Approximating the ϵ-Efficient Set of an MOP with Stochastic Search Algorithms

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Abstract. In this paper we develop a framework for the approximation of the entire set of ϵ-efficient solutions of a multi-objective optimization problem with stochastic search algorithms. For this, we propose the set of interest, investigate its topology and state a convergence result for a generic stochastic search algorithm toward this set of interest. Finally, we present some numerical results indicating the practicability of the novel approach.

1 Introduction

Since the notion of ϵ-efficiency for multi-objective optimization problems (MOPs) has been introduced more than two decades ago ([6]), this concept has been studied and used by many researchers, e.g. to allow (or tolerate) nearly optimal solutions ([6], [13]), to approximate the set of optimal solutions ([9]), or in order to discretize this set ([5], [11]). ϵ-efficient solutions or approximate solutions have also been used to tackle a variety of real world problems including portfolio selection problems ([14]), a location problem ([1]), or a minimal cost flow problem ([9]). The explicit computation of such approximate solutions has been addressed in several studies (e.g., [13], [1], [2]), and in all of them scalarization techniques have been used.

The scope of this paper is to develop a framework for the approximation of the entire set of ϵ-efficient solutions (denote by E_ϵ) with stochastic search algorithms such as evolutionary multi-objective (EMO) algorithms. This calls for the design of a novel archiving strategy to store the 'required' solutions found by a stochastic search process (though the investigation of the set of interest will be the major part in this work). One interesting fact is that the solution set (the *Pareto set*) is included in E_ϵ for all (small) values of ϵ, and thus the resulting archiving strategy for EMO algorithms can be regarded as an alternative to existing methods for the approximation of this set (e.g, [3], [7], [4], [8]).

The remainder of this paper is organized as follows: in Section 2, we give the required background for the understanding of the sequel. In Section 3, we propose a set of interest, analyze its topology, and state a convergence result. We present numerical results on two examples in Section 4 and conclude in Section 5.

2 Background

In the following we consider continuous multi-objective optimization problems

$$\min_{x\in\mathbb{R}^n}\{F(x)\}, \qquad \text{(MOP)}$$

where the function F is defined as the vector of the objective functions $F:\mathbb{R}^n\to\mathbb{R}^k,\quad F(x)=(f_1(x),\ldots,f_k(x))$, and where each $f_i:\mathbb{R}^n\to\mathbb{R}$ is continuously differentiable. Later we will restrict the search to a compact set $Q\subset\mathbb{R}^n$, the reader may think of an n-dimensional box.

Definition 1. *(a) Let $v,w\in\mathbb{R}^k$. Then the vector v is* less than *w ($v<_p w$), if $v_i<w_i$ for all $i\in\{1,\ldots,k\}$. The relation $\leq_p$ is defined analogously.*
(b) $y\in\mathbb{R}^n$ is dominated *by a point $x\in\mathbb{R}^n$ ($x\prec y$) with respect to (MOP) if $F(x)\leq_p F(y)$ and $F(x)\neq F(y)$, else y is called nondominated by x.*
(c) $x\in\mathbb{R}^n$ is called a Pareto point *if there is no $y\in\mathbb{R}^n$ which dominates x.*
(d) $x\in\mathbb{R}^n$ is weakly Pareto optimal *if there does not exist another point $y\in\mathbb{R}^n$ such that $F(y)<_p F(x)$.*

We now define a weaker concept of dominance, called ϵ-dominance, which is the basis of the approximation concept used in this study.

Definition 2. *Let $\epsilon=(\epsilon_1,\ldots,\epsilon_k)\in\mathbb{R}^k_+$ and $x,y\in\mathbb{R}^n$. x is said to ϵ-dominate y ($x\prec_\epsilon y$) with respect to (MOP) if $F(x)-\epsilon\leq_p F(y)$ and $F(x)-\epsilon\neq F(y)$.*

Theorem 1 ([10]). *Let (MOP) be given and $q:\mathbb{R}^n\to\mathbb{R}^n$ be defined by $q(x)=\sum_{i=1}^k\hat\alpha_i\nabla f_i(x)$, where $\hat\alpha$ is a solution of*

$$\min_{\alpha\in\mathbb{R}^k}\left\{\|\sum_{i=1}^k\alpha_i\nabla f_i(x)\|_2^2;\alpha_i\geq 0,i=1,\ldots,k,\sum_{i=1}^k\alpha_i=1\right\}.$$

Then either $q(x)=0$ or $-q(x)$ is a descent direction for all objective functions $f_1,\ldots,f_k$ in x. Hence, each x with $q(x)=0$ fulfills the first-order necessary condition for Pareto optimality.

In case $q(x)\neq 0$ it obviously follows that $q(x)$ is an ascent direction for all objectives. Next, we need the following distances between different sets.

Definition 3. *Let $u\in\mathbb{R}^n$ and $A,B\subset\mathbb{R}^n$. The semi-distance $dist(\cdot,\cdot)$ and the* Hausdorff distance *$d_H(\cdot,\cdot)$ are defined as follows:*

(a) $dist(u,A):=\inf_{v\in A}\|u-v\|$
(b) $dist(B,A):=\sup_{u\in B}\ dist(u,A)$
(c) $d_H(A,B):=\max\{dist(A,B),dist(B,A)\}$

Denote by $\overline{A}$ the closure of a set $A\in\mathbb{R}^n$, by $\mathring{A}$ its interior, and by $\partial A=\overline{A}\backslash\mathring{A}$ the boundary of A.

Algorithm 1 gives a framework of a generic stochastic multi-objective optimization algorithm, which will be considered in this work. Here, $Q\subset\mathbb{R}^n$ denotes the domain of the MOP, P_j the candidate set (or population) of the generation process at iteration step j, and A_j the corresponding archive.

Algorithm 1 Generic Stochastic Search Algorithm

1: $P_0 \subset Q$ drawn at random
2: $A_0 = ArchiveUpdate(P_0, \emptyset)$
3: **for** $j = 0, 1, 2, \ldots$ **do**
4: $\quad P_{j+1} = Generate(P_j)$
5: $\quad A_{j+1} = ArchiveUpdate(P_{j+1}, A_j)$
6: **end for**

3 The Archiving Strategy

In this section we define the set of interest, investigate the topology of this object, and finally state a convergence result.

Definition 4. *Let $\epsilon \in \mathbb{R}^k_+$ and $x, y \in \mathbb{R}^n$. x is said to $-\epsilon$-dominate y ($x \prec_{-\epsilon} y$) with respect to (MOP) if $F(x) + \epsilon \leq_p F(y)$ and $F(x) + \epsilon \neq F(y)$.*

This definition is of course analogous to the 'classical' ϵ-dominance relation but with a value $\tilde{\epsilon} \in \mathbb{R}^k_-$. However, we highlight it here since it will be used frequently in this work. While the ϵ-dominance is a weaker concept of dominance, $-\epsilon$-dominance is a stronger one.

Definition 5. *A point $x \in Q$ is called $-\epsilon$* weak Pareto point *if there exists no point $y \in Q$ such that $F(y) + \epsilon <_p F(x)$.*

Now we are able to define the set of interest. Ideally, we would like to obtain the 'classical' set

$$P^c_{Q,\epsilon} := \{x \in Q | \exists p \in P_Q : x \prec_\epsilon p\}^3, \tag{1}$$

where P_Q denotes the Pareto set (i.e., the set of Pareto optimal solutions) of $F\big|_Q$. That is, every point $x \in P^c_{Q,\epsilon}$ is 'close' to at least one efficient solution, measured in objective space. However, since this set is not easy to catch – note that the efficient solutions are used in the definition –, we will consider an enlarged set of interest (see Lemma 2):

Definition 6. *Denote by $P_{Q,\epsilon}$ the set of points in $Q \subset \mathbb{R}^n$ which are not $-\epsilon$-dominated by any other point in Q, i.e.*

$$P_{Q,\epsilon} := \{x \in Q | \nexists y \in Q : y \prec_{-\epsilon} x\}^4 \tag{2}$$

Example 1. (a) Figure 1 shows two examples for sets $P_{Q,\epsilon}$, one for the single-objective case (left), and one for $k = 2$ (right). In the first case we have $P_{Q,\epsilon} = [a, b] \cup [c, d]$.

[3] $P^c_{Q,\epsilon}$ is closely related to set E^1 considered in [13].
[4] $P_{Q,\epsilon}$ is closely related to set E^5 considered in [13].

(b) Consider the MOP $F : \mathbb{R} \to \mathbb{R}^2, \quad F(x) = ((x-1)^2, (x+1)^2)$. For $\epsilon = (1,1)$ and Q sufficiently large, say $Q = [-3,3]$, we obtain $P_Q = [-1,1]$ and $P_{Q,\epsilon} = (-2,2)$. Note that the boundary of $P_{Q,\epsilon}$, i.e. $\partial P_{Q,\epsilon} = \overline{P_{Q,\epsilon}} \backslash \mathring{P}_{Q,\epsilon} = [-2,2] \backslash (2,2) = \{-2,2\}$, is given by $-\epsilon$ weak Pareto points which are not included in $P_{Q,\epsilon}$ (see also Lemma 1): for $x_1 = -2$ and $x_2 = 2$ it is $F(x_1) = (9,1)$ and $F(x_2) = (1,9)$. Since there exists no $x \in Q$ with $f_i(x) < 0, i = 1,2$, there is also no point $x \in Q$ where all objectives are less than at x_1 or x_2. Further, since $F(-1) = (4,0)$ and $F(1) = (0,4)$ there exist points which $-\epsilon$-dominate these points, and they are thus not included in $P_{Q,\epsilon}$.

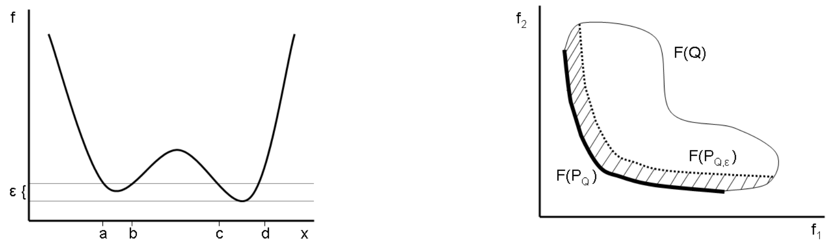


Fig. 1. Two different examples for sets $P_{Q,\epsilon}$. Left for $k = 1$ and in parameter space, and right an example for $k = 2$ in image space.

Lemma 1. *(a) Let $Q \subset \mathbb{R}^n$ be compact. Under the following assumptions*

(A1) Let there be no weak Pareto point in $Q \backslash P_Q$, where P_Q denotes the set of Pareto points of $F|_Q$.

(A2) Let there be no $-\epsilon$ weak Pareto point in $Q \backslash \overline{P_{Q,\epsilon}}$,

(A3) Define $\mathcal{B} := \{x \in Q | \exists y \in P_Q : F(y) + \epsilon = F(x)\}$. Let $\mathcal{B} \subset \mathring{Q}$ and $q(x) \neq 0$ for all $x \in \mathcal{B}$, where q is as defined in Theorem 1,

it holds:

$$
\begin{aligned}
\overline{P_{Q,\epsilon}} &= \{x \in Q |\ \nexists y \in Q : F(y) + \epsilon <_p F(x)\} \\
\mathring{P}_{Q,\epsilon} &= \{x \in Q |\ \nexists y \in Q : F(y) + \epsilon \leq_p F(x)\} \\
\partial P_{Q,\epsilon} &= \{x \in Q | \exists y_1 \in P_Q : F(y_1) + \epsilon \leq_p F(x) \wedge \nexists y_2 \in Q : F(y_2) + \epsilon <_p F(x)\}
\end{aligned} \tag{3}
$$

(b) Let in addition to the assumptions made above be $q(x) \neq 0\ \forall x \in \partial P_{Q,\epsilon}$. Then

$$\overline{\mathring{P}_{Q,\epsilon}} = \overline{P_{Q,\epsilon}} \tag{4}$$

Proof. Define $W := \{x \in Q |\ \nexists y \in Q : F(y) + \epsilon <_p F(x)\}$. We show the equality $\overline{P_{Q,\epsilon}} = W$ by mutual inclusion. $W \subset \overline{P_{Q,\epsilon}}$ follows directly by assumption (A2).

To see the other inclusion assume that there exists an $x \in \overline{P_{Q,\epsilon}} \backslash W$. Since $x \notin W$ there exists an $y \in Q$ such that $F(y) + \epsilon <_p F(x)$. Further, since F is continuous there exists further a neighborhood U of x such that $F(y) + \epsilon <_p F(u), \quad \forall u \in U$. Thus, y is $-\epsilon$-dominating all $u \in U$ (i.e., $U \cap P_{Q,\epsilon} = \emptyset$), a contradiction to the assumption that $x \in \overline{P_{Q,\epsilon}}$. Thus, we have $\overline{P_{Q,\epsilon}} = W$ as claimed.
Next we show that the interior of $P_{Q,\epsilon}$ is given by

$$I := \{x \in Q | \ \nexists y \in Q : F(y) + \epsilon \leq_p F(x)\}, \tag{5}$$

which we do again by mutual inclusion. To see that $\overset{\circ}{P}_{Q,\epsilon} \subset I$ assume that there exists an $x \in \overset{\circ}{P}_{Q,\epsilon} \backslash I$. Since $x \notin I$ we have

$$\exists y_1 \in Q : \quad F(y_1) + \epsilon \leq_p F(x). \tag{6}$$

Since $x \in \overset{\circ}{P}_{Q,\epsilon}$ there exists no $y \in Q$ which $-\epsilon$-dominates x, and hence, equality holds in equation (6). Further, by assumption (A1) it follows that y_1 must be in P_Q. Thus, we can reformulate (6) by

$$\exists y_1 \in P_Q : \quad F(y_1) + \epsilon = F(x) \tag{7}$$

Since $x \in \overset{\circ}{P}_{Q,\epsilon}$ there exists a neighborhood $\tilde{U}$ of x such that $\tilde{U} \subset \overset{\circ}{P}_{Q,\epsilon}$. Further, since $q(x) \neq 0$ by assumption (A1), there exists a point $\tilde{x} \in \tilde{U}$ such that $F(\tilde{x}) >_p F(x)$. Combining this and (7) we obtain

$$F(y_1) + \epsilon = F(x) <_p F(\tilde{x}), \tag{8}$$

and thus $y_1 \prec_{-\epsilon} \tilde{x} \in \tilde{U} \subset \overset{\circ}{P}_{Q,\epsilon}$, which is a contradiction. It remains to show that $I \subset \overset{\circ}{P}_{Q,\epsilon}$: assume there exists an $x \in I \backslash \overset{\circ}{P}_{Q,\epsilon}$. Since $x \notin \overset{\circ}{P}_{Q,\epsilon}$ there exists a sequence $x_i \in Q \backslash P_{Q,\epsilon}, i \in \mathbb{N}$, such that $\lim_{i \to \infty} x_i = x$. That is, there exists a sequence $y_i \in Q$ such that $y_i \prec_{-\epsilon} x_i$ for all $i \in \mathbb{N}$. Since all the y_i are inside Q, which is a bounded set, there exists a subsequence $y_{i_j}, j \in \mathbb{N}$, and an $y \in Q$ such that $\lim_{j \to \infty} y_{i_j} = y$ (Bolzano-Weierstrass). Since $F(y_{i_j}) + \epsilon \leq_p F(x_{i_j}), \quad \forall j \in \mathbb{N}$, it follows for the limit points that also $F(y) + \epsilon \leq_p F(x)$, which is a contradiction to $x \in I$. Thus, we have $\overset{\circ}{P}_{Q,\epsilon} = I as desired.$

For the boundary we obtain

$$\begin{aligned} \partial P_{Q,\epsilon} &= \overline{P_{Q,\epsilon}} \backslash \overset{\circ}{P}_{Q,\epsilon} \\ &= \{x \in Q | \exists y_1 \in Q : F(y_1) + \epsilon \leq_p F(x) \text{ and } \nexists y_2 \in Q : F(y_2) + \epsilon <_p F(x)\} \end{aligned} \tag{9}$$

Since by (A1) the point y_1 in (9) must be in P_Q, we obtain

$$\partial P_{Q,\epsilon} = \{x \in Q | \exists y_1 \in P_Q : F(y_1) + \epsilon \leq_p F(x) \text{ and } \nexists y_2 \in Q : F(y_2) + \epsilon <_p F(x)\} \tag{10}$$

It remains to show the second claim. It is $\overline{P_{Q,\epsilon}} = \overset{\circ}{P}_{Q,\epsilon} \cup \partial P_{Q,\epsilon}$. Assume that $\overline{\overset{\circ}{P}_{Q,\epsilon}} \neq \overline{P_{Q,\epsilon}}$, i.e., that there exists an $x \in \partial P_{Q,\epsilon}$ and a neighborhood U of x such that $U \cap \overset{\circ}{P}_{Q,\epsilon} = \emptyset$. Since $x \in \partial P_{Q,\epsilon}$ there exists a point $y \in P_Q$ such that $F(y) + \epsilon \leq_p F(x)$. By assumption it is $q(x) \neq 0$, and thus there exists an $\bar{x} \in U$ such that $F(\bar{x}) <_p F(x)$. Since $\bar{x} \notin \overset{\circ}{P}_{Q,\epsilon}$ there exists an $\bar{y} \in Q$ such that $F(\bar{y}) + \epsilon \leq_p F(\bar{x}) <_p F(x)$, which contradicts the assumption that $x \in \partial P_{Q,\epsilon}$. Thus, we have that the closure of the interior of $P_{Q,\epsilon}$ is equal to its closure as claimed.

Remark 1. (a) Note that in general $P_{Q,\epsilon}$ is neither an open nor a closed set, and that $\overline{P_{Q,\epsilon}}$ gets 'completed' by $-\epsilon$ weak Pareto points (see also Example 1).

(b) Since for x and y_1 in equation (10) it must hold that there exists an index $j \in \{1, \ldots, k\}$ such that $f_j(y_1) + \epsilon_j = f_j(x)$. Thus, the boundary of $P_{Q,\epsilon}$ can be characterized by the set of $-\epsilon$ weak Pareto points which are bounded in objective space from P_Q by ϵ.

The next example shows that the closure of the interior of $P_{Q,\epsilon}$ does in general not have to be equal to its closure, which causes trouble to approximate $\partial P_{Q,\epsilon}$ using stochastic search algorithms. However, the following Lemma shows that this is – despite for theoretical investigations – not problematic since $\overset{\circ}{P}_{Q,\epsilon}$, which can be approximated in any case, already contains all the interesting parts.

Example 2. Figure 2 shows an example which is a modification of the MOP in Example 1 (a). We have $P_{Q,\epsilon} = \{x^*\} \cup [c, d]$ and hence $\overline{\overset{\circ}{P}_{Q,\epsilon}} = [c, d] \neq \overline{P_{Q,\epsilon}}$. Note that here we have $f'(x^*) = 0$, and thus that (A3) is violated. The problem with the approximation of the entire set $P_{Q,\epsilon}$ in this case is the following: assume that $\text{argmin} f$ is already a member of the archive, then every candidate solution near x^* will be rejected by all further archives. Thus, the entire set $P_{Q,\epsilon}$ can only be approximated if x^* is a member of a population $P_i, i \in \mathbb{N}$, and the probability for this event is zero. Such problems do not occur for points in $\overset{\circ}{P}_{Q,\epsilon}$ (see proof of Theorem 2).

Lemma 2. $P^c_{Q,\epsilon} \subset \overset{\circ}{P}_{Q,\epsilon}$

Proof. Assume there exists an $x \in P^c_{Q,\epsilon} \backslash \overset{\circ}{P}_{Q,\epsilon}$. Since $x \in P^c_{Q,\epsilon}$ there exists a Pareto optimal point $p \in P_Q$ with $p \prec_\epsilon p$. Further, since $x \notin \overset{\circ}{P}_{Q,\epsilon}$ there exists an $y \in Q$ such that $F(y) + \epsilon \leq_p F(x)$. Combining both we obtain

$$\begin{aligned} F(y) \leq_p F(x) - \epsilon \leq F(p), \quad \text{and} \\ \exists j \in \{1, \ldots, k\} : f_j(y) \leq f_j(x) - \epsilon < f_j(p) \quad (\Rightarrow\ F(y) \neq F(p)), \end{aligned} \tag{11}$$

which means that $y \prec p$, a contradiction to $p \in P_Q$, and we are done.

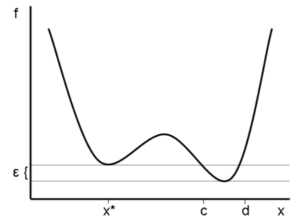


Fig. 2. Example of a set $P_{Q,\epsilon}$ where the closure of its interior is not equal to its closure.

Having analyzed the topology of $P_{Q,\epsilon}$ we are now in the position to state the following result. The archiving strategy is simply the one which keeps all obtained points which are not $-\epsilon$-dominated by any other test point.

Theorem 2. *Let an MOP $F : \mathbb{R}^n \to \mathbb{R}^k$ be given, where F is continuous, let $Q \subset \mathbb{R}^n$ be a compact set and $\epsilon \in \mathbb{R}^k_+$. Further let*

$$\forall x \in Q \text{ and } \forall \delta > 0 : \quad P\,(\exists l \in \mathbb{N} \ : \ P_l \cap B_\delta(x) \cap Q \neq \emptyset) = 1 \tag{12}$$

Then, under the assumptions made in Lemma 1, an application of Algorithm 1, where

$$ArchiveUpdateP_{Q,\epsilon}(P, A) := \{x \in P \cup A : y \not\prec_{-\epsilon} x \ \forall y \in P \cup A\}, \tag{13}$$

is used to update the archive, leads to a sequence of archives $A_l, l \in \mathbb{N}$, with

$$\lim_{l \to \infty} d_H(P_{Q,\epsilon}, A_l) = 0, \quad \text{with probability one.} \tag{14}$$

Proof. Since $dist(A, B) = dist(\overline{A}, B)$ for all sets $A, B \subset \mathbb{R}^n$ and since $\overline{\overset{\circ}{P}_{Q,\epsilon}} = \overline{P_{Q,\epsilon}}$ (see Lemma 1), it is sufficient to show that the Hausdorff distance between A_l and $\overset{\circ}{P}_{Q,\epsilon}$ vanishes in the limit with probability one.
First we show that $dist(A_l, \overset{\circ}{P}_{Q,\epsilon}) \to 0$ with probability one for $l \to \infty$. It is

$$dist(A_l, \overset{\circ}{P}_{Q,\epsilon}) = \max_{a \in A_l} \inf_{p \in \overset{\circ}{P}_{Q,\epsilon}} \|a - p\|.$$

We have to show that every $x \in Q \backslash \overline{P_{Q,\epsilon}}$ will be discarded (if added before) from the archive after finitely many steps, and that this point will never been added further on.
Let $x \in Q \backslash \overline{P_{Q,\epsilon}}$. Since x is by assumption (A2) not a $-\epsilon$-weak Pareto point, there exists a point $p \in P_{Q,\epsilon}$ such that $F(p) + \epsilon <_p F(x)$. Further, since F is continuous there exists a neighborhood U of x such that

$$F(p) + \epsilon <_p F(u), \quad \forall u \in U. \tag{15}$$

By (12) it follows that there exists with probability one a number $l_0 \in \mathbb{N}$ such that there exists a point $x_{l_0} \in P_{l_0} \cap U \cap Q$. Thus, by construction of $ArchiveUpdateP_{Q,\epsilon}$, the point x will be discarded from the archive if it is a member of A_{l_0}, and never be added to the archive further on.

Now we consider the limit behavior of $dist(\overset{\circ}{P}_{P,\epsilon}, A_l)$. It is

$$dist(\overset{\circ}{P}_{Q,\epsilon}, A_l) = \sup_{p \in \overset{\circ}{P}_{Q,\epsilon}} \min_{a \in A_l} \|p - a\|.$$

Let $\bar{p} \in \overset{\circ}{P}_{Q,\epsilon}$. For $i \in \mathbb{N}$ there exists by (12) a number l_i and a point $p_i \in P_{l_i} \cap B_{1/i}(\bar{p}) \cap Q$, where $B_\delta(p)$ denotes the open ball with center p and radius $\delta \in \mathbb{R}_+$. Since $\lim_{i\to\infty} p_i = \bar{p}$ and since $\bar{p} \in \overset{\circ}{P}_{Q,\epsilon}$ there exists an $i_0 \in \mathbb{N}$ such that $p_i \in \overset{\circ}{P}_{Q,\epsilon}$ for all $i \geq i_0$. By construction of $ArchiveUpdateP_{Q,\epsilon}$, all the points $p_i, i \geq i_0$, will be added to the archive (and never discarded further on). Thus, we have $dist(\bar{p}, A_l) \to 0$ for $l \to \infty$ as desired, which completes the proof.

Remark 2. In order to obtain a 'complete' convergence result we have postulated some (mild) assumptions in order to guarantee that $\overline{\overset{\circ}{P}_{Q,\epsilon}} = \overline{P_{Q,\epsilon}}$, which is in fact an important topological property needed for the proof. However, if we drop the assumptions we can still expect that the interior of $P_{Q,\epsilon}$ – the 'interesting' part (see Lemma 2) – will be approximated in the limit. To be more precise, regardless of assumptions (A1)–(A3) it holds in the above theorem that

$$\lim_{l\to\infty} dist(\overline{\overset{\circ}{P}_{Q,\epsilon}}, A_l) = 0, \quad \text{with probability one.}$$

4 Numerical Results

Here we demonstrate the practicability of the novel archiver on two examples. For this, we compare $ArchiveUpdateP_{Q,\epsilon}$ against the 'classical' archiving strategy which stores all nondominated solutions obtained during the search procedure ($ArchiveUpdateND$). To obtain a fair comparison of the two archivers we have decided to take a random search operator for the generation process (the same sequence of points for all settings).

4.1 Example 1

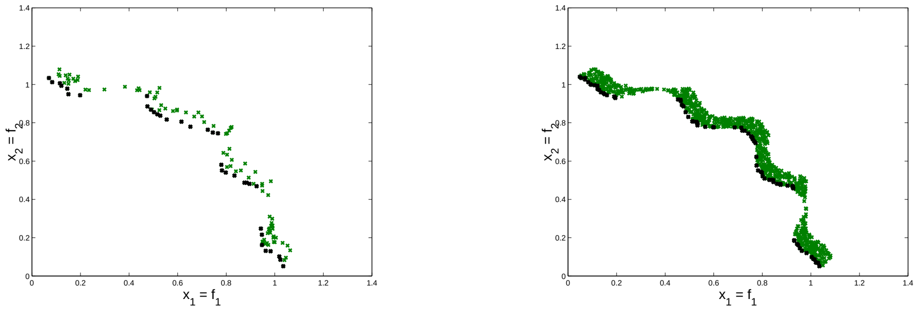
First we consider the MOP suggested by Tanaka ([12]):

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F(x_1, x_2) = (x_1, x_2) \tag{16}$$

where

$$\begin{aligned} C_1(x) &= x_1^2 + x_2^2 - 1 - 0.1\cos(16\arctan(x_1/x_2)) \geq 0 \\ C_2(x) &= (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 0.5 \end{aligned}$$

Figure 3 shows two comparisons for $N = 10,000$ and $N = 100,000$ points within $Q = [0, \pi]^2$ as domain[5], indicating that the method is capable of finding all approximate solutions.



(a) $N = 10,000$ (b) $N = 100,000$

Fig. 3. Numerical result for MOP (16) using $\epsilon = (0.1, 0.1)$.

4.2 Example 2

Finally, we consider the following MOP proposed in [8]:

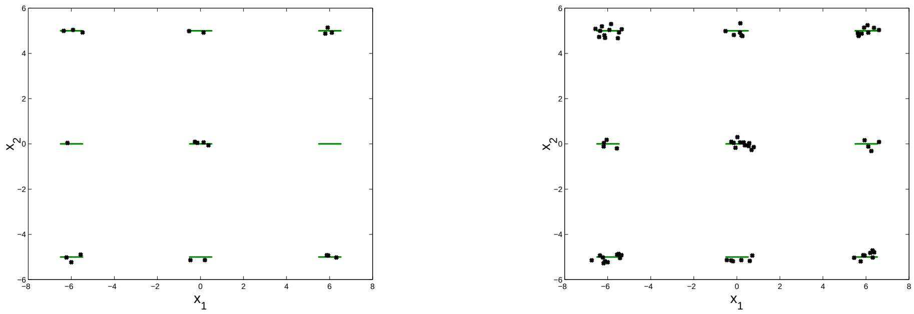
$$\begin{aligned} F &: \mathbb{R}^2 \to \mathbb{R}^2 \\ F(x_1, x_2) &= \begin{pmatrix} (x_1 - t_1(c + 2a) + a)^2 + (x_2 - t_2 b)^2 \\ (x_1 - t_1(c + 2a) - a)^2 + (x_2 - t_2 b)^2 \end{pmatrix}, \end{aligned} \tag{17}$$

where

$$t_1 = \operatorname{sgn}(x_1) \min \left(\left\lceil \frac{|x_1| - a - c/2}{2a + c} \right\rceil, 1 \right), t_2 = \operatorname{sgn}(x_2) \min \left(\left\lceil \frac{|x_2| - b/2}{b} \right\rceil, 1 \right).$$

The Pareto set consists of nine connected components of length a with identical images. We have chosen the values $a = 0.5$, $b = c = 5$, $\epsilon = (0.1, 0.1)$, the domain $Q = [-20, 20]^2$, and $N = 10,000$ randomly chosen points within Q. Figures 4 and 5 display two typical results in parameter space and image space respectively. Seemingly, the approximation quality of the Pareto set obtained by the limit set of $ArchiveUpdateP_{Q,\epsilon}$ is better than by the one obtained by $ArchiveUpdateND$, measured in the Hausdorff sense. This example should indicate that it can be advantageous to store more than just non-dominated points in the archive, even when 'only' aiming for the efficient set.

[5] To fit into our framework, we consider in fact the (compact) domain $Q' := [0, \pi]^2 \cap \{x \in \mathbb{R}^n : C_1(x) \geq 0 \text{ and } C_2(x) \leq 0.5\}$.



(a) $ArchiveUpdateND$ (b) $ArchiveUpdateP_{Q,\epsilon}$

Fig. 4. Numerical result for MOP (16) in parameter space.

5 Conclusion and Future Work

We have proposed and investigated a novel archiving strategy for stochastic search algorithms which allows – under mild assumptions on the generation process – to approximate the set $P_{Q,\epsilon}$ which contains all ϵ-efficient solutions within a compact domain Q. We have proven the convergence of the algorithm toward this set in the probabilistic sense, and have given two examples indicating the usefulness of the approach.
Since the set of approximate solutions forms an n-dimensional object, a suitable finite size representation of $P_{Q,\epsilon}$ and the related archiving strategy are of major interest for further investigations.

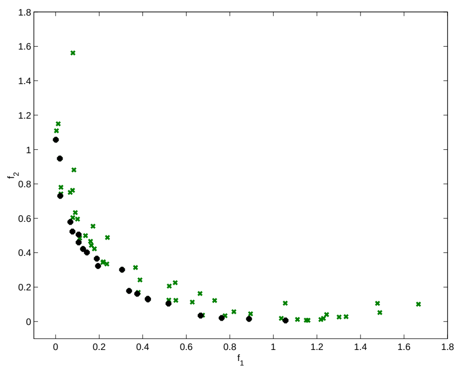


Fig. 5. Comparison of the result of both archivers in objective space.

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