Computing and Selecting ϵ-Efficient Solutions of {0,1}-Knapsack Problems

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Summary. This work deals with the computation and the selection of approximate – or ϵ-efficient – solutions of $\{0, 1\}$-knapsack problems. By allowing approximate solutions in general a much larger variety of possibilities for the underlying problem is offered to the decision maker. We enlighten the gap that can occur when passing ϵ-approximate solutions from the objective space into the parameter space (in terms of neighborhood). In this paper, we propose a novel adaptive ϵ-approximation based stochastic algorithm for the computation of the entire set of ϵ-efficient solutions, state a convergence result, and address the related decision making problem. For the latter we propose an interactive selection process which is intended to help the decision maker to understand the landscape of the obtained solutions.

Key words: {0,1}-knapsack problems, epsilon-adaptive method, approximate solutions, interactive selection procedure.

1 Introduction

In a multi-objective optimization problem (MOP) several objectives have to be optimized concurrently. Based on the standard dominance relation for optimality, the set of optimal solutions (the *Pareto set*) typically forms a $(k-1)$-dimensional object, where k denotes the number of objectives involved in the MOP. Though the trustworthy approximation of this set is already a challenging task in practice, it can make sense in certain situations to consider even a superset of the Pareto set. Using a weaker concept of optimality, nearly optimal solutions or approximate solutions can be defined. This can e.g. be done via the use of ϵ-dominance (Loridan, 1984), where the value of ϵ determines the quality of the approximation. The main advantage of allowing approximate solutions is that by this, in general, a larger flexibility can be offered to the decision maker (DM) whose task is to select an 'adequate' solution according to the given problem. In this work we aim for the numerical treatment of {0,1}-knapsack problems which have a wide range of real-world applications, e.g. capital budgeting (Rosenblatt and Sinunany-Stern, 1989), relocation problems (Kostreva *et al.*, 1999), or planning remediation (Jenkins, 2002). Moreover in all of them the value of ϵ has a physical meaning, and thus, the potential loss compared to possible exact solutions is computable.

The explicit computation of approximate solutions has been addressed in several studies, most of them employing scalarization methods, e.g. (Blanquero and Carrizosa, 2002; Engau and Wiecek, 2007; White, 1986), or aiming for robust approximations of the ϵ-efficient front (Deb *et al.*, 2005; Laumanns *et al.*, 2004; Schütze *et al.*, 2008, 2009), without providing several preimages for the same objective functions range. Recently, archiving strategies have been proposed (Schütze *et al.*, 2008; Schütze *et al.*, 2007) to maintain the entire set of ϵ-efficient solutions (denote by E_ϵ) in the limit using stochastic search algorithms. On the basis of this work we propose a novel population based search procedure which is designed to compute the approximate solutions of the {0,1}-knapsack problems. The novelty of the approach consists - besides the approximation of the entire set of ϵ-efficient solutions - of the proposed mechanism used for adapting the values of ϵ during the search as to ensure convergence towards the desired level of accuracy, in the limit and in a probabilistic sense. Furthermore, we propose an interactive procedure which should help the DM to explore the landscape of E_ϵ, and which should thus ease his or her task to find the 'right' solution according to the current situation.

The remainder of this paper is organized as follows: in Section 2, we give the required background for the understanding of the sequel. In Section 3 we state the problem and motivate why we have chosen to tackle it with stochastic search algorithms. In Section 4 we propose such an algorithms and give some numerical results. In Section 5 an interactive selection procedure is proposed, and finally we conclude in Section 6.

2 Background

In the following we consider multi-objective optimization problems

$$\min_{x\in Q}\{F(x)\}, \qquad \text{(MOP)}$$

where the function F is defined as the vector of the objective functions $F : Q \to \mathbb{R}^k, \quad F(x) = (f_1(x), \ldots, f_k(x))$, and where $Q \subset \mathbb{R}^n$ is finite.

Definition 1. *(a) Let $v, w \in \mathbb{R}^k$. Then the vector v is* less than *w ($v <_p w$), if $v_i < w_i$ for all $i \in \{1, \ldots, k\}$. The relation $\leq_p$ is defined analogously.*
(b) $y \in \mathbb{R}^n$ is dominated *by a point $x \in \mathbb{R}^n$ ($x \prec y$) with respect to (MOP) if $F(x) \leq_p F(y)$ and $F(x) \neq F(y)$, else y is called nondominated by x.*
(c) $x \in \mathbb{R}^n$ is called a Pareto point *if there is no $y \in \mathbb{R}^n$ which dominates x. Denote by P_Q the set of Pareto points of a given MOP.*

Definition 2. *Let $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \mathbb{R}^k_+$ and $x, y \in \mathbb{R}^n$.*

(a) x is said to ϵ-dominate y ($x \prec_\epsilon y$) with respect to (MOP) if $F(x) - \epsilon \leq_p F(y)$ and $F(x) - \epsilon \neq F(y)$.
(b) x is said to $-\epsilon$-dominate y ($x \prec_{-\epsilon} y$) with respect to (MOP) if $F(x) + \epsilon \leq_p F(y)$ and $F(x) + \epsilon \neq F(y)$.

The definition in (b) is of course analogous to the 'classical' ϵ-dominance relation in (a) but with a value $\tilde{\epsilon} \in \mathbb{R}^k_-$. However, we highlight it here since it will be used frequently in this work. While the ϵ-dominance is a weaker concept of dominance, $-\epsilon$-dominance is a stronger one. We now define the set of interest

Definition 3. *(Schütze* et al.*, 2007) Denote by $P_{Q,\epsilon}$ the set of points in $Q \subset \mathbb{R}^n$ which are not $-\epsilon$-dominated by any other point in Q, i.e.*

$$P_{Q,\epsilon} := \{x \in Q | \nexists y \in Q : y \prec_{-\epsilon} x\} \tag{1}$$

Algorithm 1 gives a framework of a generic stochastic multi-objective optimization algorithm, which will be considered in this work. Here, $Q \subset \mathbb{R}^n$ denotes the domain of the MOP, P_j the candidate set (or population) of the generation process at iteration step j, and A_j the corresponding archive.

Algorithm 1 Generic Stochastic Search Algorithm
1: $P_0 \subset Q$ drawn at random
2: $A_0 = ArchiveUpdate(P_0, \emptyset)$
3: **for** $j = 0, 1, 2, \ldots$ **do**
4: $\quad P_{j+1} = Generate(P_j)$
5: $\quad A_{j+1} = ArchiveUpdate(P_{j+1}, A_j)$
6: **end for**

3 The Problem

In this section we present the class of MOPs – bi-objective {0,1}-knapsack problems – which is being considered, and make further on some discussions on it:

$$f_1, f_2 : \{0,1\}^n \to \mathbb{R}, \ f_1(x) = \sum_{j=1}^{n} c_j^1 x_j, \ f_2(x) = \sum_{j=1}^{n} c_j^2 x_j \tag{2}$$

s.t.

$$\sum_{j=1}^{n} w_j x_j \leq W, \ x_j \in \{0,1\}, \ j = 1, \ldots, n,$$

where c_j^i represents the value of item j on criterion $i, i = 1, 2$; $x_j = 1$, $j = 1, \ldots, n$, if item j is included in the knapsack, else $x_j = 0$. w_j is the weight of item j, and W the overall knapsack capacity.

Here we are particularly interested in instances where the items have 'similar' values – i.e., where some c_j^i's (not necessarily all) are within a relatively small range

– since in that case the set of ϵ-efficient solutions can become large, even for small values of ϵ.

Table 1 shows some results for $n = 500$ items, and where the values c^i_j are chosen within the interval $[10 - d, 10 + d]$, $d = 1, 2, 3$. We can observe that the magnitudes of $\tilde{P_Q}$ – the number of nondominated solutions found by the search procedure– is nearly independent from the choice of the interval. This does not hold for the magnitudes of $\tilde{P_{Q,\epsilon}}$, i.e., the set of points which are not $-\epsilon$ dominated by any other test point. We see that $|\tilde{P_{Q,\epsilon}}|$ gets larger the closer the values of the items are, and in all cases we have $|\tilde{P_{Q,\epsilon}}| > |\tilde{P_Q}|$. However, in case the values of the items vary a lot, it can happen that $P_Q = P_{Q,\epsilon}$, even for large values of ϵ (see e.g. (Laumanns *et al.*, 2004) or (Tantar, 2009)).

	$\lvert\tilde{P_Q}\rvert$	$\lvert\tilde{P_{Q,\epsilon}}\rvert$
$c^i_j \in [9, 11]$	8.7	144.93
$c^i_j \in [8, 12]$	8.87	42.8
$c^i_j \in [7, 13]$	9.07	26.93

Table 1. Some numerical results for MOP (2) with $n = 500$, averaged over 30 test runs. We have taken the algorithm described in Section 4 using a population of 100 individuals ,with a number of 10,000 generations. $\tilde{P_Q}$ denotes the set of nondominated solutions and $\tilde{P_{Q,\epsilon}}$ the set of points which are not $-\epsilon$ dominated by any other test point generated by the algorithm, for $\epsilon = (5, 5)$.

The next example shows that $P_{Q,\epsilon}$ can be highly disconnected, which motivates to tackle such problems with stochastic search algorithms since 'classical' exact methods designed to locate P_Q and which utilize the locality of such MOPs, can probably not easily be tuned in order to solve the problem adequately (however, the authors do not foreclose that such algorithms will not exist in future).

Example 1. For this example, we consider $n = 6$, $w = 1$, $W = 3$, and the costs

$$\begin{aligned} c^1 &= (95, 120, 80, 98, 105, 87) \\ c^2 &= (107, 75, 115, 97, 90, 108) \end{aligned} \tag{3}$$

Here, P_Q consists of 18 points- two pairs of solutions having the same values in the image space - including $x_1 = (1, 1, 1, 0, 0, 0)$ with $F(x_1) = (295, 297)$ (see Figure 1). When choosing $\epsilon = (5, 5)$ – the value of $\epsilon_i = 5$ relates to approximately 5 percent of the average weight of *one* item – we see that $x_2 = (0, 0, 0, 1, 1, 1)$ with $F(x_2) = (290, 295)$ is an ϵ-efficient solution since it is ϵ-dominating x_1 (and only this point). The Hamming distance is 6, thus the maximal possible value. There exists also $x_4 = (0, 1, 1, 1, 0, 0)$ which is an ϵ-efficient solution since it is ϵ-dominating $x_3 = (1, 0, 0, 1, 1, 0)$, with $F(x_3) = (298, 294)$, $x_3 \in P_Q$. The Hamming distance between x_3 and x_4 is 4.

Further instances with larger distances of approximate solutions to P_Q can be constructed, see also the example in Section 5 or (Tantar, 2009).

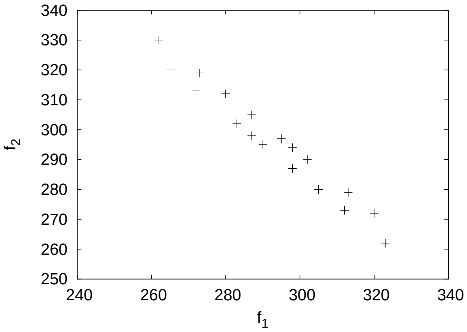

Fig. 1. The set of feasible solutions for the considered example. It can be observed that $P_{Q,\epsilon}$ contains one solution with $d_H = 6$ and several solutions with $d_H = 4$.

4 A Stochastic Search Algorithm

4.1 The Algorithm

The algorithm is a population based evolutionary technique designed for providing $P_{Q,\epsilon}$ approximation sets. It is intended as the first component of an exploratory process which offers to the users information about $P_{Q,\epsilon}$ approximation sets, required in studying the solutions landscape. The idea behind the exploration strategy is to improve the performances obtained using the archiving strategy by adapting the ϵ values to the feasible solutions landscape.

The adaptation of ϵ values during the exploration appeared in the context of providing P_Q approximations (Grosan, 2006). It consisted of a gradual decrement by 1 of the value of ϵ each time the number of consecutive generations without improvement attained a specific value. Although efficient, the process could not guarantee the convergence towards a desired value of ϵ in a finite number of steps and necessitated also large values for the initial ϵ, which implies a large number of iterations.

In this paper we propose the general assumptions required to prove convergence toward $P_{Q,\epsilon}$. These assumptions are taken into account in constructing a specific decrease function together with a way of considering the distance criterion between two consecutive archives in the adaptive process. For simplicity, we assume that all components of ϵ are identical, i.e., $\epsilon = (\epsilon^*, \ldots, \epsilon^*)$. The value of ϵ^* is specified by the user, as well as a maximal starting value, $\epsilon_{\max}$. The maximal value can be deduced for specific problems by performing bounds computations for the objective functions.

Algorithm 2 exposes the generic components of the proposed adaptive ϵ-approximate searching strategy. The technique facilitates the reduction of the number of generations required in order to attain a $P_{Q,\epsilon}$ final set.

As regards the termination criterion, for a given t, the two conditions $t \leq MaxNoGenerations$ and $\epsilon_t \geq \epsilon_{\min}$ must be met in order to continue the loop. This implies that in the worst case the algorithm will terminate after the maximal number of generations by employing the $Decrease(t)$ adaptation for ϵ_t.

Algorithm 2 Generic Adaptive ϵ-Approximation Search

$t = 0$;
$\epsilon_0 = \epsilon_{\max}$
$A_0 = \emptyset$
$P_0 \subset Q$ drawn at random
$dist = 0$
while $\neg$ Termination_Criterion(P_t) **do**
  $P_{t+1} =$ Generate(A_t,P_t);
  Evaluate(P_{t+1});
  $A_{t+1} = ArchiveUpdate(\epsilon_t, P_t, A_t)$;
  $\Delta = dist(A_{t+1}, A_t)$;
  if $\Delta < MinimalQualityIncrease$ **then**
    $\epsilon_{t+1} = Decrease(t + Increase(\Delta))$
  else
    $\epsilon_{t+1} = min(\epsilon_t, Decrease(t))$;
  end if
  $t = t + 1$;
end while

4.2 Discussion and Analysis

Adaptation of ϵ

The distance criterion consists in computing a comparative metric between A_{t+1} and A_t - $dist(A_{t+1}, A_t) = Metric(A_{t+1}, A_t)$. If the value of the improvement falls below a specified threshold - denoted as $MinimalQualityIncrease$ - the length of the step used in decreasing the value of ϵ increases by Δ. For our purposes the C-metric, proposed in (Zitzler and Thiele, 1999), is computed between A_{t+1} and A_t. It was chosen by its ability of providing the percent of solutions from A_{t+1} which are dominating the ones in A_t. Also it has the advantage of being computed independently, without considering external factors, as a specified point. Other comparative metrics can be similarly employed.

Let $Decrease : \mathbb{N} \to [\epsilon^*, \epsilon_{max}]$ be a monotonically decreasing function which defines the value of ϵ in the adaptive process. The following assumption on *Decrease* is necessary in order to ensure convergence in the limit toward $P_{Q,\epsilon}$:

$$\exists t_0 \in N : \epsilon(t) = \epsilon^*, \forall t \geq t_0. \tag{4}$$

For our computations we have used the following function:

$$Decrease(t) := \epsilon_{\max} - \exp^{-\gamma\left(\frac{\beta}{MaxNoGenerations} t\right)^2} *(\epsilon_{\max} - \epsilon^*), \quad \text{for } t \leq t_0, \tag{5}$$

where β represents an arbitrarily large value.

ArchiveUpdate

Here we use the archiving strategy proposed in (Schütze *et al.*, 2007) and which was designed to maintain the entire set of ϵ-efficient solutions with generic stochastic search algorithms. The archiving strategy is simply the one which keeps all obtained points which are not $-\epsilon$-dominated by any other test point, i.e.

$$ArchiveUpdateP_{Q,\epsilon}(\epsilon, P, A) := \{x \in P \cup A : y \not\prec_{-\epsilon} x \ \forall y \in P \cup A\}, \tag{6}$$

The following theorem states a result on the underlying abstract algorithm of the procedure proposed above.

Theorem 1. *Let an MOP of the form (2) be given and $\epsilon \in \mathbb{R}^k_+$. Further let*

$$\forall x \in \{0,1\}^n : \quad P\left(\exists l \in \mathbb{N} \ : \ x \in P_l\right) = 1 \tag{7}$$

Then an application of Algorithm 1, where $ArchiveUpdateP_{Q,\epsilon}()$ is used to update the archive, leads to a sequence of archives $A_l, l \in \mathbb{N}$, with

$$\lim_{l \to \infty} d_H(P_{Q,\epsilon}, A_l) = 0, \quad \textit{with probability one,} \tag{8}$$

where d_H denotes the Hausdorff distance.

Proof. This is a direct consequence of a result from (Schütze *et al.*, 2007), which holds for the continuous case.

Remark 1. **a)** The crucial assumption required to obtain convergence is (7). This is e.g. fulfilled if the sequence $(P_t)_{t\geq 0}$ of candidate sets obtained by *Generate()* is a homogeneous finite Markov chain with irreducible transition matrix (Iosifescu, 1980; Rudolph and Agapie, 2000).

b) By (4) it is assured that $P_{Q,\epsilon}$ is computed in the limit. In the first steps, where larger values of ϵ_i are used (in order to increase the performance of the algorithm), outer approximations of $P_{Q,\epsilon}$ are generated since for all $\epsilon_1, \epsilon_2 \in \mathbb{R}^k_+$ with $\epsilon_1 \leq_p \epsilon_2$ it follows that $P_{Q,\epsilon_1} \subset P_{Q,\epsilon_2}$. Condition (4) has to be added for theoretical purposes since the function $f : \mathbb{R}^+_0 \to \mathbb{R}^+_0$,

$$f(\Delta) = dist(P_{Q,\epsilon+1\Delta}, P_{Q,\epsilon}) = \sup_{p \in P_{Q,\epsilon+1\Delta}} \inf_{q \in P_{Q,\epsilon}} ||p - q||, \tag{9}$$

does not have to be continuous (e.g., if $F(Q)$ is not convex).

4.3 Numerical results

Employing the $ArchiveUpdate_{P_{Q,\epsilon}}$ has the advantage of preserving a larger spectrum of alternatives for a nondominated candidate solution. It can be observed from Figure 2 that the solutions obtained by the adaptive technique include all the solutions provided by the $ArchiveUpdateND$. For the adaptive process we used $\epsilon_{max} = 5$ and $\epsilon^* = 2$ (and thus $\epsilon = (2, 2)$). For both algorithms a comparable number of evaluations has been performed, each of the algorithms being executed with the same maximal number of generations, namely 10,000 generations. The size of the population was set to 100 individuals.

Also, a comparative study between the non-adaptive version of the algorithm and the adaptive technique has been entailed - see Table 2. The same configuration of the parameters as before has been kept and the maximum allowed number of generations was reduced to 1000, for simplifying the statistical process.

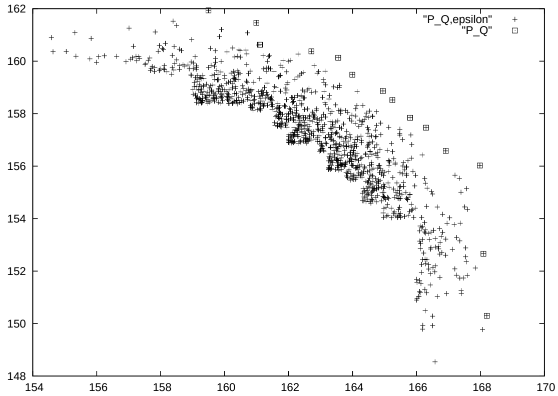

Fig. 2. Approximations of P_Q and $P_{Q,\epsilon}$ for an instance with $n = 30$ and for $\epsilon = (2,2)$. Though ϵ is relatively small, the set of approximate solutions is much larger than the set of nondominated points, offering thus more possibilities for the DM.

Table 2. Comparative results between the adaptive and non-adaptive version.

		$\|P_{Q,\epsilon}\|$		$\|P_Q\|$		C-metric	
Nb. objects	Instance	Adaptiv	Non-adaptiv	Adaptiv	Non-adaptiv	Adaptiv	Non-adaptiv
300	inst1	84.7	65.4	25.5	20	0.44	0.52
300	inst2	79.4	75.4	21.6	22.4	0.44	0.48
300	inst3	61.7	61	19.9	18.5	0.29	0.55
400	inst1	70.2	65.6	23	21	0.33	0.45
400	inst2	76.4	67.2	24.9	21.2	0.46	0.33
400	inst3	59.1	56.2	19.5	19.1	0.44	0.51
500	inst1	62.8	57.5	22.3	21.6	0.35	0.5
500	inst2	51.2	54.4	18.4	18.3	0.4	0.47
500	inst3	54.5	52.5	18.9	19.3	0.46	0.4

5 Interactive Selection Method

Having computed an approximation of $P_{Q,\epsilon}$ (denote by $\tilde{P_{Q,\epsilon}}$), the question naturally arises how to select a suitable point out of this (large) set according to the given application. The scope of this section is to propose such a selection mechanism.

The selection mechanism is intended as the second step of the exploration process. The target users are developers who want to focus on specific interest regions in order to gather knowledge about the topology of the landscape described by subsets of $P_{Q,\epsilon}$ and $F(P_{Q,\epsilon})$ sets. The selection mechanism starts in the image space due to its low dimensionality. The main steps of the Interactive Selection Method are exposed in the following. $\tilde{P_Q}, \epsilon$ - the given approximation of the set which has to be explored - is computed using the Algorithm 2.

Algorithm 3 Interactive component

1: $\mathcal{F}$:= nondominated solutions of $P_{\tilde{Q},\epsilon}$
2: **while** user $\neg$ satisfied **do**
3: R = User Input Interest Region($\mathcal{F}$)
4: ObjectiveSpaceDisplay ($P_{Q,\epsilon}(R), P^{\mathcal{F}}_{Q,\epsilon}$)
5: DecisionalSpaceDisplay ($P_{Q,\epsilon}(R)$)
6: **end while**

The user visualizes a filtered front - further denoted as $\mathcal{F}$ - composed only of the Pareto non-dominated solutions from the $P_{Q,\epsilon}$. Further he specifies a region by employing graphical tools and/or by specifying a tolerance value for ϵ. The solutions from $P_{Q,\epsilon}$ contained in the specified interest region, R - further denoted as $P_{Q,\epsilon}(R)$ - are graphically depicted in both the objective and the decisional space, see Fig. 3.

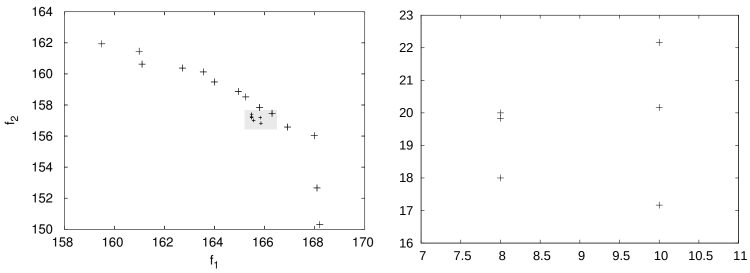

Fig. 3. Interactive Selection: ObjectiveSpaceDisplay (left), DecisionSpaceDisplay (right) for the problem depicted in Section 3. Decision space (right): representation of the selected interest region, having first axis - d_H *(Pareto point, current point)* , the second axis - average (d_H(*neighborhood*)).

6 Conclusions

In this paper we have addressed the computation of ϵ-efficient solutions for {0,1}-knapsack problems by means of an ϵ-adaptive process and have shown the convergence in the limit of the technique towards the $P_{Q,\epsilon}$ set. The success of the technique relies on the cooperation between the archiver and the adaptive choice of the values of ϵ, this allowing a better spreading of the final archive.

For future work the design of a specific comparison metric should be addressed in order to speed up the adaptive process. Another topic of interest consists in the design of benchmarks for which the ϵ-efficient set is difficult to reach, useful in stressing adaptive methods and archiving techniques which are ϵ-approximate oriented.

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