An Archiving Strategy Based on the Convex Hull of Individual Minima for MOEAs

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***Abstract*— Diversity plays an important role in evolutionary multi-objective optimization. Because of this, a number of density estimators (i.e., mechanisms that help to maintain diversity) have been proposed since the early days of multi-objective evolutionary algorithms (MOEAs). Fitness sharing and niching were among the most popular density estimator used with non-elitist MOEAs, but their main drawback was their high dependence on the niche radius, which was normally difficult to set. In recent years, the use of external archives to store the nondominated solutions found by an elitist MOEA has become popular. This has motivated an important amount of research related to archiving techniques for MOEAs. In this paper, we contribute to such literature by introducing a new archiving strategy based on the Convex Hull of Individual Minima (CHIM). Our proposed approach is compared with respect to two competitive MOEAs (NSGA-II and SPEA2) using standard test problems and performance measures taken from the specialized literature.**

I. Introduction

Since their origins, multi-objective evolutionary algorithms (MOEAs) have had two main goals [2]: (1) maximize the number of elements of the Pareto optimal set that are generated and (2) distribute such solutions as uniformly as possible (such uniform distribution is normally enforced in objective function space—i.e., along the Pareto front). Here, we focus on the second of these goals, since we propose a new method to distribute non-dominated solutions. Over the years, a number of such methods have been proposed, going from the use of fitness sharing and niching [6], to the incorporation of adaptive grids [13] and clustering techniques [22]. Our proposed method is not based on any of these mechanisms. Instead, it is based on the *Convex Hull of Individual Minima* (CHIM) to maintain solutions well-distributed along the Pareto front. The goal of this strategy is to achieve convergence towards the Pareto optimal set by maintaining a suitable representation of the Pareto front.

The remainder of the paper is organized as follows. In Section II, we provide the basic background required for understanding the rest of the paper. In Section III, we briefly review the previous related work. In Section IV, we explain in detail our proposed approach. In Section V, we show the results of our comparative study. Finally, in Section VI we give our conclusion and a brief description of some possible future work.

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II. Basic Concepts

A *multi-objective optimization problem* (MOP) can be formally defined as to find a vector $x \in \mathcal{Q}$ which minimizes:

$$\begin{array}{c} F : \Omega \rightarrow \mathbb{R}^k \\ F(x) = [f_1(x), \ldots, f_k(x)]^T \end{array} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ defines the feasible region of the problem.

The notation and basic concepts used within the multi-objective optimization (assuming minimization problems) are provided next.

Definition 1. Let $v, w \in \mathbb{R}^k$, we say that v is *less than* w ($v \leq_p w$) if $v_i \leq w_i, \quad \forall i = 1, \ldots, k$, and v *dominates* w ($v \prec w$), if $v \leq_p w$ and $v \neq w$.

Definition 2. A point $x \in \Omega \subset \mathbb{R}^n$ is called *Pareto optimal* or a *Pareto point* if there is no $y \in \mathbb{R}^n$ such that $y \prec x$.

Definition 3. The *Pareto Optimal Set* $\mathcal{P}^*$ is defined as:

$$\mathcal{P}^* = \{x \in \mathbb{R}^n | x \text{ is Pareto optimal}\}$$

Definition 4. The *Pareto Front* $\mathcal{F}^*$ is defined as:

$$\mathcal{F}^* = \{F(x) \in \mathbb{R}^k | x \in \mathcal{P}^*\}$$

Thus, when solving MOPs, we are interested in finding the best possible *trade-offs* among the objectives, such that no objective can be improved without worsening another. Since the number of Pareto optimal solutions can be very large, we are also interested in obtaining a well-distributed set of solutions, since the size of our approximation (produced by a MOEA) will be normally small (e.g., 100 solutions).

A. The Convex Hull of Individual Minima

The *Convex Hull of Individual Minima* concept was initially introduced by Das in [3] and it has been the basis of some multi-objective optimization techniques available in the mathematical programming literature, such as the *Normal Boundary Intersection* [5] (NBI) method and the *Recursive Knee Approach* [4] (RKA). The following definitions are taken from [3].

Definition 5. Let x_i^* be the respective global minimizers of $f_i(\vec{x})$, $i = 1, \ldots, k$ over $x \in \Omega$. Let $F_i^* = F(x_i^*)$, $i = 1, \ldots, k$. Let Φ be the $k \times k$ matrix whose i^{th} column is $F_i^* - F^*$ (sometimes known as the *pay-off* matrix). Then, the set of points in $\mathbb{R}^k$ that are convex combinations of $F_i^* - F^*$, that is:

$$\mathcal{H} = \{\Phi\beta : \beta \in \mathbb{R}^k, \sum_{i=1}^{k} \beta_i = 1, \beta_i \geq 0\} \tag{2}$$

is referred to as the *Convex Hull of Individual Minima* (CHIM).

The set of all attainable objective vectors $\mathcal{F} \subset \mathbb{R}^k$ determines the *objective space*, and the *boundary intersection* of $\mathcal{F}$ is denoted by $\partial\mathcal{F}$, see Fig. 1.

Definition 6. Let $CHIM_\infty$ be the affine subspace of lowest dimension that contains the $CHIM$. Then, $CHIM_\infty$ is defined as the smallest simply connected set that contains every point in the intersection of $\partial\mathcal{F}$ and $CHIM_\infty$. More informally, consider extending (or withdrawing) the boundary of the $CHIM$ simplex to touch $\partial\mathcal{F}$; the 'extension' of $CHIM$ thus obtained is defined as $CHIM_\infty$. Here, it is denoted by $\mathcal{H}_\infty$.

Since the hyperplane defined by $\mathcal{H}_\infty$ is an affine subspace, there exists a unique normal vector n for each point h in the surface $\mathcal{H}_\infty$. That is:

$$\forall h \in \mathcal{H}_\infty, \exists_1 n \in \mathbb{R}^k \setminus \{\mathbf{0}\} : \langle h, n \rangle = 0 \tag{3}$$

where $||n|| = 1$ and $\langle \cdot, \cdot \rangle$ denotes the inner product.

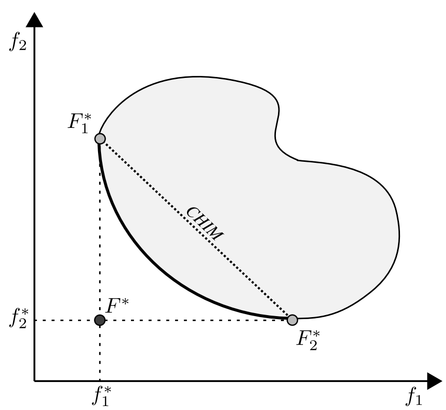


Fig. 1. Convex Hull of Individual Minima

III. Previous Related Work

As indicated before, the interest in developing strategies to maintain diversity among the solutions generated by a MOEA, dates back to the origins of such algorithms. *Niching* and *fitness sharing* [16], [11] are among the oldest mechanisms to maintain diversity in MOEAs, although they were originally developed for dealing with multimodal functions [6]. Niching and fitness sharing were very popular as density estimators during the early days of evolutionary multiobjective optimization [10], [19], [12]. However, over the years, a variety of density estimators have been proposed, including the use of adaptive grids [14], entropy [9], clustering techniques [22], and crowding [7], among others. Most of these approaches, however, rely on the definition of some critical parameter (e.g., the niche radius).

More recently, the use of archiving strategies that rely on relaxed forms of Pareto dominance were introduced. From them, ε-dominance is probably the best known [15]. This approach also relies on a critical parameter: the value of ε, which, in the general case, cannot be known beforehand and has to be empirically estimated through some preliminary sampling of the search space. However, the nice mathematical properties of ε-dominance, as well as its good performance in practice, made it a relatively popular archiving mechanism and has motivated the development of other (more elaborate) archiving techniques for MOEAs (see for example [18]).

To the authors' best knowledge, CHIM had not be used before as an archiving technique in MOEAs, as we do here. The only related work that we could find is the proposal of Cococcioni et al. [1], which adopts the concept of convex hull for binary classification using a MOEA, but does not incorporate it into any archiving technique.

IV. Our Proposed Approach

In this section, we describe our proposal which is called *Convex Hull Multi-objective Evolutionary Algorithm* (CH-MOEA). This approach refers to a MOEA which uses an archiving mechanism based on the convex hull of individual minima as its strategy to store well-distributed non-dominated solutions.

A. The Multi-Objective Evolutionary Algorithm

In CH-MOEA, the initial population $\mathcal{P}_0$ is defined by N uniformly distributed random solutions. Then, the evolutionary process is carried out. Our approach uses a selection mechanism ($\mu + \lambda$) based on Pareto ranking (it is used in the $UpdatePopulation$ procedure). In this way, the best solutions obtained from the evolutionary process are maintained at the current population. If two solutions are non-dominated, we randomly choose any of them. Our proposal uses an archive $\mathcal{L}$ which stores all the non-dominated solutions found by the MOEA at each iteration. The archive is also used during the selection process. The crossover and mutation operators are the same used by the NSGA-II [7] (Simulated Binary Crossover (SBX) and Parameter-Based Mutation (PBM)). The full pseudocode of our proposed MOEA is shown in Algorithm 1 and the details of the archiving strategy are presented next.

B. Archiving the Convex Hull of Individual Minima

Considering $\mathcal{H}_\infty$ as an affine subspace, then, there exists a unique normal vector for each point in $\mathcal{H}_\infty$. A well-distributed set of solutions along the Pareto front can be achieved by the use of non-dominated solutions which are orthogonal to each point in $\mathcal{H}_\infty$. However, to obtain an analytical expression for $\mathcal{H}_\infty$, is equivalent to obtain an analytical expression for the Pareto curve (i.e., it is impossible in most cases). In order to obtain the points of our interest, we rely on the CHIM (as before, it is denoted by $\mathcal{H}$) as the basis for our archiving strategy. Thus, to produce a well-distributed set of solutions in the archive $\mathcal{L}$, we will take the non-dominated solutions which are orthogonal to each point $h \in \mathcal{H}$. However, it is well-known that, for $k > 2$, there do not exist orthogonal points in $\mathcal{H}$ for all the elements in the Pareto front (see [5]). In order to deal with this drawback, an alternative strategy is required. In our case, we use the direction vector with respect to the utopian vector F^*. With this, the non-dominated solutions into the archive $\mathcal{L}$ are

Algorithm 1 *CH-MOEA*

$t = 0$;
$\mathcal{L} = \emptyset$;
Generate initial population $\mathcal{P}_t$ of size N;
$Evaluate(\mathcal{P}_t)$;
$\mathcal{L} = UpdateArchive(p, \mathcal{L}), \forall p \in \mathcal{P}_t$;
while $(t < max_{gen})$ **do**
  $\mathcal{Q} = \emptyset$;
  for all $p_1, p_2 \in \mathcal{P}_t$ **do**
    Choose $l_1, l_2 \in \mathcal{L} : p_1 \neq l_1$ and $p_2 \neq l_2$;
    $q_1 = mutation(crossover(p_1, l_1))$;
    $q_2 = mutation(crossover(p_2, l_2))$;
    $\mathcal{L} = UpdateArchive(q_i, \mathcal{L}), i = 1, 2$;
    $\mathcal{Q} = \mathcal{Q} \cup \{q_1, q_2\}$;
  end for
  $Evaluate(\mathcal{Q})$;
  $\mathcal{P}_{t+1} = UpdatePopulation(\mathcal{P}_t, \mathcal{Q})$; `// keep` N `individuals using Pareto ranking`
  $t = t + 1$;
end while

properly allocated. The main idea in the archiving strategy, is to store into the archive a single non-dominated solution $p \leftarrow F(x)$ for each direction defined by the utopian vector F^* and a point h_i in the CHIM.

Let W be the finite set of vectors that are well-distributed in $\mathbb{R}^k$, such that, each vector $w_i \in W$ is a convex combination of weights (where k is the number of objectives). Then, each point $h_i \in \mathcal{H}$ is defined according to:

$$h_i(w_i) = \Phi w_i \tag{4}$$

Therefore, all the h_i's represent a finite set of well-distributed points in the hyperplane $\mathcal{H}$.

The uniformity of the solutions along the Pareto front depend directly of the uniformity of the weight vectors w_i. Therefore, a good distribution of the vectors in $\mathcal{H}$ implies a good distribution of solutions along the Pareto front. The construction of well-distributed convex combinations of weights can be achieved either as proposed by Das in [3] or by Zhang and Li in [20].

We consider the archive $\mathcal{L}$ as the set of solutions accepted by the archiving mechanism. Then, a solution p is accepted by the archive, if and only if, p is non-dominated with respect to all solutions stored in $\mathcal{L}$. If the solution p is not dominated, and there is no other solution $q \in \mathcal{L}$, such that the direction q is closer than direction p with respect to h_i (for any $h_i \in \mathcal{H}$), then the solution p is stored in the archive; otherwise, the solution is rejected. Thus, we prefer the solution p, such that it satisfies:

$$\frac{h_i(w_i)}{||h_i(w_i)||} = \frac{p}{||p||} \tag{5}$$

for any $h_i(w_i) \in \mathcal{H}$.

In practice, to satisfy equation (5) is unlikely. Thus, we prefer a solution p that minimizes:

$$\mathcal{A}(p, w_i) = \left|\left| \frac{h_i(w_i)}{||h_i(w_i)||} - \frac{p}{||p||} \right|\right| \tag{6}$$

Since $\mathcal{H}$ is defined by F^*, the compromise solution p should be translated to F^* in order to get a correct direction. Thus, we assume that $p \leftarrow F(x) - F^*$.

In CH-MOEA, the utopian vector F^* is constructed with the minimum value of each objective function. Furthermore, the individual minima are defined for each x_i^* $(i = 1, \ldots, k)$, such that the following *augmented weighted Tchebycheff* function is minimized:

$$\min_{x_i^* \in \mathcal{D}} \max_{j=1,\ldots,k} \{w_j |f_j(x_i^*) - F_i^\star|\} + \rho \sum_{j=1}^{k} |f_i(x_i^*) - F_j^\star| \tag{7}$$

where, ρ is a sufficiently small positive scalar, $\mathcal{D}$ is the set of all solutions found so far by a MOEA and w_j is the j^{th} canonical vector in $\mathbb{R}^k$.

According to equation (6), for each $w_i \in W$, it is expected to find a single solution p such that it minimizes equation (6). Therefore, the archive $\mathcal{L}$ is defined by a finite number of solutions which correspond to the total number of weight vectors w_i in W. The complete archiving strategy is shown in Algorithm 2.

C. Final Comments about the Archiving Strategy

1) The set of obtained solutions: The solution set obtained by our archiving strategy contains only non-dominated solutions. According to Algorithm 2, the first filter (based on the use of Pareto dominance) guarantees to have only non-dominated solutions into the archive $\mathcal{L}$. This set of solutions has a finite size which is defined by the number of weighted vectors w's. Our approach heavily relies on the minimum values of each objective function. However, in practice, this is not a serious drawback, since we can adopt the extreme points of the Pareto front obtained by the MOEA. It is worth noting, however, that, because of its nature, our archiving strategy is expected to have a poor behavior when dealing with disconnected Pareto fronts.

2) Computational Complexity: Algorithm 2 shows our archiving strategy in detail. The first part of the algorithm decides whether or not a solution is non-dominated. This procedure has a complexity $O(m)$, where m is the size of the archive $\mathcal{L}$. The second part refers to the filtering of each solution p using its direction to the utopian vector F^*. Considering k as the number of vectors in W, the computational effort to find the vector $w' \in W$ such that it minimizes equation (6) is linear with respect to the number of weighted vectors in W, that is $O(k)$. Then, the archiving algorithm finds a solution q (if it exists) in the archive $\mathcal{L}$, such that $w'' \in W$ minimizes equation (6) regarding q and $w' = w''$; therefore, the overall computational cost is bounded by $O(mk)$. However, this procedure has to be done for each element of the current archive $\mathcal{L}$ for deleting solutions which have the same direction that the candidate

Algorithm 2 *UpdateArchive*($p, \mathcal{L}_0$)

// *Filtering out solutions using Pareto dominance*
if $(\mathcal{L}_0 = \emptyset)$ **then**
  $\mathcal{L} = \{p\}$;
  return $\mathcal{L}$;
end if
if $(\exists l \in \mathcal{L}_0 : l \prec p)$ **then**
  // *Rejecting solution* p;
  return $\mathcal{L}_0$;
else
  $\mathcal{L} = \mathcal{L}_0 \setminus \mathcal{D}$, where: $\mathcal{D} = \{l \in \mathcal{L}_0 : p \prec l\}$
end if
if $(\mathcal{L} = \emptyset)$ **then**
  $\mathcal{L} = \{p\}$;
  return $\mathcal{L}$;
end if
// *Filtering out solutions based on their direction*
$\mathcal{L}_0 = \mathcal{L}_0 \cup \{p\}$;
// *Considering* $\mathcal{L}_0 = \{l_1, \ldots, l_m\}$
$\mathcal{L} = l_1$;
for all $l_i \in \mathcal{L}_0, i = 2, \ldots, m$ **do**
  $w' = w \in W,$ such that minimizes: $\mathcal{A}(l_i, w)$;
  if $(\exists q \in \mathcal{L} | w'' = w \in W,$ such that minimizes: $\mathcal{A}(q, w)$ and $w' \neq w'')$ **then**
    if $\mathcal{A}(l_i, v') < \mathcal{A}(q, v')$ **then**
      $\mathcal{L} = \mathcal{L} \cup \{l_i\}$;
    end if
  else
    $\mathcal{L} = \{\mathcal{L} \setminus \{q\}\} \cup \{l_i\}$;
  end if
end for
return $\mathcal{L}$;

solution p. Hence, the filtering by directions requires a total computational effort bounded by $O(m(k + mk))$.

Note however that, in practice, our procedure has really a lower computational complexity, because the above complexity is required only when the utopian vector or the individual minima have changed. If this does not happen, the vector w which minimizes equation (6) will be the same in each call from the archiving algorithm. Thus, we save this time and the complexity of filtering out solutions based on their directions is reduced from $O(m(k+mk))$ to $O(mk)$. Considering this fact, the total computational complexity for our archiving strategy is given by $O(m) + O(mk)$ which is considered as the average case.

V. Comparison of Results

A. Performance Measures

In order to evaluate the performance of our proposal, we compared its performance with respect to NSGA-II [7] and SPEA2 [22] using the following performance measures:

1) Hypervolume: Hypervolume (Δ) was proposed in [23]. This measure quantifies the approximation of non-dominated solutions to the Pareto front. We can say that a set of solutions A has a better approximation to the Pareto front than a set B if:

$$\Delta(A) > \Delta(B)$$

This performance measure uses a reference vector r which can be defined by the user. Here, we define the reference vector according to:

$$r = [f_1^+, f_2^+, \ldots, f_k^+]$$

where k is the number of objective functions, f_i^+ is the maximum value $f_i(x)$ found in the set $\mathcal{T} = \{A_1 \cup \cdots A_l\}$ and A_j is the set of solutions found by the j^{th} algorithm applied to a given MOP.

2) Spacing: Spacing ($\mathcal{S}$) was proposed by Schott [17] and it quantifies the spread of solutions in the obtained approximation of the Pareto front. The $\mathcal{S}$ performance measure can be calculated as:

$$\mathcal{S} = \sqrt{\frac{1}{|P|-1} \sum_{i=1}^{|P|} (\overline{d} - d_i)^2} \qquad (8)$$

where d_i and $\overline{d}$ are defined as:

$$d_i = \min_{i, i \neq j} \left\{ \sum_{k=1}^{M} |f_k^i - f_k^j| \right\}$$

$$\overline{d} = \frac{\sum_{i=1}^{|P|} d_i}{|P|}$$

A value of zero for this performance measure indicates that all the solutions are uniformly spread (i.e., the best possible performance).

3) Set Coverage: Set Coverage ($\mathcal{SC}$) was proposed by Zitzler et al. [21], and it compares a set of non-dominated solutions A with respect to another set B, using Pareto dominance. This performance measure is defined as:

$$\mathcal{SC}(A, B) = \frac{|\{b \in B | \exists a \in A : a \preceq b\}|}{|B|} \qquad (9)$$

If all points in A dominate or are equal to all points in B, this implies that $\mathcal{SC}(A, B) = 1$. Otherwise, if no point of A dominates some point in B then $\mathcal{SC}(A, B) = 0$. When $\mathcal{SC}(A, B) = 1$ and $\mathcal{SC}(B, A) = 0$ then, we say that A is better than B. Since the Pareto dominance relation is not symmetric, we need to calculate both $\mathcal{SC}(A, B)$ and $\mathcal{SC}(B, A)$.

B. Test Problems

We adopted test problems having 2 and 3 objective functions, and taken from two well-known benchmarks: the Zitzler-Deb-Thiele (ZDT) test suite [21] (except for ZDT5, which was omitted because is a binary problem) and the first four problems from the Deb-Thiele-Laumanns-Zitzler (DTLZ) test suite [8]. The description of these problems is presented in Table I.

TABLE I

TEST PROBLEMS

Problem	n	Domain	Objective functions
ZDT1	30	$x_i \in [0,1]$	$f_1(x) = x_1$ $f_2(x) = g(x)(1 - \sqrt{x_1/g(x)})$ $g(x) = 1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i$
ZDT2	30	$x_i \in [0,1]$	$f_1(x) = x_1$ $f_2(x) = g(x)(1 - (x_1/g(x))^2)$ $g(x) = 1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i$
ZDT3	30	$x_i \in [0,1]$	$f_1(x) = x_1$ $f_2(x) = g(x)(1 - \sqrt{x_1/g(x)}$ $-(x_1/g(x)\sin(10\pi x_1))$ $g(x) = 1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i$
ZDT4	10	$x_1 \in [0,1]$ $x_i \in [-5,5]$ $i = 2,\ldots,n$	$f_1(x) = x_1$ $f_2(x) = g(x)(1 - \sqrt{x_1/g(x)})$ $g(x) = 1 + 10(n-1)$ $+\sum_{i=2}^{n} x_i^2 - 10\cos(4\pi x_i)$
ZDT6	10	$x_i \in [0,1]$	$f_1(x) = 1 - e^{-4x_1}\sin^6(4\pi x_1)$ $f_2(x) = g(x)(1 - (f_1(x)/g(x))^2)$ $g(x) = 1 + 9(\sum_{i=2}^{n} x_i/(n-1))^{0.25}$
DTLZ1	12	$x_i \in [0,1]$	$f_1(x) = \frac{1}{2}x_1x_2(1+g(x))$ $f_2(x) = \frac{1}{2}x_1(1-x_2)(1+g(x))$ $f_3(x) = \frac{1}{2}(1-x_1)(1+g(x))$ $g(x) = 100(n + \sum_{i=3}^{n}[(x_i - 0.5)^2$ $-\cos(20\pi(x_i - 0.5))])$
DTLZ2	12	$x_i \in [0,1]$	$f_1(x) = \cos(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)h(x)$ $f_2(x) = \cos(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2)h(x)$ $f_3(x) = \sin(\frac{\pi}{2}x_1)h(x)$ $g(x) = \sum_{i=3}^{n}(x_i - 0.5)^2$ $h(x) = (1+g(x))$
DTLZ3	12	$x_i \in [0,1]$	$f_1(x) = \cos(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)h(x)$ $f_2(x) = \cos(\frac{\pi}{2}x_1)\sin(\frac{\pi}{2}x_2)h(x)$ $f_3(x) = \sin(\frac{\pi}{2}x_1)h(x)$ $g(x) = 100[10 + \sum_{i=3}^{n}((x_i - 0.5)^2$ $-\cos(20\pi(x_i - 0.5))]$ $h(x) = (1+g(x))$
DTLZ4	12	$x_i \in [0,1]$	$f_1(x) = \cos(\frac{\pi}{2}x_1^{\alpha})\cos(\frac{\pi}{2}x_2^{\alpha})h(x)$ $f_2(x) = \cos(\frac{\pi}{2}x_1^{\alpha})\sin(\frac{\pi}{2}x_2^{\alpha})h(x)$ $f_3(x) = \sin(\frac{\pi}{2}x_1^{\alpha})h(x)$ $g(x) = \sum_{i=3}^{n}(x_i - 0.5)^2$ $h(x) = (1+g(x))$ with $\alpha = \pi$

C. Experimental Setup

For each MOP, we performed 25 independent runs with each algorithm. Each run was restricted to 100 generations. For the problems with 2 objectives, we used a population size $N = 100$ and a population size $N = 300$ was adopted for the problems having 3 objectives. Therefore, we performed $10,000$ (for the bi-objective problems) and $30,000$ (for the three-objective problems) fitness function evaluations.

For the algorithms (CH-MOEA, NSGA-II and SPEA2), the parameters were set as follows: $P_c = 1.0$ (crossover probability) and $P_m = \frac{1}{n}$ (mutation probability), where n represents the number of decision variables of the MOP. Since our approach adopts the same genetic operators included in the NSGA-II, we used the same parameter values for these operators in both algorithms, that is: crossover index $\eta_c = 20$ and mutation index $\eta_m = 20$.

For each MOP, the algorithms were evaluated using the three performance measures previously indicated (Δ, $\mathcal{S}$ and $\mathcal{SC}$). The results are summarized in Tables III to VIII. Each table displays both the average and the standard deviation (σ) of each performance measure, for each of the test problems adopted. For an easier interpretation, the best results are presented in **boldface** for each performance measure and test problem adopted.

1) Discussion of Results: As shown in Tables III and VI, our proposed approach (CH-MOEA) outperformed both NSGA-II and SPEA2 in all the test problems with respect to the hypervolume (Δ). This indicates that our algorithm has produced a better approximation to the Pareto front in all the test problems adopted. As we said before, the reference vector is defined by the maximal vectors found in all the sets of solutions produced by the algorithms (CH-MOEA, NSGA-II and SPEA2). In this specific case, the reference vectors for each MOP are shown in Table II

Our approach also achieved better results in most of the test problems with respect to the spacing performance measure ($\mathcal{S}$), as can be seen in Tables IV and VII. The exceptions were ZDT4 and DTLZ4, but in those cases, our approach achieved better convergence.

Finally, in Tables V and VIII we can see that our approach obtained the best results with respect to the set coverage performance measure ($\mathcal{SC}$) in all the test functions adopted. This implies that our approach obtained more solutions that dominate those generated by both NSGA-II and SPEA2 and which are not dominated by any of them.

TABLE II

REFERENCE VECTORS FOR THE HYPERVOLUME PERFORMANCE MEASURE.

Problem	Reference vector
$ZDT1$	$(0.9998, 5.7764)^T$
$ZDT2$	$(0.9999, 5.8258)^T$
$ZDT3$	$(0.8595, 6.8002)^T$
$ZDT4$	$(0.9992, 98.9079)^T$
$ZDT6$	$(1.0000, 6.2133)^T$
$DTLZ1$	$(438.643624, 394.795995, 401.006598)^T$
$DTLZ2$	$(2.327085, 2.183336, 2.048173)^T$
$DTLZ3$	$(1597.412149, 1648.946164, 1678.494421)^T$
$DTLZ4$	$(2.539207, 2.158534, 2.303041)^T$

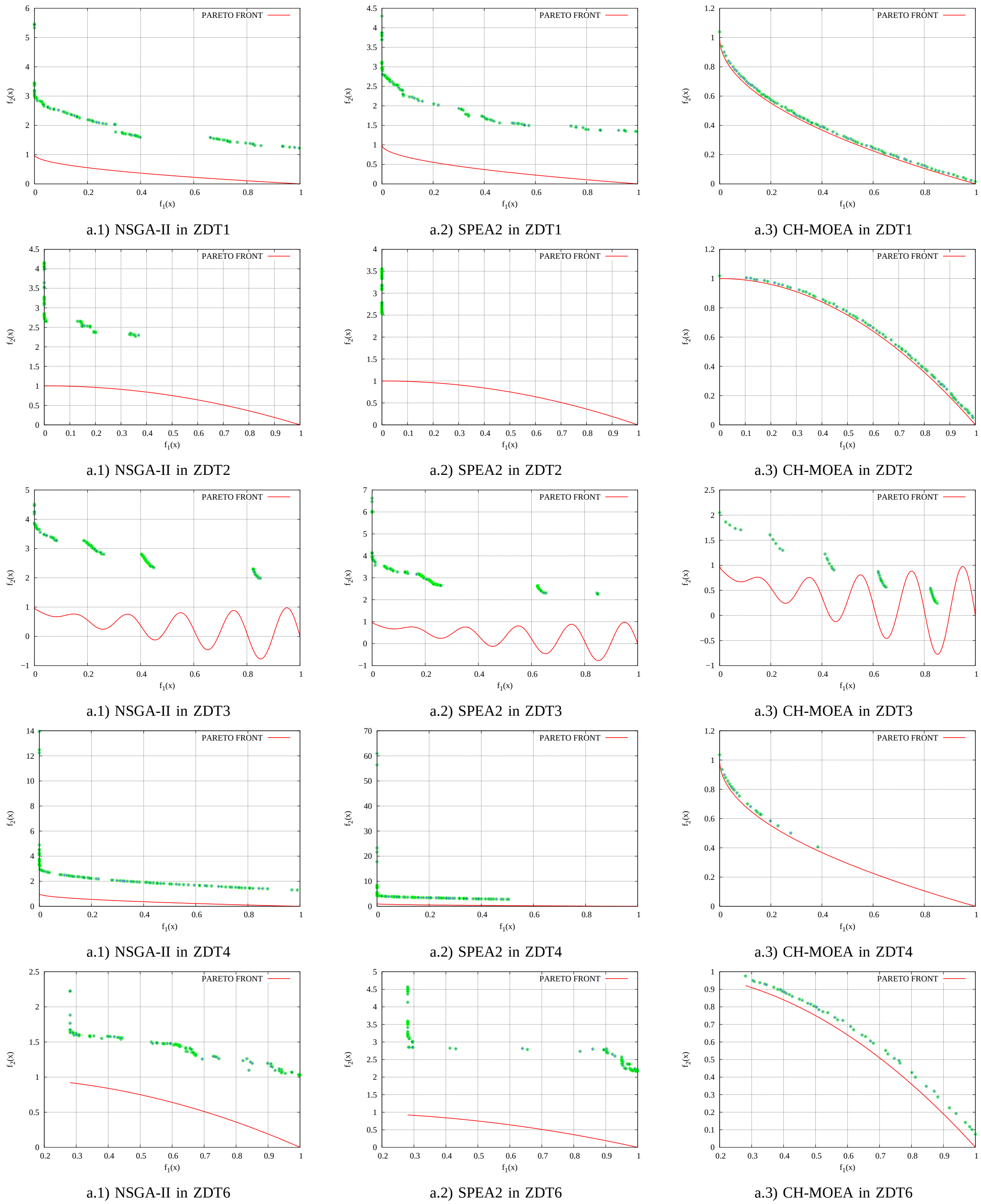
TABLE III

RESULTS OF THE Δ PERFORMANCE MEASURE FOR THE ZDT TEST SUITE.

	$NSGA-II$	$SPEA2$	$CH-MOEA$
	$average$ (σ)	$average$ (σ)	$average$ (σ)
ZDT1	4.0585 (0.1111)	3.9930 (0.1067)	**5.4173** (0.0036)
ZDT2	3.3378 (0.1935)	3.3316 (0.1824)	**5.1127** (0.0340)
ZDT3	3.4864 (0.0801)	3.4863 (0.1327)	**4.8770** (0.0131)
ZDT4	95.7671 (0.8096)	95.9258 (0.7529)	**98.2674** (0.1401)
ZDT6	3.0712 (0.3327)	3.0382 (0.2549)	**4.0187** (0.0052)

VI. CONCLUSIONS AND FUTURE WORK

We have presented a new archiving strategy, which is based on mathematical programming concepts, rather than on niching, ε-dominance, adaptive grids or any other concept traditionally adopted by MOEAs. Our proposed approach was able to outperform both NSGA-II and SPEA2 in several



a.1) NSGA-II in ZDT1 a.2) SPEA2 in ZDT1 a.3) CH-MOEA in ZDT1

a.1) NSGA-II in ZDT2 a.2) SPEA2 in ZDT2 a.3) CH-MOEA in ZDT2

a.1) NSGA-II in ZDT3 a.2) SPEA2 in ZDT3 a.3) CH-MOEA in ZDT3

a.1) NSGA-II in ZDT4 a.2) SPEA2 in ZDT4 a.3) CH-MOEA in ZDT4

a.1) NSGA-II in ZDT6 a.2) SPEA2 in ZDT6 a.3) CH-MOEA in ZDT6

Fig. 2. ZDT test problems

test problems taken from the specialized literature, with respect to three performance measures (hypervolume, spacing and set coverage).

Our proposed archiving strategy obtained very good results for problems having a fully connected Pareto front, but it performs poorly with disconnected Pareto fronts. Nevertheless, we will be looking into this issue as part of our future work.

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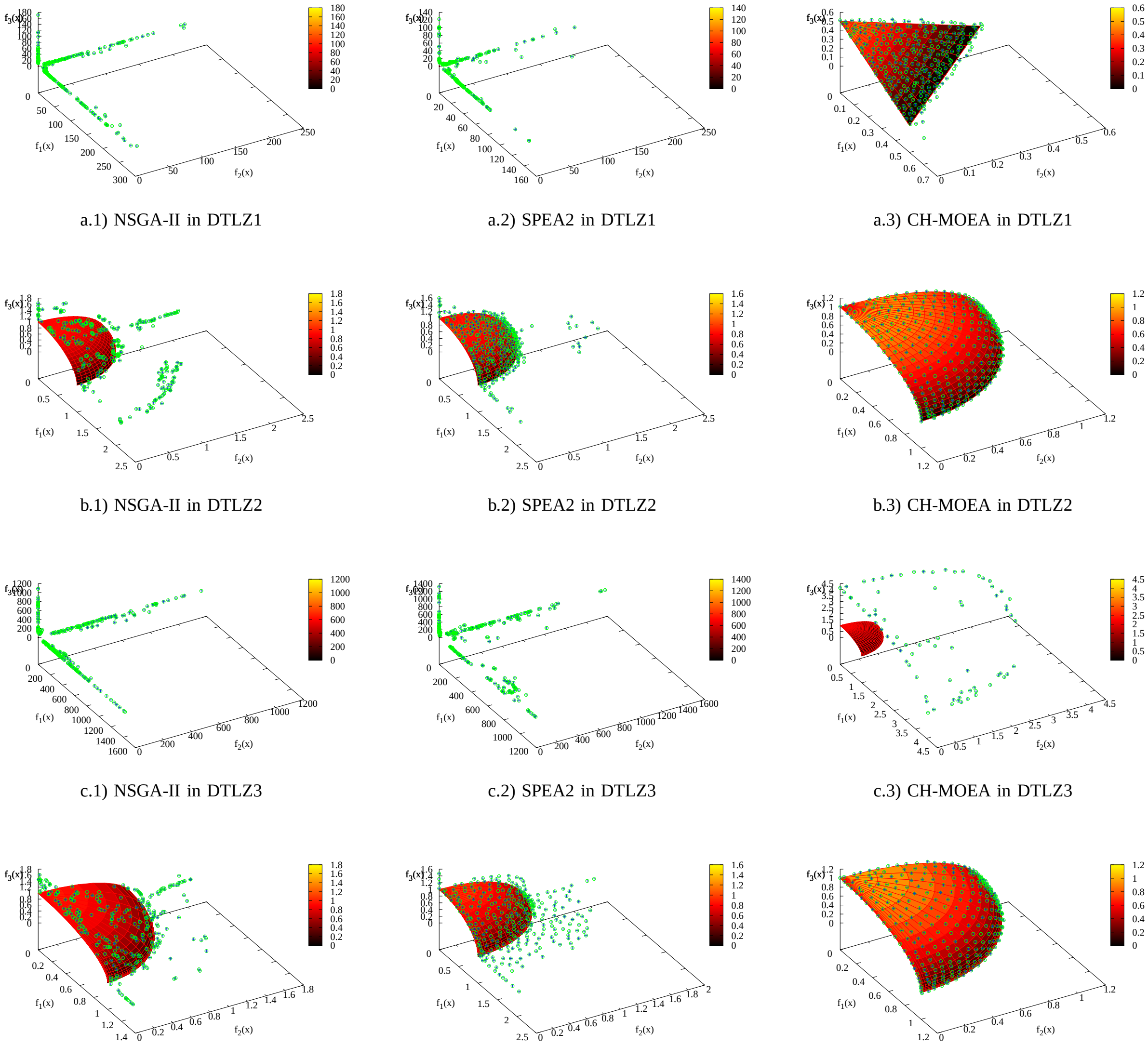

a.1) NSGA-II in DTLZ1 a.2) SPEA2 in DTLZ1 a.3) CH-MOEA in DTLZ1

b.1) NSGA-II in DTLZ2 b.2) SPEA2 in DTLZ2 b.3) CH-MOEA in DTLZ2

c.1) NSGA-II in DTLZ3 c.2) SPEA2 in DTLZ3 c.3) CH-MOEA in DTLZ3

d.1) NSGA-II in DTLZ4 d.2) SPEA2 in DTLZ4 d.3) CH-MOEA in DTLZ4

Fig. 3. DTLZ test problems

TABLE IV

RESULTS OF THE $\mathcal{S}$ METRIC FOR THE ZDT TEST SUITE.

	$NSGA-II$	$SPEA2$	$CH-MOEA$
	$average$ (σ)	$average$ (σ)	$average$ (σ)
ZDT1	0.0506 (0.0472)	0.0464 (0.0568)	**0.0125** (0.0009)
ZDT2	0.0247 (0.0180)	0.0334 (0.0326)	**0.0095** (0.0056)
ZDT3	0.0362 (0.0232)	0.0565 (0.0606)	**0.0351** (0.0040)
ZDT4	**0.0140** (0.0189)	0.0524 (0.0856)	0.0617 (0.0286)
ZDT6	0.0640 (0.0470)	0.0421 (0.0190)	**0.0145** (0.0019)

TABLE V

RESULTS OF THE $\mathcal{SC}$ PERFORMANCE MEASURE FOR THE ZDT TEST SUITE.

	$\mathcal{SC}$(NSGA-II, CH-MOEA)	$\mathcal{SC}$(CH-MOEA, NSGA-II)	$\mathcal{SC}$(SPEA2, CH-MOEA)	$\mathcal{SC}$(CH-MOEA, SPEA2)
	$average$ (σ)	$average$ (σ)	$average$ (σ)	$average$ (σ)
ZDT1	0.0000 (0.0000)	**0.9652** (0.0196)	0.0000 (0.0000)	**0.9788** (0.0134)
ZDT2	0.0000 (0.0000)	**0.9608** (0.1379)	0.0000 (0.0000)	**0.9692** (0.1056)
ZDT3	0.0000 (0.0000)	**0.9424** (0.0302)	0.0000 (0.0000)	**0.9536** (0.0267)
ZDT4	0.0000 (0.0000)	**1.0000** (0.0000)	0.0000 (0.0000)	**1.0000** (0.0000)
ZDT6	0.0000 (0.0000)	**1.0000** (0.0000)	0.0000 (0.0000)	**1.0000** (0.0000)

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TABLE VI

RESULTS OF THE Δ PERFORMANCE MEASURE FOR THE DTLZ TEST SUITE.

	$NSGA-II$	$SPEA2$	$CH-MOEA$
	$average$ (σ)	$average$ (σ)	$average$ (σ)
DTLZ1	69443230.11 (504.5391)	69443606.37 (336.0847)	**69444215.57** (0.2412)
DTLZ2	9.66 (0.0252)	9.76 (0.0070)	**9.83** (0.0009)
DTLZ3	4420267982.96 (349613.6541)	4420376426.68 (279674.9673)	**4421232511.26** (168.2344)
DTLZ4	11.89 (0.0199)	11.93 (0.0514)	**12.05** (0.0009)

TABLE VII

RESULTS OF THE $\mathcal{S}$ PERFORMANCE MEASURE FOR THE DTLZ TEST SUITE.

	$NSGA-II$	$SPEA2$	$CH-MOEA$
	$average$ (σ)	$average$ (σ)	$average$ (σ)
DTLZ1	5.9779 (2.4594)	7.0523 (3.6706)	**0.0688** (0.0682)
DTLZ2	0.0368 (0.0066)	0.0328 (0.0074)	**0.0304** (0.0008)
DTLZ3	23.8986 (14.9786)	22.3405 (6.2647)	**0.4729** (0.2826)
DTLZ4	0.0293 (0.0161)	**0.0284** (0.0149)	0.0302 (0.0007)

TABLE VIII

RESULTS OF THE $\mathcal{SC}$ PERFORMANCE MEASURE FOR THE DTLZ TEST SUITE.

	$\mathcal{SC}$(NSGA-II, CH-MOEA)	$\mathcal{SC}$(CH-MOEA, NSGA-II)	$\mathcal{SC}$(SPEA2, CH-MOEA)	$\mathcal{SC}$(CH-MOEA, SPEA2)
	$average$ (σ)	$average$ (σ)	$average$ (σ)	$average$ (σ)
DTLZ1	0.0000 (0.0000)	**0.8772** (0.1245)	0.0000 (0.0000)	**0.8972** (0.1109)
DTLZ2	0.0001 (0.0007)	**0.6544** (0.0416)	0.0000 (0.0000)	**0.7520** (0.0147)
DTLZ3	0.0000 (0.0000)	**0.9931** (0.0184)	0.0000 (0.0000)	**0.9987** (0.0025)
DTLZ4	0.0000 (0.0000)	**0.8689** (0.0399)	0.0003 (0.0009)	**0.9040** (0.0264)

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