec{b}, \vec{z})$ for $\vec{a} \prec \vec{b}$. The IGD$^+$ indicator overcomes the main drawbacks of both IGD and Δ_p.

III. Previous Related Work

Previously, we said that the performance indicator which has been most commonly used for the selection mechanism of a MOEA, is the hypervolume [2], [10]. This indicator encapsulates in a single unary value a measure of both the convergence to the true Pareto front and the maximum spread of our Pareto front approximation. One of the best hypervolume-based selection mechanisms currently available is the one incorporated into the *S Metric Selection-Evolutionary Multi-Objective Optimization Algorithm* (SMS-EMOA) [11]. However, the high cost involved in computing exact hypervolume contributions limits the practical use of SMS-EMOA in many-objective problems. In order to address these limitations, some researchers have developed different alternatives, such as the use of reference sets. Next, we will briefly discuss some indicator-based MOEAs which rely on the use of reference sets. Our discussion will focus specifically on approaches that adopt either IGD^+ or Δ_p, since this last indicator combines both GD and IGD (both of which are based on the use of reference sets).

The first MOEA based on Δ_p was called DDE [12], and uses differential evolution as its search engine. Although promising for many-objective optimization, this approach has some limitations related to the use of Δ_p (e.g., it doesn't properly work with multi-frontal problems). Another approach based on this indicator is Δ_p-EMOA [13], which is inspired on SMS-EMOA and incorporates a novel mechanism for building the reference set. The core idea in this case is to linearize the nondominated (piecewise linear) front of the current population. This approach was designed to solve only bi-objective problems. An extension of this approach was proposed in [14] for solving three-objective problems, but the generalization to any number of objectives is very difficult to implement.

Another approach based on this indicator is the *Reference Indicator-Based Evolutionary Multi-Objective Algorithm* (RIB-EMOA) [15]. This MOEA integrates a novel mechanism for building a reference set by using a family of curves and incorporates a selection mechanism based on the exclusive contribution of a solution (as done in SMS-EMOA). RIB-EMOA uses a weight vector set for approximating the reference set and it can solve many-objective optimization problems.

Another MOEA based on Δ_p was proposed in [16]. The authors proposed to create a reference set at each generation using ϵ-dominance. This novel algorithm was validated using standard test functions, having from three to six objective functions.

Recently, a new MOEA based on the IGD$^+$ indicator, called IGD$^+$-EMOA, was proposed in [17]. This MOEA adopts a transformation to a linear assignment problem into its selection mechanism, and IGD$^+$ is also used as the cost function for the linear assignment problem. In IGD$^+$-EMOA, its authors also proposed a novel technique for approximating the reference set, which is built using a set of γ-supersphere curves using a dynamic procedure. IGD$^+$-EMOA uses Newton's method for approximating the $\gamma-$superspheres curve. In spite of the good performance of IGD$^+$-EMOA in many-objective problems, it had difficulties for solving problems with degenerate Pareto fronts. Here, we extend this work by providing a more detailed analysis of the effect of using the aforemention transformation into a linear assignment problem when combined with IGD, Δ_p and IGD^+.

IV. Our Proposed Approach

A. General Framework

The general framework of a MOEA starts with a population $\mathcal{P}_t$ which contains N randomly generated individuals. A new offspring is created by choosing two different parents from $\mathcal{P}$. The parents are recombined using evolutionary operators. In this work, we adopted Simulated Binary Crossover (SBX) and Polynomial-based Mutation [18]. Thereafter, the resulting offspring are added to the new set. This process is repeated until having a total of λ offspring. After that, the algorithm combines the parents and the offspring populations to form the so-called Q set. The new population at generation $t+1$ is generated using different selection mechanisms. Next, we will provide more details of the technique that we propose for selecting the new population.

B. Selection Mechanism

We propose to use a combination of selection techniques. The first technique transforms the selection mechanism into a Linear Assignment Problem (LAP), which uses different cost functions for defining the LAP. In order to explain our technique, we need to provide first more details about the LAP. The LAP is the problem of choosing an optimal assignment of n items (e.g., jobs) to m machines (or workers). Mathematically, the LAP can be expressed as: Given two sets, $A = \{a_1, \dots, a_n\}$ and $T = \{t_1, \dots, t_n\}$ with the same cardinality, and a cost function $C : A \times T \to \mathbb{R}$ and having $\Phi : A \to T$ as the set of all bijections between A and T. So, the LAP can be formulated as follows:

$$\min_{\phi\in\Phi} \sum_{a\in A} C(a, \Phi(a)) \tag{9}$$

Normally, the cost of the problem is also described as a squared matrix C, where each element $C_{i,j} = C(a_i, t_j)$ represents the relationship between a_i and t_j. The cost matrix in terms of a MOP is created as:

$$C_{i,j} = d+(a_i, z_j), \quad i = 1, \dots, |\mathcal{A}|, \quad j = 1, \dots, |\mathcal{Z}|. \tag{10}$$

where $a_i \in \mathcal{Q}$ is the i^{th} point from the population $\mathcal{Q}$, $z_j \in \mathcal{Z}$ is the reference point and $d+$ is the modified Euclidean distance. Analogously, that process can be combined with the normal Euclidean distance.

In order to solve the LAP, we can make use of the Kuhn-Munkres Algorithm, also known as the Hungarian algorithm, which is able to solve LAP instances in polynomial time $\mathcal{O}(n^3)$ [19] for squared matrices. An extension of this algorithm for rectangular matrices was introduced by Bourgeois [20]. The extension to rectangular matrices allows the algorithm to operate in situations where the numbers of reference points and individuals from the population are not equal. The optimal solution to our assignment problem is found by identifying the combination of values in the cost matrix C resulting in the smallest sum. This solution corresponds to the best relationship between the current points of the

$$\begin{pmatrix} 0.350 & 0.727 & 0.007 & 0.165 & 0.221 \\ 0.007 & 0.943 & 0.223 & 0.381 & 0.064 \\ 0.485 & 0.567 & 0.138 & 0.006 & 0.356 \\ 0.663 & 0.011 & 0.317 & 0.183 & 0.534 \\ 0.130 & 0.884 & 0.163 & 0.322 & 0.005 \\ 0.671 & 0.061 & 0.325 & 0.191 & 0.542 \\ 0.677 & 0.025 & 0.331 & 0.197 & 0.548 \\ 0.349 & 0.726 & 0.006 & 0.164 & 0.219 \\ 0.360 & 0.722 & 0.014 & 0.160 & 0.231 \end{pmatrix}$$

Fig. 1. Initial Cost matrix with 9 solutions and 5 reference points, which was generated using the modified Euclidean distance for DTLZ2.

$$\begin{pmatrix} 0.343 & 0.717 & 0.002 & 0.159 & 0.216 \\ 0.000 & 0.933 & 0.217 & 0.375 & 0.059 \\ 0.478 & 0.556 & 0.133 & 0.000 & 0.351 \\ 0.656 & 0.000 & 0.311 & 0.177 & 0.529 \\ 0.124 & 0.873 & 0.157 & 0.316 & 0.000 \\ 0.665 & 0.050 & 0.320 & 0.185 & 0.538 \\ 0.671 & 0.014 & 0.325 & 0.191 & 0.544 \\ 0.342 & 0.715 & 0.000 & 0.158 & 0.215 \\ 0.353 & 0.711 & 0.008 & 0.153 & 0.226 \end{pmatrix}$$

Fig. 2. Final Cost matrix with 9 solutions and 5 reference points, which contains the optimal solution of the LAP for DTLZ2.

population and the elements of a reference set. Example 1 (see Figure 1) shows a cost matrix with 9 rows and 5 columns, where the number of columns expresses the cardinality of the reference set. The optimal solution of the LAP is obtained by selecting the solutions 1, 2, 3, 4 and 7 from the final cost matrix. The smallest sum of the cost matrix is $0.007 + 0.006 + 0.011 + 0.005 + 0.006 = 0.035$, which represents the best relationship between the reference set and the objective solutions.

Fig. 3 shows an example of how the selection technique works. In this case, each reference point is represented by a white square and the optimal solution of the LAP is indicated using black circles. Our proposed selection mechanism works in the same manner as the one proposed in [17], but with one important difference: here, the construction of the reference set is performed at the beginning of the search and remains static (i.e., without changes), whereas in [17], the reference set is re-computed at each generation.

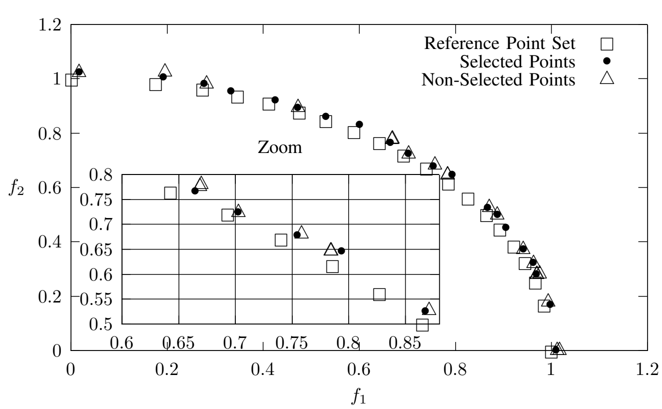

Fig. 3. Example of our proposed selection mechanism based on IGD$^+$ and the Hungarian Algorithm for DTLZ1 with 20 reference points.

Finally, we adopt another way to reduce the size of the current population $\mathcal{Q}$ (for more details see [15]). This selection mechanism chooses the N best individuals from $\mathcal{Q}$ using any performance indicator such as IGD, Δ_p or IGD$^+$. This selection mechanism works as the one adopted by SMS-EMOA, which discards one solution from the population. In SMS-EMOA, the individual that is removed is the one that minimizes the exclusive contribution of the hypervolume indicator. The exclusive contribution of a solution using any reference set indicator (I_x) is described as:

$$EI_x(a, \mathcal{A}, \mathcal{Z}) = I_x(\mathcal{A} \backslash \{a\}, \mathcal{Z}). \tag{11}$$

where $\mathcal{A}$ is the population and $\mathcal{Z}$ is the reference set. However, this process works only if the cardinality of the offspring population is one.

C. Approximating the Reference Set

There are several methods available for building the reference set. One of them was proposed by Menchaca et al. in [16], and uses ϵ-dominance to establish a lower region. They proposed to use an identification vector for splitting the space into hypercubes. Each component of the vector keeps the ϵ distance, which is established for each dimension of the space. This is a novel approach but, unfortunately, it does not provide solutions with a good (i.e., uniform) distribution. There is another approach, which was proposed in [17], where the authors tried to approximate the reference set using super-spheres. In this case, the authors departed from the assumption that the Pareto fronts are smooth convex or concave surfaces. They showed that it is possible to calculate the curve by applying Newton's method. Although that approach solves MOPs in which the Pareto front is either linear, concave or convex, it does not work with degenerate Pareto fronts.

In this paper, we aim to analyze the impact of using different selection mechanisms based on reference sets. For this reason, we used a sample of the true Pareto front, which was randomly generated with 3000 points. The size of the reference set was also reduced to N points. Indeed, we selected the N points that maximize the hypervolume indicator. In order to do this, we adopted a greedy algorithm based on the hypervolume contributions to reduce the number of reference points to a certain specific size.

V. Experimental results

We compare the performance of the selection mechanisms previously discussed. For this sake, each selection mechanism was included into a general MOEA (see subsection IV-A). The parameters of each MOEA used in our study were chosen in such a way that we could do a fair comparison among them

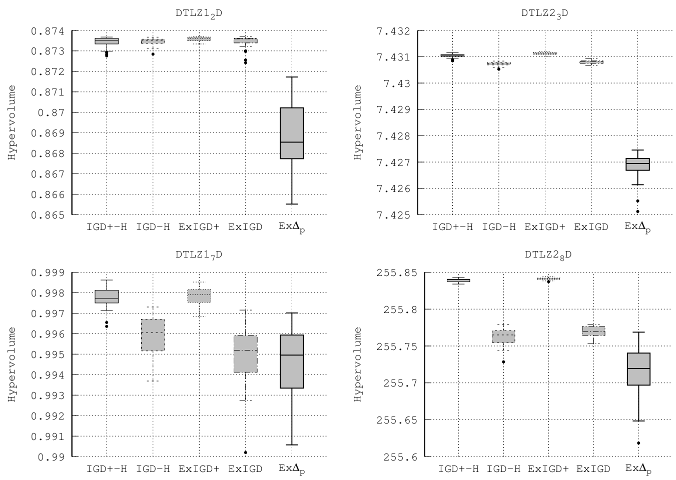

Fig. 4. Performance comparison among MOEAs, where each plot was obtained from 30 independent executions solving DTLZ1 and DTLZ2 with 2 to 8 objectives.

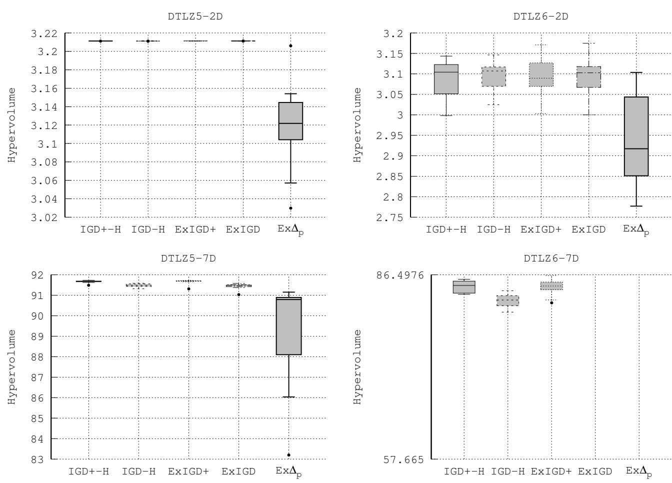

Fig. 5. Performance comparison among MOEAs, where each plot was obtained by 30 independent executions solving DTLZ5 and DTLZ6 with 2 to 7 objectives.

and we could adopt the same evolutionary operators for each version of the MOEA.

A. Test problems

For our comparative study, we adopted the Deb-Thiele-Laumanns-Zitzler (DTLZ) test suite [21]. This set of problems includes aspects such as separability and multifrontality which make them more difficult to solve. We selected problems in such a way that we had different Pareto front shapes such as linear, concave and degenerate linear, since we aimed to observe the impact of each of the selection mechanisms previously discussed.

B. Methodology

For our comparative study, we decided to adopt the hypervolume indicator, which assesses both convergence to the true Pareto front and maximum spread along it. Mathematically, if Λ denotes the Lebesgue measure, the hypervolume can be described as follows:

TABLE I
REFERENCE POINTS USED TO COMPUTE THE HYPERVOLUME INDICATOR FOR EACH DTLZ TEST PROBLEM.

Problem	**Reference points**
DTLZ1	$(1, 1, 1, \ldots, 1)$
DTLZ2-6	$(2, 2, 2, \ldots, 2)$

$$I_H(\mathcal{A}, \vec{y_{ref}}) = \Lambda\left(\bigcup_{\vec{y}\in\mathcal{A}} \{\vec{x} | \quad \vec{y} \prec \vec{x} \prec \vec{y_{ref}}\}\right) \tag{12}$$

where $\mathcal{A}$ is the approximation of the Pareto optimal front and $\vec{y_{ref}} \in \mathbb{R}^k$ denotes the reference point. To compute I_H, we used the reference points shown in Table I. Additionally, we also compared the running time of each version of MOEA, which was measured in minutes.

C. Parameterization

In the DTLZ test suite, the total number of decision variables is given by $n = m + k - 1$, where m is the number of objectives and k was set to 5 for DTLZ1 and to 10 for DTLZ2 to DTLZ6. The number of objectives was set from 2 to 8.

The parameters of each MOEA used in our study were chosen in such a way that the MOEAs were able to converge to the true Pareto Front of the test instances adopted. The distribution indexes for the SBX and polynomial-based mutation operators [18], adopted by each MOEA, were set as: $\eta_c = 20$ and $\eta_m = 20$, respectively. The crossover probability was set to $p_c = 0.9$ and the mutation probability was set to $p_m = 1/L$, where L is the number of decision variables. The total number of function evaluations was not allowed to exceed 50,000. We used a population size of 110 individuals and we iterated during 450 generations. For each MOEA, we used the same reference set, which was generated for each test problem (DTLZ1 to DTLZ6). Our experiments were run on a computer with an Intel Core i5-3930k processor running at 2.70 GHz, with 8GB of RAM.

D. Discussion of Results

Table II provides the average hypervolume over the 30 independent executions of each compared MOEA for each instance of the DTLZ test suite. The best results are presented in **boldface** and grey-colored cells show the second best results. The running time is shown in parentheses. It is clear that the winner in this experimental study is the IGD$^+$-based selection mechanism since the IGD$^+$-based MOEAs were able to outperform both the IGD-based MOEAs and the Δ_p-based MOEA in all the test problems in terms of the hypervolume indicator.

We can observe in Figures 4 and 5 that the medians and variances of all results are different, which means that the differences obtained are statistically significant. Likewise, we can see that the Δ_p-based MOEA has a lower hypervolume value than both the IGD$^+$H-based MOEA and the IGD-H-based MOEA. It is worth noting that, since Δ_p adopts both IGD and GD, the GD indicator affects the performance of the Δ_p indicator. The reason is that GD calculates the average distance from each solution to its closest reference point, and this causes the selection mechanism based on the Δ_p indicator to select dominated solutions. For this reason, the Δ_p-based MOEA is not able to converge. As can be observed in Figures 4 and 5, the Δ_p-based selection mechanism incorporates a change between GD an IGD which makes the variance to increase, since these two selection schemes perform differently.

As shown in Table II, IGD$^+$-H-MOEA and ExIGD$^+$-MOEA were able to outperform IGD-H-EMOA, ExIGD-EMOA and ExΔ_p-EMOA in all cases. These two approaches (IGD$^+$-H-MOEA and ExIGD$^+$-MOEA) obtained similar results. Although IGD-H-EMOA works similarly to IGD$^+$-H-EMOA (since both incorporate the same selection mechanism based on LAP, but have a different cost function), IGD-H-EMOA was not able to converge to the true Pareto front, whereas IGD$^+$-H-EMOA was able to do it. The main reason for this is that the use of the Euclidean distances affects the dominance relation because the calculation of the Euclidean distance is inconsistent with the Pareto dominance relation when the reference point does not dominate the solutions. This makes the IGD-based selection mechanism to choose solutions which are close to the reference set, and avoids selecting nondominated solutions. ExIGD-EMOA and ExΔ_p-EMOA are unable to converge to the Pareto front in DTLZ5 and DTLZ6 with 5, 6, 7 and 8 objectives since the use of Euclidean distances affects their selection mechanisms.

The main reason for which the IGD$^+$-based MOEAs showed a better performance than MOEAs based on IGD and Δ_p, is the incorporation of the modified Euclidean distance since this distance adopts an inferiority vector, which can be viewed as the minimum amount of the increase from a z reference point so that the result vector is weakly dominated by the objective point. That modification makes possible to consider non-dominated points when the IGD$^+$ indicator is used as our selection mechanism. The modified Euclidean distance solves the drawbacks of IGD and Δ_p. We showed that the use of the modified Euclidean distance significantly improves the performance of the selection mechanism. Notwithstanding, the running time of ExIGD$^+$-EMOA is higher than that of IGD$^+$-H-EMOA.

Table II indicates that IGD$^+$-H-EMOA has the lowest running times. However, ExIGD$^+$-EMOA was able to solve the test problems adopted in a reasonably low running time (particularly for the instances having 2 and 3 objectives).

Figures 6 and 7 present a graphical representation of the running time of each MOEA for DTLZ1 and DTLZ5. These plots correspond to the average time value from 30 independent executions. We can observe that the LAP reduces the running time of a MOEA since the optimal solution of the LAP guarantees the best relationship between the reference

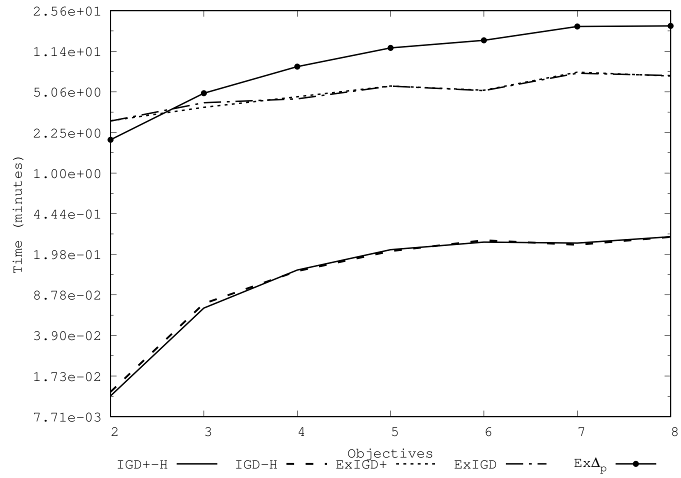

Fig. 6. Graph showing the running time of each MOEA for DTLZ1.

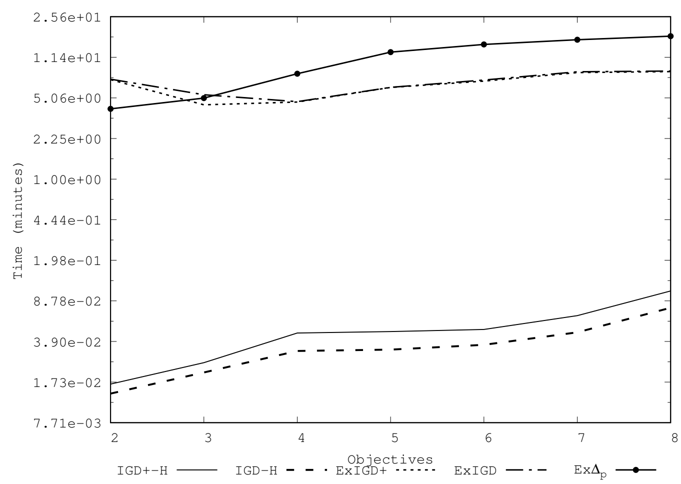

Fig. 7. Graph showing the running time of each MOEA for DTLZ5.

point and the solutions. This allows this selection mechanism to converge faster than the use of the Exclusive-based selection mechanism.

Thus, IGD$^+$-H-EMOA is computationally cheaper than ExIGD-EMOA, ExIGD$^+$-EMOA and ExΔ_p-EMOA. It is worth indicating that IGD$^+$-H-EMOA is able to achieve a significant speed up (of up to 200x) with respect to ExIGD$^+$-EMOA. This confirms that the use of a selection mechanism based on IGD$^+$ is an effective way to solve some many-objective problems.

VI. Conclusions and Future Work

We have proposed several selection mechanisms for indicator-based MOEAs which use a reference set. The core idea of our proposed algorithm is to adopt the IGD$^+$ performance indicator in the selection mechanism of a MOEA. Here, we showed that the use of the modified Euclidean distance significantly improves the performance of the selection

TABLE II

RESULTS OBTAINED IN THE DTLZ TEST PROBLEMS BY EACH MOEA. WE COMPARED THE PERFORMANCE OF EACH MOEA USING THE HYPERVOLUME INDICATOR. THE VALUES IN PARENTHESES CORRESPOND TO THE COMPUTATIONAL TIME (MEASURED IN MINUTES) REQUIRED BY EACH EXECUTION OF THE MOEAS COMPARED. THIS TABLE WAS OBTAINED FROM 30 INDEPENDENT RUNS. ALL THE ALGORITHMS WERE COMPILED USING THE GNU C COMPILER AND THEY WERE EXECUTED ON THE SAME COMPUTER.

	IGD+-H-EMOA	**IGD-H-EMOA**	**ExIGD+-EMOA**	**ExIGD-EMOA**	**ExΔ_p-EMOA**
m	DTLZ1				
2	0.87343 (0.0117 min)	0.87345 (0.0126 min)	**0.87358** (2.8401 min)	0.87345 (2.8238 min)	0.86872 (1.9446 min)
3	0.97371 (0.0674 min)	0.97338 (0.0738 min)	**0.97386** (3.7193 min)	0.97351 (4.075 min)	0.97266 (4.9237 min)
4	0.99379 (0.1442 min)	0.99353 (0.1411 min)	**0.99386** (4.5933 min)	0.99308 (4.3992 min)	0.99121 (8.3803 min)
5	0.99371 (0.2163 min)	0.99275 (0.2103 min)	**0.99404** (5.6831 min)	0.99223 (5.6811 min)	0.99045 (12.1779 min)
6	0.99054 (0.2517 min)	0.98847 (0.2612 min)	**0.99057** (5.2241 min)	0.98726 (5.1929 min)	0.98831 (14.1643 min)
7	0.99776 (0.2471 min)	0.99592 (0.2386 min)	**0.99783** (7.4982 min)	0.99479 (7.3405 min)	0.99462 (18.6716 min)
8	**0.99586** (0.2801 min)	0.99344 (0.2792 min)	0.9958 (6.9452 min)	0.9921 (7.0153 min)	0.99168 (18.8965 min)
	DTLZ2				
2	3.21147 (0.0125 min)	3.21151 (0.0113 min)	**3.21156** (4.7688 min)	3.21153 (4.7642 min)	3.12112 (3.8215 min)
3	7.43104 (0.0301 min)	7.43073 (0.0276 min)	**7.43113** (5.9651 min)	7.4308 (7.2221 min)	7.42677 (8.182 min)
4	15.58708 (0.0451 min)	15.5851 (0.0396 min)	**15.58732** (6.2784 min)	15.5852 (6.3927 min)	15.58049 (9.9203 min)
5	31.69208 (0.0559 min)	31.68537 (0.0477 min)	**31.69251** (8.1911 min)	31.68585 (8.4751 min)	31.68244 (13.9195 min)
6	63.76177 (0.0608 min)	63.74695 (0.0482 min)	**63.76286** (9.2453 min)	63.74789 (9.2507 min)	63.71765 (18.4489 min)
7	127.81248 (0.0725 min)	127.78877 (0.054 min)	**127.81384** (10.7463 min)	127.7894 (10.9737 min)	127.76065 (20.8394 min)
8	255.83892 (0.0756 min)	255.76216 (0.059 min)	**255.84114** (11.7623 min)	255.76964 (18.0689 min)	255.71304 (21.9143 min)
	DTLZ5				
2	3.21127 (0.0167 min)	3.21123 (0.0138 min)	3.21131 (7.2897 min)	**3.21131** (7.3568 min)	3.132 (4.0653 min)
3	6.1043 (0.0255 min)	6.10427 (0.021 min)	**6.10436** (4.4098 min)	6.10441 (5.388 min)	5.92074 (5.0434 min)
4	12.00938 (0.0462 min)	12.00646 (0.0323 min)	**12.00975** (4.6598 min)	12.00786 (4.6972 min)	11.92451 (8.201 min)
5	23.82496 (0.0476 min)	23.81727 (0.0332 min)	**23.82717** (6.24 min)	23.81817 (6.2444 min)	23.75192 (12.6418 min)
6	47.39216 (0.0496 min)	47.35934 (0.0365 min)	**47.39692** (7.0745 min)	47.36511 (7.2339 min)	46.70794 (14.7261 min)
7	**91.66916** (0.0653 min)	91.47108 (0.0468 min)	91.66848 (8.3923 min)	91.45174 (8.5274 min)	89.56395 (16.1746 min)
8	**145.92603** (0.107 min)	145.63295 (0.0765 min)	145.87716 (8.555 min)	135.55127 (8.6648 min)	126.21945 (17.3731 min)
	DTLZ6				
2	3.08569 (0.0303 min)	3.0933 (0.031 min)	**3.09634** (3.7122 min)	3.08765 (3.6532 min)	2.9441 (6.0076 min)
3	**5.9579** (0.2248 min)	5.91691 (0.1842 min)	5.94636 (2.4423 min)	5.85907 (2.4041 min)	5.72483 (5.5973 min)
4	11.50411 (0.3443 min)	11.53968 (0.2563 min)	**11.62126** (3.9025 min)	0 (3.9697 min)	11.92451 (7.6791 min)
5	21.77326 (0.3251 min)	21.6063 (0.2376 min)	**22.29951** (5.6863 min)	0 (5.6109 min)	0 (12.0119 min)
6	41.68516 (0.3037 min)	40.5511 (0.2284 min)	**41.98876** (6.5304 min)	0.56907 (6.7096 min)	2.36595 (13.8672 min)
7	**84.29418** (0.3171 min)	81.75814 (0.2262 min)	84.2936 (7.5512 min)	0 (7.441 min)	0 (15.1196 min)

mechanism. Additionally, the transformation of the selection mechanism into a LAP reduces the running time, which makes possible a significant speed up (of up to 200x). Our experimental results showed that a selection mechanism based on Δ_p has some drawbacks when it tries to solve problems with degenerate Pareto fronts. This selection mechanism was not able to solve degenerate multi-objective problems with more than 5 objectives. As can be observed, the Pareto compliant property between two-objective vectors is of utmost importance and improves the performance of the selection mechanism of a MOEA. Our preliminar experimental results proved that IGD$^+$-based selection mechanism is an effective way to solve some many-objective problems and it would be interesting to test with more multi-objective test suite. As part of our future work, we are interested in developing a new technique for building the reference set during the evolutionary process.

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