

Optimal Design of Reinforced Concrete Beams using Genetic Algorithms

Carlos Coello Coello *

Filiberto Santos Hernández

Francisco Alonso Farrera

Escuela de Ingeniería Civil, Universidad Autónoma de Chiapas, México

Abstract: *This paper presents a method for optimizing the design of reinforced concrete beams subject to a specified set of constraints. A new model of optimization is proposed, leading to more realistic and practical designs. As there are an infinite number of possible beam dimensions and reinforcement ratios that yield the same moment of resistance, it becomes difficult to achieve the least-cost design by conventional iterative methods. We present a method based upon a search technique using genetic algorithms. Several applications show how our system provides more realistic designs than other methods based on mathematical programming techniques. Also, we show our results of experimenting with several representation schemes for the genetic algorithm, and the methodology that we used to adjust its parameters—i.e., population size, crossover and mutation rates, and maximum number of generations—so that it produces a reasonable answer in a short period of time. A prototype of this system is currently being tested at our school, to see its potential use as a tool for real-world applications.*

Keywords: structural optimization, genetic algorithms, design optimization, artificial intelligence.

1 Introduction

The design of reinforced concrete elements plays a very important role in México, because of its extended use by civil engineers. In the traditional design methodology, a solution is proposed and then corroborated by mathematical analysis in order to verify that the problem requirements are satisfied. If the requirements are not satisfied, a new solution is proposed. In this trial and error process the engineer gains experience, but at a very high cost in terms of time and effort (see Figure 1). As time is always a constraint in

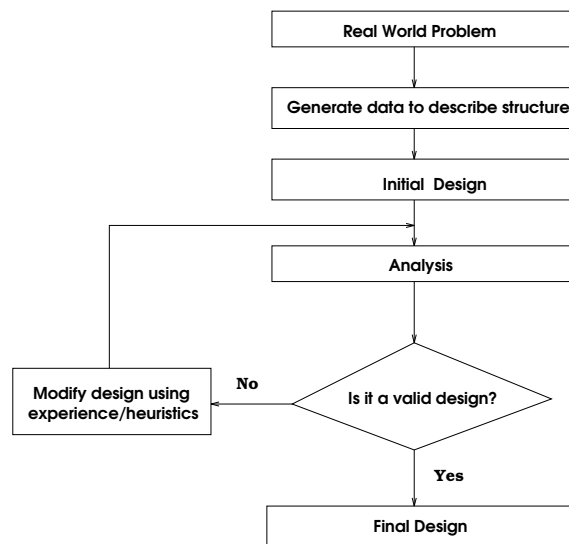


Figure 1: Traditional design process.

real design, a reasonable sub-optimal solution is normally adopted. Computers have been recently used to help engineers automate this process. However, their use has concentrated mainly in performing the tedious mathematical calculations that are required.

Alternatively, the optimal design approach consists of changing the design based on a certain “optimality condition” (see Figure 2). However, the general optimal design problem is highly nonlinear and nonconvex [1]. As a result, structural optimization problems are characterized by having multiple local optima.

This paper focuses on the use of an artificial intelligence (AI) technique based on the mechanics of natural selection, called the *genetic algorithm* [16] [13]. The design process based on this technique is very similar to the optimal design process previously shown (see Figure 3). The main difference is the notion of a *fitness function* instead of a cost function, and the fact that the adaptation of the design is dependent upon

* Currently at the Department of Computer Science of Tulane University, in New Orleans, LA (USA)

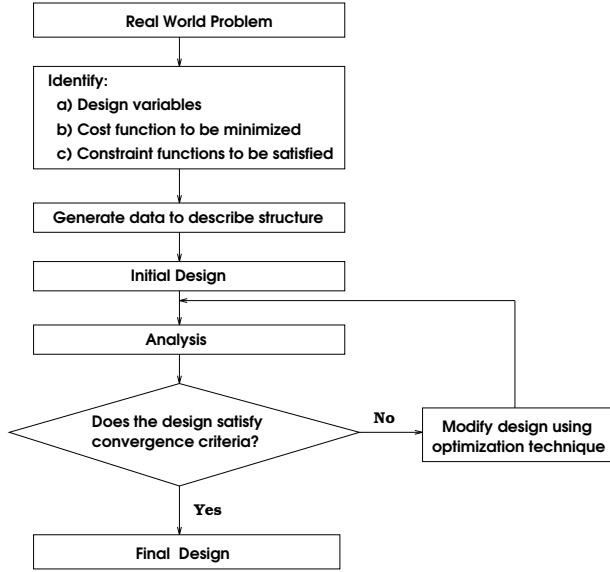


Figure 2: Optimal design process [1].

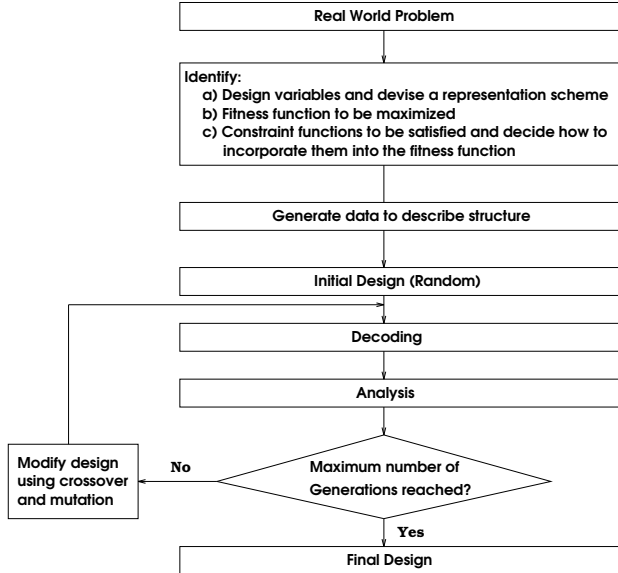


Figure 3: Optimal design process using a Genetic Algorithm.

neither (a) the engineer nor (b) the gradient of the cost function, as in the two previous cases. Even more interesting is that initial designs are randomly generated, without any human intervention, and nevertheless the technique converges to an optimal design in a reasonable amount of time.

The design of a reinforced concrete beam is normally an iterative process like the one shown in Figure 1, in which the engineer assumes the self-weight of the beam beforehand, and a trial section is chosen. Then, the moment of resistance of this section is determined, to check its suitability against the given applied bending moment. This process is repeated until a trial section is found suitable. This procedure often creates difficulty in exactly matching the moment of resistance of the section with the total applied bending moment due to the self-weight of the beam, which may be quite substantial in many cases. Therefore, the design process of a beam is not only slow, but also has a complete lack of economics, since the only concern is to find any section suitable for the given conditions, instead of looking for the most economic one.

In this paper, we will present a model for optimal design which minimizes the cost of a reinforced concrete beam based not only on the allowable stresses of the element, but also in the costs of concrete, steel and shuttering. Our model follows the one proposed by Chakrabarty [2] [3], with certain modifications (i.e., additional constraints) that makes it suitable for practical applications. In the next section, we will introduce some general concepts from reinforced concrete design. Then, our model will be shown and the genetic algorithm approach will be described. Finally, we will present the results found by our model when solving some problems found in the literature, and we will briefly discuss some of the issues that arose when using genetic algorithms in this kind of application.

2 Basic Concepts

For the purposes of this research, we adopted strength design procedures, because they have, among others, the following advantages [9]:

- Strength design better predicts the strength of a section because of the recognition of the non-linearity of the stress-strain diagram at high stress levels.
- Because the dead loads to which a structure is subjected are more certainly determined than the live loads, it is unreasonable to apply the same

factor of safety to both. Therefore, this approach allows the use of different safety factors for them.

The basic assumptions that are taken when using strength design are the following [9]:

- Plane sections before bending remain plane after bending.
- At ultimate capacity, strain and stress are not proportional.
- Strain in the concrete is proportional to the distance from the neutral axis.
- Tensile strength of concrete is neglected in flexural computations.
- The ultimate concrete strain is 0.003.
- The modulus of elasticity of the reinforcing steel is 29,000,000 psi (200,000 MPa).
- The average compressive stress in the concrete is $0.85f'_c$.
- The average tensile stress in the reinforcement does not exceed f_y .

According to this design method, the nominal moment capacity M_n of a rectangular beam with tension reinforcement only is given by [9]:

$$M_n = bd^2 f'_c w (1 - 0.59w) \quad (1)$$

where b is the width of the beam, d is the distance from the extreme compressive fibre to the centroid of tension reinforcement, f'_c is the compressive strength of concrete, $w = (A_s f_y / b d f'_c)$, f_y is the yield strength of reinforcement and A_s is the area of tension reinforcement.

There is an infinite amount of solutions to equation (1) that yield the same value of M_n [9]. In the traditional design process, the values of b and/or d are assumed, and the remaining parameters are calculated based on them, iterating until a suitable section is found. An obvious restriction of this approach is that only a few sections can be evaluated in this manner. Since equation (1) does not incorporate any cost parameter, there is no way of achieving a least-cost design. Therefore, we need to include certain cost parameters combined with the design parameters in our optimal design model, so that we can produce least-cost suitable designs.

3 Previous Work

The optimal design of beams was first proposed by Galileo [10], even though his calculations were wrong. Apparently, the doctoral dissertation by E. J. Haug Jr. [15] (see also [14]) in 1966 is one of the first modern attempts to use a digital computer as a tool for the optimal design of this structural element. Haug reduced the non-linear optimal design problem to a Lagrange problem in the Calculus of Variations. His model includes restrictions and tries to minimize the weight of the beam in several different situations.

Venkayya [21] developed a method based on an energy criteria and a search procedure for design of structures subjected to static loading. He argues that his method can handle very efficiently, (a) design for multiple load conditions, (b) stress constraints, (c) constraints on displacements, (d) constraints on sizes of the elements. His method has been successfully applied to the design of trusses, frames and beams.

Osyczka [18] [19] has applied multi-objective optimization techniques to beam design problems. Also, Prakash et al. [20] proposed a model for optimal design of reinforced concrete sections in which the costs of steel, concrete and shuttering were included. Chakrabarty's model [2] [3] has some similarities with Prakash's model, but the former is more complete and detailed. That's the main reason why we decided to use Chakrabarty's model as a basis for our implementation, even though we had to slightly modify it in order to produce designs that fall into Mexico's standard regulations for reinforced concrete design, since the original model lead in some cases to inconsistent designs.

4 The Optimal Design Model

A schematic section of a rectangular singly reinforced concrete beam is shown in Figure 4. The cost per unit length of the beam will be given by the following expression [3]:

$$y(x) = c_1 x_1 + c_2 x_2 x_3 + c_3 x_2 + c_4 x_3 \quad (2)$$

where $y(x)$ is the cost per unit length of the beam (\$/cm), c_1 is the cost coefficient due to volume of tensile steel reinforcement in the beam (\$/cm³), c_2 is the cost coefficient due to volume of concrete in the beam (\$/cm³), c_3 is the cost coefficient due to shuttering along the vertical surfaces of the beam (\$/cm²), c_4 is the cost coefficient due to shuttering along the bottom horizontal surface of the beam (\$/cm²), x_1 is the variable giving the area of tensile steel reinforcement as

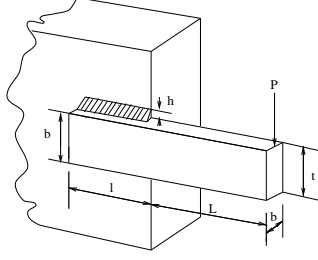


Figure 4: Schematic section of a singly reinforced rectangular beam. Taken from [[4].

shown in Figure 4 (cm^2), x_2 is the variable giving the depth of the beam as shown in Figure 4 (cm) and x_3 is the variable giving the width of the beam, as shown in Figure 4 (cm).

The variables x_1 , x_2 and x_3 not only affect the cost of a beam, but will also determine its moment of resistance. Since x_1 may be calculated if we know x_2 and x_3 [9], we'll propose different values for these two variables so that the total cost of the beam is minimum, verifying at the same time that our section has a proper resistant moment. Then, our optimal design model is the following:

$$\text{minimize: } f(x) = c_1x_1 + c_2x_2x_3 + c_3x_2 + c_4x_3$$

subject to:

$$a_1x_1^{-1}x_3x_5 < 1 \quad (\text{equilibrium constraint}) \quad (3)$$

$$a_2x_4^{-1} + a_3x_2x_3x_4^{-1} < 1$$

(bending moment compatibility constraint) (4)

$$0.25 \leq x_3/x_2 \leq 0.6$$

(width – height ratio constraint) (5)

$$Q(x_2 - a_5x_5)(f_r f'_c x_5 x_3 + x_1 f_y) a_5 / x_4 \geq 1$$

(acting moment constraint) (6)

$$a_6/x_3 < 1 \quad (\text{minimum width constraint}) \quad (7)$$

$$x_1, x_2, x_3, x_4, x_5 > 0 \quad (\text{non – negativity constraint}) \quad (8)$$

Here x_4 is a variable defining the total applied bending moment including the bending moment due to self-weight of the beam; x_5 is a variable defining the depth of the equivalent rectangular stress block. Additionally, we have the following formulas:

$$c_1 = w_s \times c_s \quad (\$/cm^3) \quad (9)$$

where $w_s = 0.00785 \text{ kg/cm}^3$ (assumed value) is the unit weight of steel reinforcement, and c_s is the unit cost of steel reinforcement ($\$/kg$).

$$c_2 = (1 + r)c_c \times 10^{-6} \quad (\$/cm^3) \quad (10)$$

where c_c is the unit cost of concrete ($\$/m^3$) and r is the cover ratio.

$$c_3 = 2(1 + r)c_r \times 10^{-4} \quad (\$/cm^2) \quad (11)$$

where c_r is the unit cost of shuttering ($\$/m^2$).

$$c_4 = c_r \times 10^{-4} \quad (\$/cm^2) \quad (12)$$

$$a_1 = 0.85f'_c/f_y \quad (13)$$

where f_y is the yield strength of steel reinforcement (N/cm^2) and f'_c is the compressive strength of concrete (N/cm^2).

$$a_3 = D(1 + r)w_c k L^2 \quad (14)$$

where $D = 1.4$ (assumed) is the load factor for dead load, $w_c = 0.0228 \text{ N/cm}^3$ is the unit weight force of concrete, k is the moment coefficient for the design section ($= 1.8$ for simply supported beam) and L is the span of the beam (cm).

$$a_4 = 1/(f_r Q f'_c) \quad (15)$$

where Q is the capacity reduction factor ($= 0.90$ for flexure) and $f_r = 0.85$ (assumed) is the reduction factor of concrete. Also, a_2 is the applied bending moment ($N - cm$), $a_5 = \frac{1}{2}$ (assuming the centroid of compressive force at half the depth of equivalent rectangular stress block), and a_6 is the minimum acceptable width of the beam.

To determine x_4 (total bending moment, including self-weight of the beam), we use:

$$x_4 = a_2 + a_3 x_2 x_3 \quad (16)$$

To calculate x_1 (area of reinforcement steel), we use:

$$x_1 = \omega x_2 x_3 f'_c / f_y \quad (17)$$

$$\text{where } \omega = \frac{1 - \sqrt{1 - \frac{4(0.59)x_4}{0.9x_3x_2^2f'_c}}}{1.18}$$

This last expression can be derived from equation (1). Finally, x_5 (depth of the equivalent stress block) is given by:

$$x_5 = x_1 / (a_1 x_3) \quad (18)$$

5 Use of Genetic Algorithms

To solve this optimization problem, we used the Simple Genetic Algorithm (SGA) proposed by Goldberg [13], and we experimented with several representation schemes. We have previously used binary representation [7] and we have tried Gray coding [6] for structural optimization problems with a continuous search space like this one. For this particular application, we decided to experiment also with floating point representation. We won't talk much about the genetic algorithm (GA), since we have done so in previous publications [4] [8] [5]. Instead, we'll give some details about the different representation schemes that we used in our experiments.

The traditional representation used by the genetic algorithms community is the binary scheme according to which a chromosome is a string of the form $\langle b_1, b_2, \dots, b_m \rangle$, where b_1, b_2, \dots, b_m are called *allele* (either zeros or ones). Since the binary alphabet offers the maximum number of schemata per bit of information of any coding [13], its use has become very popular among scientists. This coding also facilitates theoretical analysis of the technique and allows elegant genetic operators. However, since the "implicit parallelism" property of GAs does not depend on using bit strings [17] it may be worthwhile to experiment with larger alphabets and even with new genetic operators. In particular, for optimization problems in which the parameters to be adjusted are continuous, a floating point representation scheme seems a logical choice. According to this representation, a chromosome is a string of the form $\langle d_1, d_2, \dots, d_m \rangle$, where d_1, d_2, \dots, d_m are digits (numbers between zero and nine).

One of the advantages of floating point representation is that it has the property that two points close

to each other in the representation space must also be close in the problem space, and vice versa [17]. This is not generally true in the binary approach, where the distance in a representation is normally defined by the number of different bit positions. However, it is possible to reduce such discrepancy by using Gray coding. The Gray code representation has the property that any two points next to each other in the problem space differ by only one bit [17]. In other words, an increase of one step in the parameter value corresponds to a change of a single bit in the code. This is a well known technique used to reduce the distance of two points in the problem space, and it is argued to bring some benefit because of their adjacency property, and the small perturbation caused by many single mutations. However, the use of Gray codes didn't help much in this particular application, as we'll see in the next section.

Finally, we should mention that we used a two-point crossover, and binary tournament selection in all our tests. The only operator that had to be redefined was *mutation*, which in the floating point representation consists on selecting a random number between 0 and 9. Our fitness function was given by equation (2), using a penalty function of the form $fitness = 1 / (cost * (v * 500 + 1))$ where v depends on the number of constraints violated. Whenever the design doesn't violate any constraint, the fitness function is just the inverse of the cost (the GA only maximizes, and we require a minimization in this case).

6 Examples

The following example was taken from Everard and Tanner [9]:

Design a least-cost reinforced concrete rectangular beam simply supported over a span of 10 m supporting a uniform dead load of 15 kN/m and a uniform live load of 20 kN/m. The concrete strength $f'_c = 30 \text{ MPa}$ and the steel yield strength $f_y = 300 \text{ MPa}$. The unit cost of steel (CS), concrete (CC) and shuttering (CSH) are \$ 0.72/kg, \$ 64.5/m³ and \$ 2.155/m², respectively. Assume a cover ratio (r) of 0.10, unit weight of concrete of 2323 kg/m³ and capacity reduction factor as 0.90.

The ultimate uniform load is

$$= 1.4 \times 15 + 1.7 \times 20 = 55 \text{ kN/m.}$$

The ultimate applied bending moment is

$$= 55 \times 10^2 / 8 = 687.5 \text{ kN, } m = 687.5 \times 10^5 \text{ N-cm.}$$

Using this information, we can get the values of the cost coefficients and the other model constants:

Table 4: Comparison of the geometric programming approach used by Chakrabarty [5] and the GA using floating point representation. We assume $b = 62.50$ cm. Notice how the constraint imposed by equation (5) is violated by Chakrabarty’s design.

Parameter	Chakrabarty	GA (Binary)	GA (Gray Coding)	GA (FP)
x_1 (cm^2)	37.6926	36.1893	41.5905	37.5205
x_2 (cm)	86.0629	89.5402	78.6177	86.4776
x_3 (cm)	30.0000	30.0162	30.0447	30.0022
x_4 ($N - cm$)	80'064, 711.73	80'540, 242.0620	79'111, 846.5650	80'131, 661.9160
x_5 (cm)	14.7814	14.1842	16.2857	14.7128
$cost$ (\$/cm)	0.4435	0.4442	0.4464	0.4436

Table 1: Comparison of the geometric programming approach used by Chakrabarty [4] and the GA using binary and floating point representation.

Parameter	Chakrabarty	GA (FP)
x_1 (cm^2)	57.0072	50.2583
x_2 (cm)	59.8678	66.7029
x_3 (cm)	40.000	40.0033
$cost$ (\$/cm)	0.3680	0.3716

Table 5: Comparison of the geometric programming approach used by Chakrabarty [5] and the GA using floating point representation. $b = 40$ cm, $CS = 0.36$, $CC = 64.5$ and $CSH = 2.155$. Notice how the constraint imposed by equation (5) is violated by Chakrabarty's design.

Parameter	Chakrabarty	GA (FP)
x_1 (cm^2)	37.2006	37.0318
x_2 (cm)	89.5455	90.0205
x_3 (cm)	40.000	40.0010
$cost$ (\$/cm)	0.6206	0.6207

Table 6: Comparison of the geometric programming approach used by Chakrabarty [5] and the GA using floating point representation. $b = 40$ cm, $CS = 1.08$, $CC = 64.5$ and $CSH = 2.155$.

fore, in practice an engineer would prefer our design even when it's more expensive, for the reasons previously exposed. In all the remaining examples, it will always be the case that when our results are not equals to those produced by Chakrabarty's model (or almost equal, should we say, since there is always a difference in the last digit due to rounding-off errors) it's because his design is violating some constraint—normally that defined by equation (5).

Finally, we tested different values for the costs of reinforcement steel, concrete and shuttering. The results are shown in Tables 5 to 11. Again, the dis-

Parameter	Chakrabarty	GA (FP)
x_1 (cm^2)	33.2691	33.2279
x_2 (cm)	101.1724	101.3565
x_3 (cm)	40.000	40.0001
$cost$ (\$/cm)	0.7198	0.7199

Table 7: Comparison of the geometric programming approach used by Chakrabarty [5] and the GA using floating point representation. $b = 40$ cm, $CS = 1.44$, $CC = 64.5$ and $CSH = 2.155$.

Parameter	Chakrabarty	GA (FP)
x_1 (cm^2)	33.0372	35.0698
x_2 (cm)	95.5279	95.4719
x_3 (cm)	40.000	40.0001
$cost$ (\$/cm)	0.3875	0.3876

Table 8: Comparison of the geometric programming approach used by Chakrabarty [5] and the GA using floating point representation. $b = 40$ cm, $CS = 0.72$, $CC = 32.25$ and $CSH = 2.155$.

Parameter	Chakrabarty	GA (FP)
x_1 (cm^2)	55.4240	49.9278
x_2 (cm)	61.2698	67.0981
x_3 (cm)	40.000	40.0050
$cost$ (\$/cm)	0.6987	0.7035

Table 9: Comparison of the geometric programming approach used by Chakrabarty [5] and the GA using floating point representation. $b = 40$ cm, $CS = 0.72$, $CC = 129.0$ and $CSH = 2.155$. Notice how the constraint imposed by equation (5) is violated by Chakrabarty's design.

Parameter	Chakrabarty	GA (FP)
x_1 (cm^2)	42.3510	42.5568
x_2 (cm)	78.3650	78.0625
x_3 (cm)	40.000	40.0001
$cost$ ($\$/cm$)	0.4847	0.4848

Table 10: Comparison of the geometric programming approach used by Chakrabarty [5] and the GA using floating point representation. $b = 40$ cm , $CS = 0.72$, $CC = 64.5$ and $CSH = 1.0775$.

Parameter	Chakrabarty	GA (FP)
x_1 (cm^2)	45.9454	45.9012
x_2 (cm)	72.4085	72.5103
x_3 (cm)	40.000	40.0003
$cost$ ($\$/cm$)	0.5511	0.5512

Table 11: Comparison of the geometric programming approach used by Chakrabarty [5] and the GA using floating point representation. $b = 40$ cm , $CS = 0.72$, $CC = 64.5$ and $CSH = 4.31$.

crepancies between our results and those produced by Chakrabarty's method will indicate some violation of the constraints imposed by our model.

Future Work and Conclusions

There is still plenty of work to do, and we are considering the possibility of experimenting with other techniques for adjusting the parameters of the GA, such as fuzzy logic. Also, we are interested on doing a theoretical analysis of the search space of this optimization problem, so that we can devise some strategies to solve it more efficiently. However, so far the results seem very promising, and the system has called the attention of more than one engineer in our University, and even when it has been used only for academic purposes so far, we are considering the possibility of giving it commercial use.

We have been working in the use of GAs for structural optimization problems during the last two years, and now we are able to produce optimal designs of beams, columns and plane and space trusses. However, our final goal is to develop a complete structural optimization system that uses GAs, and that probably incorporates also the traditional mathematical programming techniques available, together with some other

powerful heuristics such as tabu search [11] [12]. This should provide a powerful tool for engineers involved in design, introducing considerable savings without sacrificing safety.

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