Identifying On-Line Behavior and Some Sources of Difficulty in Two Competitive Approaches for Constrained Optimization

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Abstract- In this paper, we present an empirical study whose aim is twofold: (1) to analyze the on-line behavior of two state-of-the-art approaches for constrained optimization, whose results provided in a well-known benchmark were competitive, in order to identify features of a problem which makes it difficult to solve when using an evolutionary algorithm and (2) to propose a new set of problems whose features cover those sources of difficulty. The on-line behavior analyzed consists on using three performance measures to know how fast the technique reaches the feasible region and to also know the capabilities of the algorithm to improve feasible solutions previously found. Besides, we analyze the ability of the approaches to maintain diversity (to have solutions inside and outside the feasible region as well). Based on the obtained results we propose a set of eleven test problems (either artificial or real-world problems) taken from the literature in order to re-test the approaches. The results are discussed and some conclusions are drawn.

1 Introduction

Evolutionary algorithms (EAs) have been widely used to solve optimization problems [8, 5, 4]. However, in their original versions, EAs lack of a mechanism to deal with constrained search spaces. Thus, many approaches have been proposed to incorporate the constraints of a problem into the fitness function of an EA [16, 3]. These approaches are mainly proposed to either solve an specific real-world problem [19] or to be tested on a well known set of problems [17]. However, most of the time, the approaches are only compared based on their final statistical results on a sample of independent runs. The motivation of this work is twofold: (1) To propose the use of performance measures in the comparison of different constraint-handling techniques in order to get a better knowledge of the behavior of the approaches and (2) based on these results, to propose a set of problems which contain features that have been found to be sources of difficulty in the behavior of the algorithms.

In this work, we propose to use three performance measures in order to analyze the behavior of two state-of-the-art approaches found in the literature whose performances when tested in a well-known set of 13 functions were very competitive. The performance measures will allow us: (1) to known how many evaluations of the objective function are required to reach the feasible region of the search space, (2) to quantify the improvement of feasible solutions found and (3) to assess the effectiveness of the approach to maintain diversity in the population (in constrained optimization we understand diversity as the fact of having both, feasible and infeasible solutions during all the evolutionary process). The approaches used in our study are the Simple Multimembered Evolution Strategy (SMES) [12] and the Stochastic Ranking [17]. The SMES is based on three mechanisms: (1) A diversity mechanism which allows infeasible solutions close to the feasible region and with a good value of the objective function to remain in the population for the next generation; (2) a reduction of the initial stepsize of the mutation operator and (3) a panmictic discrete-intermediate recombination operator. The SR promotes a balance between the value of the objective function and the sum of constraint violation in the selection process; allowing infeasible solutions with a promising value of the objective function to remain in the population for the next generation. Both approaches use an evolution strategy as a search engine. Based on the obtained results and the detected sources of difficulty in the current benchmark, we propose a set of eleven new test problems which are solved using both techniques with the same set of parameters defined when the previous set of thirteen problems were solved.

The paper is organized as follows: In Section 2 we provide a review of the related work. Afterwards, in Section 3 we detail the three performance measures used in this work and discuss the obtained results from each compared approach. In Section 4, we present the new 11 test functions whose features cover the sources of difficulty found in the study presented in Section 3. After that, in Section 5 the results obtained using the two compared approaches when solving this new set of problems are discussed. Finally in Section 6 some conclusions are established and the future work is presented.

2 Related Work

The use of performance measures in global constrained optimization using EAs is not very common. However, there are some authors whose reported results include other measures. For example, Lampinen [11] reported the number of evaluations required to find the first feasible solution. Mezura & Coello [12] reported the rate of feasible solutions at a certain generation.

On the other hand, the idea of having a set of constrained optimization problems with different characteristics to test evolutionary algorithms was initially proposed by Michalewicz [14, 13], and summarized by Michalewicz & Schoenauer [16]. That set consisted of eleven problems with features such as different types of objective function (linear, quadratic, nonlinear), different types of constraints (linear, nonlinear, equality or inequality) and different dimensionality. Koziel & Michalewicz [10] added one function to the original benchmark. The main feature of this new function is its disjoint feasible region. Runarsson & Yao added another function to the benchmark [17]. This function has three nonlinear equality constraints and the objective function is also nonlinear. Those two new functions [10, 17] addressed two features that the original benchmark lacked (disjoint feasible regions and a combination of linear and nonlinear equality constraints). The goal of this benchmark is to have a reliable mean to test the quality and robustness of constraint-handling techniques used with evolutionary algorithms.

Michalewicz also explored the idea of generating artificial constrained test functions [15]. He proposed a test case generator which allows to generate problems by varying several features such as: dimensionality, multimodality, number of constraints, connectivity of the feasible region, size of the feasible region with respect to the whole search space and ruggedness of the objective function. This first version had some problems because the generated functions were highly symmetric. Therefore, a new version called TCG-2 was later proposed [18] to solve this drawback and to improve its features. Both versions of the TCG were used to test a constraint-handling mechanism based on a static penalty function. They used a steady-state EA as a search engine. The results obtained suggested that the sources of difficulty for the penalty function approach were a high dimensionality and multimodality of the objective function. For the first TCG, having a disconnected feasible region also affected the performance of the approach. For the TCG-2, the width of peaks of the objective function also decreased the performance of the algorithm. The experiments also showed that for both, TCG and TCG-2, the size of the feasible region with respect to the whole search had no negative influence in the performance of the approach. In addition, for the TCG, the number of constraints and the ruggedness of the objective function did not affect the good quality of the results provided by the approach. Finally, for the TCG-2 the complexity of the function and the number of active constraints caused little impact in the performance of the approach.

3 Three Performance Measures

The first performance measure was used by Lampinen [11] to count how many evaluations are required to find the first feasible solution; we call it EVALS. The second one was proposed by Bäck [1] to measure the progress ratio for unconstrained optimization. We adapt it to measure the progress ratio just inside the feasible region, because we have detected that for constraint-handling algorithms solving constrained problems, is quite hard to improve solutions once inside the feasible region. The original expression taken from [1] is the following: $P = \ln \sqrt{\frac{f_{\min}(0)}{f_{\min}(T)}}$ where $f_{\min}(i)$ refers to the best objective function value occurring at generation i. T is the final generation of the process.

Prob.	n	Type of function	ρ	LI	NI	LE	NE
g01	13	quadratic	0.0003%	9	0	0	0
g02	20	nonlinear	99.9973%	1	1	0	0
g03	10	nonlinear	0.0026%	0	0	0	1
g04	5	quadratic	27.0079%	0	6	0	0
g05	4	nonlinear	0.0000%	2	0	0	3
g06	2	nonlinear	0.0057%	0	2	0	0
g07	10	quadratic	0.0000%	3	5	0	0
g08	2	nonlinear	0.8581%	0	2	0	0
g09	7	nonlinear	0.5199%	0	4	0	0
g10	8	linear	0.0020%	3	3	0	0
g11	2	quadratic	0.0973%	0	0	0	1
g12	3	quadratic	4.7697%	0	9^3	0	0
g13	5	nonlinear	0.0000%	0	0	0	3

Table 1: Details of the 13 well-known test problems. n is the number of decision variables, ρ is the estimated ratio between the feasible region and the whole search space, LI is the number of linear inequality constraints , NI the number of nonlinear inequality constraints, LE is the number of linear equality constraints and NE is the number of nonlinear equality constraints.

However, we are interested in measuring the progress once the feasible region is reached. Therefore the modified expression that we use is the following: $P = \left| \ln \sqrt{\frac{f_{\min}(G_{\mathrm{ff}})}{f_{\min}(T)}} \right|$ where $f_{\min}(G_{\mathrm{ff}})$ refers to the objective function value of the first feasible solution found and $f_{\min}(T)$ refers to the objective function value of the best feasible solution found in the last generation. The third measure counts the number of evaluations required by an algorithm to have its population with only feasible solutions. We call it ALL-FEASIBLE. This measure helps to understand the way of sampling the search space and also the feasible region. The aim is to analyze how the feasible region is sampled by each approach. Note that EVALS and ALL-FEASIBLE performance measures return an integer value. If a technique obtains a lower value than other one, it means that the first one finds the feasible region faster than the second one. On the other hand, the PROGRESS-RATIO performance measure returns a real value. If an approach gets a higher value for this performance measure, it means an improvement inside the feasible region.

3.1 Experiments

For these experiments, we adopted the parameters used and suggested by the authors of each approach when the set of 13 well-known set of problems was solved (see Table 1 for details of each problem). We only modified the total number of evaluations of the objective function (240000 in our experiments). Also, because of the fact that the SMES uses a dynamic adjustment for the allowable tolerance of the equality constraints and SR does not, we decided for the SMES to consider a feasible solution only when the same value for tolerance used by SR ($\epsilon = 0.0001$) was reached. 30 independent runs were performed for each approach for each test function for each performance measure. The summary of the parameters adopted is the following: (1) SMES: $(100+300)$-ES 800 generations (240000 evaluations), $S_r = 0.97$, Initial Stepsize= 40%, (2) SR: $(30, 200)$-ES 1200 generations (240000 evaluations), $P_f = 0.45$, Sweeps for ranking$= 200$. The summary of results for the three performance measures obtained by each approach for each test problem is presented in Table 2.

P	Stat	Evals		Progress-Ratio		All-feasible	
		SR	SMES	SR	SMES	SR	SMES
g01	best	3585	**1315**	**1.408**	1.399	13230	**4900**
	mean	9146	**2455**	1.286	**1.324**	19050	**6050**
	worst	15352	**3444**	1.166	**1.229**	25630	**9100**
	St. Dev	2.9E+3	**5.7E+2**	5.4E-2	**4.1E-2**	2.9E+3	**8.2E+2**
g02	best	NA	NA	**0.281**	0.276	NA	NA
	mean	NA	NA	0.260	**0.262**	NA	NA
	worst	NA	NA	0.244	**0.251**	NA	NA
	St. Dev	NA	NA	9.0E-3	**8.0E-3**	NA	NA
g03	best	1823	**868**	∗0.346	0.338	NA	NA
	mean	6674	**2745**	∗0.307	0.241	NA	NA
	worst	14026	**7554**	∗0.166	0.092	NA	NA
	St. Dev	3.3E+3	**1.7E+3**	**5.1E-2**	6.4E-2	NA	NA
g04	best	NA	NA	4.387	**4.394**	830	**400**
	mean	NA	NA	**4.182**	4.081	3877	**540**
	worst	NA	NA	**3.588**	3.553	7030	**700**
	St. Dev	NA	NA	**1.8E-1**	2.1E-1	1.9E+3	**1.5E+2**
g05	best	**24727**	51139	NA	NA	**30230**	240000
	mean	**27843**	124120	NA	NA	**40323**	240000
	worst	**31694**	240000	NA	NA	**58830**	240000
	St. Dev	**1.7E+3**	6.9E+4	NA	NA	7.6E+3	0
g06	best	**262**	674	#4.283	**4.320**	**4830**	4900
	mean	**1181**	1552	#4.124	4.045	108344	**5850**
	worst	**1711**	2395	#3.749	3.225	240000	**7600**
	St. Dev	**3.7E+2**	4.9E+2	**1.6E-1**	2.7E-1	1.2E+5	**7.3E+2**
g07	best	2087	**1211**	&2.038	**2.436**	5430	**4300**
	mean	3093	**2024**	&1.400	**1.998**	8350	**4950**
	worst	4049	**2772**	&0.873	**1.159**	33230	**5800**
	St. Dev	4.8E+2	**4.5E+2**	**2.5E-1**	2.9E-1	4.8E+3	**4.2E+2**
g08	best	3	**1**	0.079	**0.087**	1830	**1600**
	mean	110	**91**	0.042	**0.045**	2383	**1950**
	worst	428	**292**	**0.015**	0.014	3230	**2800**
	St. Dev	1.1E+2	**7.4E+1**	**1.1E-2**	1.2E-2	**3.2E+2**	3.4E+2
g09	best	6	**3**	4.564	**4.738**	2430	**1300**
	mean	**122**	132	2.387	**2.749**	3717	**1580**
	worst	375	**359**	**0.359**	0.212	5230	**2200**
	St. Dev	1.1E+2	**9.1E+1**	**1.2E+0**	1.4E+0	6.6E+2	**2.5E+2**
g10	best	7864	**1155**	†0.459	**0.625**	19430	**8500**
	mean	77222	**4630**	†0.133	**0.497**	65706	**10440**
	worst	240000	**6907**	†0.004	**0.325**	240000	**14800**
	St. Dev	9.3E+4	**1.3E+3**	1.4E-1	**7.3E-2**	7.3E+4	**1.5E+3**
g11	best	503	**30**	‡0.142	**0.144**	**8830**	240000
	mean	26463	**4290**	‡0.054	**0.098**	**35940**	240000
	worst	240000	**16204**	‡0.000	**0.004**	240000	240000
	St. Dev	7.2E+4	**4.3E+3**	5.2E-2	**4.6E-2**	6.9E+4	0
g12	best	NA	NA	**0.161**	0.147	2430	**1600**
	mean	NA	NA	**0.093**	0.092	3023	**2290**
	worst	NA	NA	0.013	**0.032**	3830	**3400**
	St. Dev	NA	NA	4.1E-2	**3.3E-2**	**3.1E+2**	4.8E+2
g13	best	**16578**	209429	NA	NA	**22030**	215500
	mean	**18949**	212579	NA	NA	**29710**	215530
	worst	**21616**	214125	NA	NA	**57030**	215800
	St. Dev	1.4E+3	**9.7E+2**	NA	NA	8.3E+3	**9.2E+1**

Table 2: Statistics obtained for the three performance measures in 30 independent runs. A number in **boldface** indicates the best result found. "NA" means: does Not Apply. The symbols in column 5 mean that in only a fraction of the 30 runs feasible solutions were found in the last generation. "∗"$=\frac{13}{30}$, "#"$=\frac{17}{30}$, "&"$=\frac{29}{30}$, "†"$=\frac{25}{30}$ and "‡"$=\frac{27}{30}$.

3.1.1 EVALS performance measure

From the results obtained we can comment the following: For functions g02 and g04 both approaches find feasible solutions from the first population randomly generated. For problem g12 both algorithms exhibit a very similar behavior, although the SMES has "better" mean, median and standard deviation values and the SR has "better" min and max values. For problem g09 there is an inverse behavior, the SMES has "better" min and max values and the SR has "better" mean and median values. The SMES consistently provided "better" results for six functions: g01, g03, g07, g08, g10 and g11. The SR obtained better results for three problems: g05, g06 and g13.

The overall discussion of the results suggests that the SMES finds the feasible region faster than the SR, except in problems where there is more than one nonlinear equality constraint (g05 and g13). On the other hand, both approaches reach the feasible region almost at the same time when the size of the feasible region is approximately more than 4% of the whole search space (g02, g04 and g12).

3.1.2 PROGRESS-RATIO performance measure

From these results, we observe very "similar" values of improvement by both approaches in four problems (g02, g05, g08, g12 and g13). It is important to assert that in functions g05 and g13 none of the algorithms was capable of improving the feasible solutions found. Both problems have three nonlinear equality constraints. For problems g08 and g12 both approaches could get a small improvement inside the feasible region. This is because for these two functions feasible solutions are found very quickly and, after that, the global optimum is found quickly, as well. This is because these two problems are "easy" to solve. The results also show that SMES was able to get a "better" improvement inside the feasible region in seven problems: g01, g03, g06 g07, g09, g10 and g11. The SR presented some problems to maintain the feasible solutions previously found. For example, in function g03 only in 13 runs out of 30, we were able to find feasible solutions in the last generation. A similar behavior was found in problems g06 (17/30), g07 (29/30), g10 (25/30) and g11 (27/30). Nevertheless, SR obtained consistently "better" results in problem g04.

We can conclude that SMES is able to have a slightly better improvement of the first feasible solution found than SR. We can also conclude that in presence of more than one nonlinear equality constraints, both approaches could not improve significantly the first feasible solution found. Finally it was observed that for problem g02, a high dimensionality (20 decision variables) combined with a nonlinear objective function generates a very rugged and complicated feasible region to improve previously found feasible solutions.

3.1.3 ALL-FEASIBLE performance measure

The SMES required less evaluations to get a fully feasible population in eight functions (g01, g04, g06, g07, g08, g09, g10 and g12). This is because the approach focuses on finding the feasible region and after that, the diversity mechanism allows the approach to return to its boundaries. The SR provided "better" results in three problems (g05, g11 and g13). It is interesting to note that the mechanism to maintain solutions with a good value of the objective function regardless of feasibility is suitable to solve these problems with a very small feasible region and in presence of one up to three nonlinear equality constraints. In function g02 both approaches got a fully feasible population in the first generation because almost all the search space (99%) is feasible. For problem g03, none of the approaches could get a fully feasible population. This problem has one nonlinear inequality constraint, 10 decision variables and a very small feasible region (approximately 0.0026% of the search space is feasible).

The results clearly confirm the way of sampling the feasible region of the problem: SMES concentrates on finding it and, after that, it looks for the boundaries. On the other hand, SR focuses on finding promising areas, regardless of feasibility. When the problem has nonlinear equality constraints, the second approach seems to be more suitable.

3.2 Confidence intervals

In order to have more certainty about of results, we performed a statistical test to estimate the confidence intervals for the mean statistic. We used a Kolmogorov-Smirnov one-sample test to verify closeness to a normal distribution of our samples, which did not occur in all cases. Therefore, we applied a Bootstrapping test in order to avoid assuming normality. The results are presented in Table 3.

As it can be seen, the results of the statistical test confirm the discussed results based on the obtained samples. In this way, we establish the following conclusions regarding the behavior of each approach and also with respect to the sources of difficulty found:

- SMES is able to generate a feasible solution faster than the SR except in problems with more than one nonlinear equality constraint.
- SMES seems to have a slightly better capability to improve the results once the feasible region is reached.
- In presence of more than one nonlinear equality constraint both approaches are unable to improve significantly a feasible solution.
- For problems with a low dimensionality (between 2 and 3 decision variables) and a quadratic objective function, the progress provided by both approaches inside the feasible region is enough as to reach, reasonably fast, the global optimum.
- Due to its emphasis on reaching the feasible region, SMES requires less evaluations to generate a fully feasible population. Therefore, its diversity mechanism is effective when active during most of the evolutionary process, since such mechanism maintains a few recently-generated infeasible solutions close to the boundaries of the feasible region.
- SR tends to have almost always a considerable number of infeasible solutions with a good value of the objective function, regardless of how close they are from the feasible region. In fact, this mechanism seems to be more adequate when solving problems with nonlinear equality constraints where the SR outperforms SMES.
- With a high dimensionality (20 decision variables), both approaches could not improve as expected the first feasible solution found.

4 A New Set of Test Functions

Based on the sources of difficulty detected on the previous study (high dimensionality and the number of nonlinear equality constraints) we added other features not included in the previous set of test problems (Table 1): more than ten nonlinear inequality constraints, and a disjoint feasible region (only one function with this feature is included in the well-known benchmark).

Prob.	n	Type of function	ρ	LI	NI	LE	NE
g14	10	nonlinear	0.00%	0	0	3	0
g15	3	quadratic	0.00%	0	0	1	1
g16	5	nonlinear	0.0204%	4	34	0	0
g17	6	nonlinear	0.00%	0	0	0	4
g18	9	quadratic	0.00%	0	12	0	0
g19	15	nonlinear	33.4761%	0	5	0	0
g20	24	linear	0.00%	0	6	2	12
g21	7	linear	0.00%	0	1	0	5
g22	22	linear	0.00%	0	1	8	11
g23	9	linear	0.00%	0	2	3	1
g24	2	linear	79.6556%	0	2	0	0

Table 4: Details of the new 11 test problems. n is the number of decision variables, ρ is the estimated ratio between the feasible region and the search space, LI is the number of linear inequality constraints, NI the number of nonlinear inequality constraints, LE is the number of linear equality constraints and NE is the number of nonlinear equality constraints.

Despite the fact that it is well known that in mathematical programming, nonlinear equality constraints are sources of difficulty [2], to the best of our knowledge, there are no empirical studies that use performance measures to show that the same could happen when using evolutionary algorithms.

Unlike the test case generators previously discussed, we do not intend to provide the user with the best possible EA to be used for a certain type of problem. Instead, we want to detect features that keep an EA from reaching the feasible region or the global optimum, so that we can gain a deeper understanding of the sort of features that make difficult a problem for the approaches compared in this paper.

Our experimental design was the following: First, we selected test functions (either artificial or from real world problems) that had at least one of the features mentioned before. We selected seven functions from Himmelblau's book [9] (g14, g15, g16, g17, g18, g19, g20), two are from heat exchanger network problems detailed in [6] (g21, g22). One more was proposed by Xia [21] (g23) and the last one was taken from Floudas et al.'s Handbook [7] (g24). Selected problems with high dimensionality are: g19, g20 and g22. Test functions with more than three nonlinear equality constraints are: g17, g20, g21 and g22. For the secondary set of features, we chose problems g16 and g18, which have more than ten nonlinear inequality constraints. Problems having a nonlinear objective function are g14, g16, g17 and g19. Finally, a test function having a feasible region consisting on two disconnected sub-regions is g24. For completeness, we also included two functions that seemed to be easy to solve because they have only one nonlinear equality constraint (g15) and a quadratic and linear objective function (g23), respectively. The characteristics of each problem of our new proposed benchmark are summarized in Table 4. We also calculated the "ρ" metric [16] for this set of new functions. The details of each functions are the following:

- **g14** [9]:

  Minimize: $f(\vec{x}) = \sum_{i=1}^{10} x_i \left(c_i - \ln \frac{x_i}{\sum_{j=1}^{10} x_j} \right)$

  subject to:

  $h_1(\vec{x}) = x_1 + 2x_2 + +2x_3 + x_6 + x_{10} - 2 = 0$

  $h_2(\vec{x}) = x_4 + 2x_5 + x_6 + x_7 - 1 = 0$

  $h_3(\vec{x}) = x_3 + x_7 + x_8 + 2x_9 + x_{10} - 1 = 0$

	Evals		Progress-Ratio		All-feasible	
	SR	**SMES**	**SR**	**SMES**	**SR**	**SMES**
g01	$[8103, 10189]$	$[\mathbf{2249}, \mathbf{2642}]$	$[1.270, 1.31]$	$[\mathbf{1.308}, \mathbf{1.338}]$	$[18043, 20169]$	$[\mathbf{5804}, \mathbf{6400}]$
g02	$[1, 1]$	$[1, 1]$	$[0.260, 0.260]$	$[0.259, 0.265]$	$[30, 30]$	$[100, 130]$
g03	$[5630, 7899]$	$[\mathbf{2286}, \mathbf{3458}]$	$\mathbf{\frac{13}{30}}[\mathbf{0.280}, \mathbf{0.340}]$	$[0.217, 0.265]$	$[240000, 240000]$	$[240000, 240000]$
g04	$[3, 4]$	$[3, 5]$	$[\mathbf{4.120}, \mathbf{4.250}]$	$[4.004, 4.158]$	$[3225, 4528]$	$[\mathbf{480}, \mathbf{590}]$
g05	$[\mathbf{27272}, \mathbf{28451}]$	$[104203, 151126]$	$[0.000, 0.000]$	$[0.000, 0.000]$	$[\mathbf{37521}, \mathbf{43125}]$	$[240000, 240000]$
g06	$[\mathbf{1038}, \mathbf{1315}]$	$[1384, 1735]$	$\mathbf{\frac{17}{30}}[\mathbf{4.050}, \mathbf{4.200}]$	$[3.948, 4.142]$	$[69251, 147501]$	$[\mathbf{5590}, \mathbf{6100}]$
g07	$[2920, 3261]$	$[\mathbf{1874}, \mathbf{2185}]$	$\frac{29}{30}[1.300, 1.500]$	$[\mathbf{1.891}, \mathbf{2.094}]$	$[7383, 12182]$	$[\mathbf{4800}, \mathbf{5100}]$
g08	$[80, 157]$	$[\mathbf{65}, \mathbf{118}]$	$[0.040, 0.050]$	$[0.040, 0.049]$	$[2262, 2504]$	$[\mathbf{1840}, \mathbf{2070}]$
g09	$[83, 162]$	$[98, 165]$	$[1.96, 2.84]$	$[\mathbf{2.235}, \mathbf{3.263}]$	$[3476, 3958]$	$[\mathbf{1490}, \mathbf{1670}]$
g10	$[49468, 113585]$	$[\mathbf{4144}, \mathbf{5116}]$	$\frac{25}{30}[0.080, 0.190]$	$[\mathbf{0.470}, \mathbf{0.524}]$	$[42366, 92455]$	$[\mathbf{9970}, \mathbf{10990}]$
g11	$[10462, 65829]$	$[\mathbf{3062}, \mathbf{6123}]$	$\frac{27}{30}[0.035, 0.074]$	$[\mathbf{0.081}, \mathbf{0.116}]$	$[\mathbf{20262}, \mathbf{72970}]$	$[240000, 240000]$
g12	$[17, 35]$	$[15, 31]$	$[0.077, 0.107]$	$[0.079, 0.104]$	$[2922, 3129]$	$[\mathbf{2120}, \mathbf{2460}]$
g13	$[\mathbf{18414}, \mathbf{19484}]$	$[212174, 212874]$	$[0.000, 0.000]$	$[0.000, 0.000]$	$[\mathbf{27583}, \mathbf{34644}]$	$[215500, 215560]$

Table 3: Confidence intervals for the mean statistics for the three measures (95% level of confidence). A number in **boldface** means a better result. The fraction in the PROGRESS-RATIO measure for the SR indicates the number of independent runs (out of 30) in which feasible solutions were found in the population of the last generation. In the remaining runs, no feasible solutions were found at the end of the process.

j	1	2	3	4	5
e_j	-15	-27	-36	-18	-12
c_{1j}	30	-20	-10	32	10
c_{2j}	-20	39	-6	-31	32
c_{3j}	-10	-6	10	-6	-10
c_{4j}	32	-31	-6	39	-20
c_{5j}	-10	32	-10	-20	30
d_j	4	8	10	6	2
a_{1j}	-16	2	0	1	0
a_{2j}	0	-2	0	4	2
a_{3j}	-3.5	0	2	0	0
a_{4j}	0	-2	0	-4	-1
a_{5j}	0	-9	-2	1	-2.8
a_{6j}	2	0	-4	0	0
a_{7j}	-1	-1	-1	-1	-1
a_{8j}	-1	-2	-3	-2	-1
a_{9j}	1	2	3	4	5
a_{10j}	1	1	1	1	1

Table 5: Data set for test problem g19

where the bounds are $0 \leq x_i \leq 10$ ($i = 1, \ldots, 10$), and $c_1 = -6.089$, $c_2 = -17.164$, $c_3 = -34.054$, $c_4 = -5.914$, $c_5 = -24.721$, $c_6 = -14.986$, $c_7 = -24.1$, $c_8 = -10.708$, $c_9 = -26.662$, $c_{10} = -22.179$,. The best known solution is at $x^* = (0.0350, 0.1142, 0.8306, 0.0012, 0.4887, 0.0005, 0.0209, 0.0157, 0.0289, 0.0751)$ where $f(x^*) = -47.751$.

- **g15** [9]:
  Minimize: $f(\vec{x}) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3$

  subject to:
  $h_1(\vec{x}) = x_1^2 + x_2^2 + +x_3^2 - 25 = 0$
  $h_2(\vec{x}) = 8x_1 + 14x_2 + 7x_3 - 56 = 0$

  where the bounds are $0 \leq x_i \leq 10$ ($i = 1, \ldots, 10$). The best known solution is at $x^* = (3.512,0.217,3.552)$ where $f(x^*) = 961.715$.

- **g16** [9]:
  Maximize: $f(\vec{x}) = 0.0000005843y_{17} - 0.000117y_{14} - 0.1365 - 0.00002358y_{13}$
  $- 0.0000011502y_{16} - 0.0321y_{12} - 0.004324y_5 - 0.0001\frac{c_{15}}{c_{16}} - 37.48\frac{y_2}{c_{12}}$

  subject to:
  $g_1(\vec{x}) = y_4 - \frac{0.28}{0.72}y_5 \geq 0$
  $g_2(\vec{x}) = 1.5x_2 - x_3 \geq 0$
  $g_3(\vec{x}) = 21 - 3496\frac{y_2}{c_{12}} \geq 0$
  $g_4(\vec{x}) = \frac{62,212}{c_{17}} - 110.6 - y_1 \geq 0$
  $g_5(\vec{x}) = y_1 - 213.1 \geq 0$
  $g_6(\vec{x}) = 405.23 - y_1 \geq 0$
  $g_7(\vec{x}) = y_2 - 17.505 \geq 0$
  $g_8(\vec{x}) = 1053.6667 - y_2 \geq 0$
  $g_9(\vec{x}) = y_3 - 11.275 \geq 0$
  $g_{10}(\vec{x}) = 35.03 - y_3 \geq 0$
  $g_{11}(\vec{x}) = y_4 - 214.228 \geq 0$
  $g_{12}(\vec{x}) = 665.585 - y_4 \geq 0$
  $g_{13}(\vec{x}) = y_5 - 7.458 \geq 0$
  $g_{14}(\vec{x}) = 584.463 - y_5 \geq 0$
  $g_{15}(\vec{x}) = y_6 - 0.961 \geq 0$
  $g_{16}(\vec{x}) = 265.916 - y_6 \geq 0$
  $g_{17}(\vec{x}) = y_7 - 1.612 \geq 0$
  $g_{18}(\vec{x}) = 7.046 - y_7 \geq 0$
  $g_{19}(\vec{x}) = y_8 - 0.146 \geq 0$
  $g_{20}(\vec{x}) = 0.222 - y_8 \geq 0$
  $g_{21}(\vec{x}) = y_9 - 107.99 \geq 0$
  $g_{22}(\vec{x}) = 273.366 - y_9 \geq 0$
  $g_{23}(\vec{x}) = y_{10} - 922.693 \geq 0$
  $g_{24}(\vec{x}) = 1286.105 - y_{10} \geq 0$
  $g_{25}(\vec{x}) = y_{11} - 926.832 \geq 0$
  $g_{26}(\vec{x}) = 1444.046 - y_{11} \geq 0$
  $g_{27}(\vec{x}) = y_{12} - 18.766 \geq 0$
  $g_{28}(\vec{x}) = 537.141 - y_{12} \geq 0$
  $g_{29}(\vec{x}) = y_{13} - 1072.163 \geq 0$
  $g_{30}(\vec{x}) = 3247.039 - y_{13} \geq 0$
  $g_{31}(\vec{x}) = y_{14} - 8961.448 \geq 0$
  $g_{32}(\vec{x}) = 26844.086 - y_{14} \geq 0$
  $g_{33}(\vec{x}) = y_{15} - 0.063 \geq 0$
  $g_{34}(\vec{x}) = 0.386 - y_{15} \geq 0$
  $g_{35}(\vec{x}) = y_{16} - 71084.33 \geq 0$
  $g_{36}(\vec{x}) = 140000 - y_{16} \geq 0$
  $g_{37}(\vec{x}) = y_{17} - 2802713 \geq 0$
  $g_{38}(\vec{x}) = 12146108 - y_{17} \geq 0$

  where:
  $y_1 = x_2 + x_3 + 41.6$
  $c_1 = 0.024x_4 - 4.62$
  $y_2 = \frac{12.5}{c_1} + 12$
  $c_2 = 0.0003535x_1^2 + 0.5311x_1 + 0.08705y_2x_1$
  $c_3 = 0.052x_1 + 78 + 0.002377y_2x_1$
  $y_3 = \frac{c_2}{c_3}$
  $y_4 = 19y_3$
  $c_4 = 0.04782(x_1 - y_3) + \frac{0.1956(x_1 - y_3)^2}{x_2}$
  $c_5 = 100x_2$
  $c_6 = x_1 - y_3 - y_4$
  $c_7 = 0.950 - \frac{c_4}{c_5}$
  $y_5 = c_6c_7$
  $y_6 = x_1 - y_5 - y_4 - y_3$
  $c_8 = (y_5 + y_4)0.995$
  $y_7 = \frac{c_8}{y_1}$
  $y_8 = \frac{c_8}{3798}$
  $c_9 = y_7 - \frac{0.0663y_7}{y_8} - 0.3153$
  $y_9 = \frac{96.82}{c_9} + 0.321y_1$
  $y_{10} = 1.29y_5 + 1.258y_4 + 2.29y_3 + 1.71y_6$
  $y_{11} = 1.71x_1 - 0.452y_4 + 0.580y_3$
  $c_{10} = \frac{12.3}{752.3}$
  $c_{11} = (1.75y_2)(0.995x_1)$
  $c_{12} = 0.995y_{10} + 1998$
  $y_{12} = c_{10}x_1 + \frac{c_{11}}{c_{12}}$
  $y_{13} = c_{12} - 1.75y_2$
  $y_{14} = 3623 + 64.4x_2 + 58.4x_3 + \frac{146.312}{y_9 + x_5}$
  $c_{13} = 0.995y_{10} + 60.8x_2 + 48x_4 - 0.1121y_{14} - 5095$
  $y_{15} = \frac{y_{13}}{c_{13}}$
  $y_{16} = 148000 - 331000y_{15} + 40y_{13} - 61y_{15}y_{13}$
  $c_{14} = 2324y_{10} - 28740000y_2$
  $y_{17} = 14130000 - 1328y_{10} - 531y_{11} + \frac{c_{14}}{c_{12}}$
  $c_{15} = \frac{y_{13}}{y_{15}} - \frac{y_{13}}{0.52}$
  $c_{16} = 1.104 - 0.72y_{15}$
  $c_{17} = y_9 + x_5$

and where the bounds are $704.4148 \leq x_1 \leq 906.3855$, $68.6 \leq x_2 \leq 288.88$, $0 \leq x_3 \leq 134.75$, $193 \leq x_4 \leq 287.0966$ and $25 \leq x_5 \leq 84.1988$. The best known solution is at $x^* = (705.06, 68.6, 102.9, 282.341, 35.627)$ where $f(x^*) = 1.905$.

- **g17** [9]:
Minimize: $f(\vec{x}) = f(x_1) + f(x_2)$

subject to:
$f_1(x_1) = \begin{cases} 30x_1 & 0 \leq x_1 < 300 \\ 31x_1 & 300 \leq x_1 < 400 \end{cases}$
$f_2(x_2) = \begin{cases} 28x_2 & 0 \leq x_2 < 100 \\ 29x_2 & 100 \leq x_2 < 200 \\ 30x_2 & 200 \leq x_2 < 1000 \end{cases}$
$h_1(\vec{x}) = x_1 = 300 - \frac{x_3 x_4}{131.078} \cos(1.48477 - x_6) + \frac{0.90798 x_3^2}{131.078} \cos(1.47588)$
$h_2(\vec{x}) = x_2 = -\frac{x_3 x_4}{131.078} \cos((1.48477 + x_6) + \frac{0.90798 x_4^2}{131.078} \cos(1.47588)$
$h_3(\vec{x}) = x_5 = -\frac{x_3 x_4}{131.078} \sin((1.48477 + x_6) + \frac{0.90798 x_4^2}{131.078} \sin(1.47588)$
$h_4(\vec{x}) = 200 - \frac{x_3 x_4}{131.078} \sin((1.48477 - x_6) + \frac{0.90798 x_3^2}{131.078} \sin(1.47588)$

where the bounds are $0 \leq x_1 \leq 400$, $0 \leq x_2 \leq 1000$, $340 \leq x_3 \leq 420$, $340 \leq x_4 \leq 420$, $-1000 \leq x_5 \leq 1000$ and $0 \leq x_6 \leq 0.5236$. The best known solution is at $x^* = (107.81, 196.32, 373.83, 420, 21.31, 0.153)$ where $f(x^*) = 8927.5888$.

- **g18** [9]:
Maximize: $f(\vec{x}) = 0.5(x_1x_4 - x_2x_3 + x_3x_9 - x_5x_9 + x_5x_8 - x_6x_7)$

subject to:
$g_1(\vec{x}) = 1 - x_3^2 - x_4^2 \geq 0$
$g_2(\vec{x}) = 1 - x_9^2 \geq 0$
$g_3(\vec{x}) = 1 - x_5^2 - x_6^2 \geq 0$
$g_4(\vec{x}) = 1 - x_1^2 - (x_2 - x_9)^2 \geq 0$
$g_5(\vec{x}) = 1 - (x_1 - x_5)^2 - (x_2 - x_6)^2 \geq 0$
$g_6(\vec{x}) = 1 - (x_1 - x_7)^2 - (x_2 - x_8)^2 \geq 0$
$g_7(\vec{x}) = 1 - (x_3 - x_5)^2 - (x_4 - x_6)^2 \geq 0$
$g_8(\vec{x}) = 1 - (x_3 - x_7)^2 - (x_4 - x_8)^2 \geq 0$
$g_9(\vec{x}) = 1 - x_7^2 - (x_8 - x_9)^2 \geq 0$
$g_9(\vec{x}) = x_1x_4 - x_2x_3 \geq 0$
$g_{10}(\vec{x}) = x_3x_9 \geq 0$
$g_{11}(\vec{x}) = -x_5x_9 \geq 0$
$g_{12}(\vec{x}) = x_5x_8 - x_6x_7 \geq 0$

where the bounds are $-10 \leq x_i \leq 10$ $(i = 1, \ldots, 8)$ and $0 \leq x_9 \leq 20$. The best known solution is at $x^* = (0.9971, -0.0758, 0.5530, 0.8331, 0.9981, -0.0623, 0.5642, 0.8256, 0.0000024)$ where $f(x^*) = 0.8660$.

- **g19** [9]:
Maximize: $f(\vec{x}) = \sum_{i=1}^{10} b_i x_i - \sum_{j=1}^{5} \sum_{i=1}^{5} c_{ij} x_{(10+i)} x_{(10+j)} - 2 \sum_{j=1}^{5} d_j x_{(10+j)}^3$

subject to:
$g_j(\vec{x}) = 2 \sum_{i=1}^{5} c_{ij} x_{(10+i)} + 3 d_j x_{(10+j)}^2 + e_j - \sum_{i=1}^{10} a_{ij} x_i \geq 0 \qquad j = 1, \ldots, 5$

where $\vec{b} = [-40, -2, -.25, -4, -4, -1, -40, -60, 5, 1]$ and the remaining data is on Table 5. The bounds are $0 \leq x_i \leq 10$ $(i = 1, \ldots, 15)$. The best known solution is at $x^* = (0, 0, 5.1740, 0, 3.0611, 11.8395, 0, 0, 0.1039, 0, 0.3, 0.3335, 0.4, 0.4283, 0.2240)$ where $f(x^*) = -32.386$.

- **g20** [9]:
Minimize: $f(\vec{x}) = \sum_{i=1}^{24} a_i x_i$

subject to:
$h_i(\vec{x}) = \frac{x_{(i+12)}}{b_{(i+12)} \sum_{j=13}^{24} \frac{x_j}{b_j}} - \frac{c_i x_i}{40 b_i \sum_{j=1}^{12} \frac{x_j}{b_j}} = 0 \qquad i = 1, \ldots, 12$
$h_{13}(\vec{x}) = \sum_{i=1}^{24} x_i - 1 = 0$
$h_{14}(\vec{x}) = \sum_{i=1}^{12} \frac{x_i}{d_i} + f \sum_{i=13}^{24} \frac{x_i}{b_i} - 1.671 = 0$
$g_i(\vec{x}) = \frac{-(x_i + x_{(i+12)})}{\sum_{j=1}^{24} x_j + e_i} \geq 0 \qquad i = 1, 2, 3$
$g_i(\vec{x}) = \frac{-(x_{(i+3)} + x_{(i+15)})}{\sum_{j=1}^{24} x_j + e_i} \geq 0 \qquad i = 4, 5, 6$

where $f = (0.7302)(530)(\frac{14.7}{40})$ and the data set is detailed on Table 6. The bounds are $0 \leq x_i \leq 10$ $(i = 1, \ldots, 24)$. The best known solution is at $x^* = (9.53E-7, 0, 4.21eE-3, 1.039E-4, 0, 0, 2.072E-1, 5.979E-1, 1.298E-1, 3.35E-2, 1.711E-2, 8.827E-3, 4.657E-10, 0, 0, 0, 0, 0, 2.868E-4, 1.193E-3, 8.332E-5, 1.239E-4, 2.07E-5, 1.829E-5)$ where $f(x^*) = 0.09670$.

- **g21** [6]:
Minimize: $f(\vec{x}) = x_1$

subject to:
$g_1(\vec{x}) = -x_1 + 35x_2^{0.6} + 35x_3^{0.6} \leq 0$
$h_1(\vec{x}) = -300x_3 + 7500x_5 - 7500x_6 - 25x_4x_5 + 25x_4x_6 + x_3x_4 = 0$
$h_2(\vec{x}) = 100x_2 + 155.365x_4 + 2500x_7 - x_2x_4 - 25x_4x_7 - 15536.5 = 0$
$h_3(\vec{x}) = -x_5 + \ln(-x_4 + 900) = 0$
$h_4(\vec{x}) = -x_6 + \ln(x_4 + 300) = 0$
$h_5(\vec{x}) = -x_7 + \ln(-2x_4 + 700) = 0$

where the bounds are $0 \leq x_1 \leq 1000$, $0 \leq x_2, x_3 \leq 40$, $100 \leq x_4 \leq 300$, $6.3 \leq x_5 \leq 6.7$, $5.9 \leq x_6 \leq 6.4$ and $4.5 \leq x_7 \leq 6.25$. The best known solution is at $x^* = (193.7783493, 0, 17.3272116, 100.0156586, 6.684592154, 5.991503693, 6.214545462)$ where $f(x^*) = 193.7783493$.

- **g22** [6]:
Minimize: $f(\vec{x}) = x_1$

subject to:
$g_1(\vec{x}) = -x_1 + x_2^{0.6} + x_3^{0.6} + x_4^{0.6} \leq 0$
$h_1(\vec{x}) = x_5 - 100000x_8 + 1 \times 10^7 = 0$
$h_2(\vec{x}) = x_6 + 100000x_8 - 100000x_9 = 0$
$h_3(\vec{x}) = x_7 + 100000x_9 - 5 \times 10^7 = 0$
$h_4(\vec{x}) = x_5 + 100000x_{10} - 3.3 \times 10^7 = 0$
$h_5(\vec{x}) = x_6 + 100000x_{11} - 4.4 \times 10^7 = 0$
$h_6(\vec{x}) = x_7 + 100000x_{12} - 6.6 \times 10^7 = 0$
$h_7(\vec{x}) = x_5 - 120x_2x_{13} = 0$
$h_8(\vec{x}) = x_6 - 80x_3x_{14} = 0$
$h_9(\vec{x}) = x_7 - 40x_4x_{15} = 0$
$h_{10}(\vec{x}) = x_8 - x_{11} + x_{16} = 0$
$h_{11}(\vec{x}) = x_9 - x_{12} + x_{17} = 0$
$h_{12}(\vec{x}) = -x_{18} + \ln(x_{10} - 100) = 0$
$h_{13}(\vec{x}) = -x_{19} + \ln(-x_8 + 300) = 0$
$h_{14}(\vec{x}) = -x_{20} + \ln(x_{16}) = 0$
$h_{15}(\vec{x}) = -x_{21} + \ln(-x_9 + 400) = 0$
$h_{16}(\vec{x}) = -x_{22} + \ln(x_{17}) = 0$
$h_{17}(\vec{x}) = -x_8 - x_{10} + x_{13}x_{18} - x_{13}x_{19} + 400 = 0$
$h_{18}(\vec{x}) = x_8 - x_9 - x_{11} + x_{14}x_{20} - x_{14}x_{21} + 400 = 0$
$h_{19}(\vec{x}) = x_9 - x_{12} - 4.60517x_{15} + x_{15}x_{22} + 100 = 0$

where the bounds are $0 \leq x_1 \leq 20000$, $0 \leq x_2, x_3, x_4 \leq 1 \times 10^6$, $0 \leq x_5, x_6, x_7 \leq 4 \times 10^7$, $100 \leq x_8 \leq 299.99$, $100 \leq x_9 \leq 399.99$, $100.01 \leq x_{10} \leq 300$, $100 \leq x_{11} \leq 400$, $100 \leq x_{12} \leq 600$, $0 \leq x_{13}, x_{14}, x_{15} \leq 500$, $0.01 \leq x_{16} \leq 300$, $0.01 \leq x_{17} \leq 400$, $-4.7 \leq x_{18}, x_{19}, x_{20}, x_{21}, x_{22} \leq 6.25$. The best known solution is at $x^* = (12812.5, 722.1602494, 8628.371755, 2193.749851, 9951396.436, 18846563.16, 11202040.4, 199.5139644, 387.979596, 114.8336587, 27.30318607, 127.6585887, 52.020404, 160, 4.871266214, 4.610018769, 3.951636026, 2.486605539, 5.075173815)$ where $f(x^*) = 12812.5$.

- **g23** [21]:
Minimize: $f(\vec{x}) = -9x_5 - 15x_8 + 6x_1 + 16x_2 + 10(x_6 + x_7)$

subject to:
$h_1(\vec{x}) = x_1 + x_2 - x_3 - x_4 = 0$
$h_2(\vec{x}) = 0.03x_1 + 0.01x_2 - x_9(x_3 + x_4) = 0$
$h_3(\vec{x}) = x_3 + x_6 - x_5 = 0$
$h_4(\vec{x}) = x_4 + x_7 - x_8 = 0$
$g_1(\vec{x}) = x_9x_3 + 0.02x_6 - 0.025x_5 \leq 0$
$g_2(\vec{x}) = x_9x_4 + 0.02x_7 - 0.015x_8 \leq 0$

where the bounds are $0 \leq x_1, x_2, x_6 \leq 300$, $0 \leq x_3, x_5, x_7 \leq 100$, $0 \leq x_4, x_8 \leq 200$ and $0.01 \leq x_9 \leq 0.03$. The best known solution has a objective function value of $f(x^*) = -400.0551$

i	a_i	b_i	c_i	d_i	e_i
1	0.0693	44.094	123.7	31.244	0.1
2	0.0577	58.12	31.7	36.12	0.3
3	0.05	58.12	45.7	34.784	0.4
4	0.2	137.4	14.7	92.7	0.3
5	0.26	120.9	84.7	82.7	0.6
6	0.55	170.9	27.7	91.6	0.3
7	0.06	62.501	49.7	56.708	
8	0.1	84.94	7.1	82.7	
9	0.12	133.425	2.1	80.8	
10	0.18	82.507	17.7	64.517	
11	0.1	46.07	0.85	49.4	
12	0.9	60.097	0.64	49.1	
13	0.0693	44.094			
14	0.0577	58.12			
15	0.05	58.12			
16	0.2	137.4			
17	0.26	120.9			
18	0.55	170.9			
19	0.06	62.501			
20	0.1	84.94			
21	0.12	133.425			
22	0.18	82.507			
23	0.1	46.07			
24	0.09	60.097			

Table 6: Data set for test problem g20

- **g24** [7]:
  Minimize: $f(\vec{x}) = -x_1 - x_2$

  subject to:
  $g_1(\vec{x}) = -2x_1^4 + 8x_1^3 - 8x_1^2 + x_2 \leq 0$
  $g_2(\vec{x}) = -4x_1^4 + 32x_1^3 - 88x_1^2 + 96x_1 + x_2 - 36 \leq 0$

  where the bounds are $0 \leq x_1 \leq 3$ and $0 \leq x_2 \leq 4$. The feasible global minimum is at $x^* = (2.3295, 3.17846)$ where $f(x^*) = -5.50796$.

Our next step was to solve the new set of 11 problems using SMES and SR, adopting the exact same parameters previously defined to solve the 13 test functions taken from [17].

Most of the previous work on constraint-handling techniques relates to the benchmark proposed in [17]. However, we know (from the No Free Lunch Theorems for search [20]) that using such a limited set of functions does not guarantee, in any way, that an algorithm that performs well on them will necessarily be competitive in a different set of problems. This motivated us to identify, based on a study of the on-line behavior of competitive approaches, new test functions in which these two approaches did not exhibit a good performance. In this way, we will show empirically that the performance measures help to compare different algorithms and also help to detect sources of difficulty.

We performed 30 independent runs for each test function. In the experimentation process we used the same parameters reported in Section 3.1 for each approach.

5 Results and Discussion

The statistical results for the new set of 11 functions are summarized in Table 7 for both approaches.

Both approaches had no difficulties to solve problem g16 despite its low value of ρ. g16 involves a considerable number of nonlinear inequalities (34) combined with 4 linear inequality constraints and a nonlinear objective function. The problem has a low dimensionality (5 decision variables). SMES and SR also solved considerably well problems g14, g15, g18 and g19. Problems g14, g18 and g19 have not nonlinear equality constraints; g14 and g19 have a nonlinear objective function and g18 have a quadratic objective function

Problem & Best Known Sol.	Stats	SR [17]	SMES [12]
g14 (min) -47.656	best	$\mathbf{-47.749}$	-47.535
	mean	$\mathbf{-47.466}$	-47.367
	worst	-46.898	$\mathbf{-47.053}$
	St. Dev	2.1E-1	**1.33E-1**
g15 (min) 961.715	best	961.715	**961.698**
	mean	**961.861**	963.922
	worst	**965.126**	967.787
	St. Dev	6.21E-1	**1.79E-1**
g16 (max) 1.905	best	**1.905**	**1.905**
	mean	**1.905**	**1.905**
	worst	**1.905**	**1.905**
	St. Dev	**0**	**0**
g17 (min) 8927.5888	best	**8853.551**	–
	mean	**8856.252**	–
	worst	**8865.584**	–
	St. Dev	**4.08E+2**	–
g18 (max) 0.866	best	0.866	0.866
	mean	**0.846**	0.716
	worst	**0.670**	0.648
	St. Dev	**5.8E-2**	8.19E-2
g19 (max) -32.386	best	$\mathbf{-33.099}$	-34.223
	mean	$\mathbf{-34.345}$	-37.208
	worst	$\mathbf{-36.638}$	-41.251
	St. Dev	**8.9E-1**	2.10E+0
g20 (min) 0.0967	best	–	–
	mean	–	–
	worst	–	–
	St. Dev	–	–
g21 (min) 193.778	best	–	–
	mean	–	–
	worst	–	–
	St. Dev	–	–
g22 (min) 12812.5	best	–	–
	mean	–	–
	worst	–	–
	St. Dev	–	–
g23 (min) -400.0551	best	–	–
	mean	–	–
	worst	–	–
	St. Dev	–	–
g24 (min) -5.507960	best	-5.508013	-5.508013
	mean	-5.487235	$\mathbf{-5.508011}$
	worst	-5.380674	$\mathbf{-5.507959}$
	St. Dev	3.82E-2	**1.0E-5**

Table 7: Statistical results obtained by the SR and SMES on the new set of test functions. A number in **boldface** means a better result. "-" means no feasible solutions found.

like g15, but g15 has only one nonlinear equality constraint. The dimensionality of these problems varies from 3 to 15 decision variables.

Problems g17, g20, g21 and g22 have one common aspect: they have more nonlinear equality constraints than any other problem (4, 12, 5 and 11, respectively). In those problems, only the SR found competitive results in problem g17. In all the remaining functions, none of the approaches could find feasible solutions. The dimensionality is different for each of these four problems (6, 24, 7 and 22, respectively). For three problems, the objective function is linear (g20, g21 and g22). Only g17 has a nonlinear objective function. All that suggests that the difficulty comes from the number of nonlinear equality constraints. It is worth reminding that none of the 13 original test functions has more than 3 nonlinear equality constraints. The obtained results suggest that the combination of an increasing dimensionality and a high number of nonlinear equality constraints make a problem more difficult to solve by the compared approaches. The exception is problem g23, which has only one nonlinear equality constraint, a moderate dimensionality (9 decision variables) and a linear objective function. None of the approaches could find feasible solutions. The possible source of difficulty may come from the combination of linear and nonlinear inequality constraints (3 and 1, respectively) with a moderate dimensionality. However, this case deserves a more in-depth analysis. Finally, Problem g24 with a disjoint and a very large feasible region but with a low dimensionality of 2 represented no problem for SMES and SR.

To summarize, the overall results suggest that the two main factors that affect the performance of the two EAs are the dimensionality (like Michalewicz & Schmidt concluded for the static penalty function approach [15, 18]) and the increasing number of nonlinear equality constraints. The factors that do not seem to decrease the performance of our EA are a high number of inequality constraints (even nonlinear), and, remarkably, the type of objective function. For some problems, including a linear objective function, the problems resulted difficult to solve (even reaching the feasible region). Finally, disjoint feasible regions with a considerable large size with respect to the search space and a low dimensionality do not seem to be difficult to reach for the compared EAs.

These conclusions seem to agree with those obtained in our analysis based on our performance measures. This study is far from being conclusive, but it provides some insights in order to design more competitive EA-based approaches to solve constrained optimization problems and to improve the way constraint-handling techniques are compared.

6 Conclusions and Future Work

We have presented an empirical study of the on-line behavior of two state-of-the-art algorithms to solve constrained optimization problems. We measured how fast they reach the feasible region, how capable they are of improving feasible solutions and how fast the population becomes completely feasible. Based on the obtained results, we detected sources of difficulty which were emphasized on 11 new test problems which were solved using both analyzed approaches. The performance provided by the approaches confirmed the increasing number of nonlinear equality constraints and a high dimensionality as sources of difficulty even for very competitive approaches. As future work we will analyze each new test function more in-depth and we will also apply the performance measures in this new set of problems. Finally, we will work in an improved technique capable of solving this new set of test functions.

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