Supplementary Material of the Paper “Uniform Mixture Design via Evolutionary Multi-Objective Optimization”

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Abstract

This supplementary material provides complementary information related to the paper entitled “Uniform Mixture Design via Evolutionary Multi-Objective Optimization.” In particular, we present proofs that support this research work.

Keywords: Uniform mixture design, evolutionary multi-objective optimization.

Appendix A. Proofs

Appendix A.1. Any solution evaluated in MOP1 maps to a mixture

We prove that given an $\mathbf{x} \in [0,1]^{M-1}$ such that $\mathbf{x} = (x_1, \dots, x_{M-1})$ and the objective vector $F(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_M(\mathbf{x}))^T$ produced by MOP1 (Equation (3)) is a mixture.

i) First, we show that given an $\mathbf{x} \in [0,1]^{M-1}$, $f_i(\mathbf{x}) \geq 0$ for each $i \in \{1, \dots, M\}$. Since $x_j \in [0,1]$ for all $j \in \{1, \dots, M-1\}$ then:
   - $\prod_{i=1}^{k} x_i \geq 0$ for $0 < k \leq M-1$, and
   - $(1 - x_j) \geq 0$

   Therefore, $f_i(\mathbf{x}) \geq 0$ for each $i \in \{1, \dots, M\}$.

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ii) Now, we show that given an $\mathbf{x} \in [0,1]^{M-1}$, $\sum_{i=1}^{M} f_i(\mathbf{x}) = 1$.

Substituting each term of the sum by its definition in MOP1 (Equation (3)) and simplifying, we obtain:

$$
\begin{aligned}
& f_1(\mathbf{x}) + f_2(\mathbf{x}) + f_3(\mathbf{x}) + \cdots + f_{M-1}(\mathbf{x}) + f_M(\mathbf{x}) \\
=& x_1 x_2 \cdots x_{M-1} + (1 - x_{M-1}) x_1 x_2 \cdots x_{M-2} + \\
& (1 - x_{M-2}) x_1 x_2 \cdots x_{M-3} + \cdots + (1 - x_2) x_1 + (1 - x_1) \\
\\
=& x_1 x_2 \cdots x_{M-1} + (x_1 x_2 \cdots x_{M-2} - x_1 x_2 \cdots x_{M-1}) + \\
& + (x_1 x_2 \cdots x_{M-3} - x_1 x_2 \cdots x_{M-2}) + \cdots + (x_1 - x_1 x_2) + \\
& (1 - x_1) = 1
\end{aligned}
\tag{A.1}
$$

Hence, from i) and ii), we have that given an $\mathbf{x} \in [0,1]^{M-1}$, $F(\mathbf{x})$ is a mixture.■

Appendix A.2. Any solution evaluated in MOP2 maps to a mixture

We prove that given an $\mathbf{x} \in [0,1]^{M-1}$ such that $\mathbf{x} = (x_1, \ldots, x_{M-1})$ and $x_1 \le x_2 \le \cdots \le x_{M-1}$, the objective vector $F(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_M(\mathbf{x}))^T$ produced by MOP2 (Equation (4)) is a mixture.

i) First, we show that given an $\mathbf{x} \in [0,1]^{M-1}$, $f_i(\mathbf{x}) \ge 0$ for each $i \in \{1, \ldots, M\}$.

Since $x_j \in [0,1]$ for all $j \in \{1, \ldots, M-1\}$ and $0 \le x_1 \le x_2 \le \cdots \le x_{M-1} \le 1$, we have that:

- $x_1 \ge 0$
- Since $1 \ge x_i \ge x_{i-1} \ge 0$ for each $i \in \{2, \ldots, M-1\}$, $x_i - x_{i-1} \ge 0$
- $1 - x_{M-1} \ge 0$

Therefore, $f_i(\mathbf{x}) \ge 0$ for each $i \in \{1, \ldots, M\}$.

ii) Now, we show that given an $\mathbf{x} \in [0,1]^{M-1}$ such that $x_1 \le x_2 \le \cdots \le x_{M-1}$, $\sum_{i=1}^{M} f_i(\mathbf{x}) = 1$.

Substituting each term of the sum by its definition in MOP2 (Equation (4)) and simplifying, we obtain:

$$
\begin{aligned}
&f_1(\mathbf{x}) + f_2(\mathbf{x}) + f_3(\mathbf{x}) + \cdots + f_{M-1}(\mathbf{x}) + f_M(\mathbf{x}) = \\
=&x_1 + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{M-1} - x_{M-2}) + \\
&(1 - x_{M-1}) = 1
\end{aligned}
$$

Hence, from i) and ii), we have that given an $\mathbf{x} \in [0,1]^{M-1}$ such that $x_1 \leq x_2 \leq \cdots \leq x_{M-1}$, $F(\mathbf{x})$ is a mixture.■

Appendix A.3. Any solution evaluated in MOP3 maps to a mixture

We prove that given an $\mathbf{x} \in [0,1]^M$ sucht that $\mathbf{x} \neq \mathbf{0}$, the objective vector $F(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_M(\mathbf{x}))^T$ produced by MOP3 (Equation (5)) is a mixture.

i) First, we show that given an $\mathbf{x} \in [0,1]^M$ such that $\mathbf{x} \neq 0$, $f_i(\mathbf{x}) \geq 0$ for each $i \in \{1, \ldots, M\}$.
   Since $x_j \in [0,1]$ and the L_1-norm is nonnegative and $\mathbf{x} \neq \mathbf{0}$, we have that $\frac{|x_i|}{||\mathbf{x}||_1}$ is well-defined and it is always a nonnegative number.
   Therefore, $f_i(\mathbf{x}) \geq 0$ for each $i \in \{1, \ldots, M\}$.

ii) Now, we show that given an $\mathbf{x} \in [0,1]^M$ such that $\mathbf{x} \neq \mathbf{0}$, $\sum_{i=1}^{M} f_i(\mathbf{x}) = 1$.
   Substituting each term of the sum by its definition in MOP3 (Equation (5)) and simplifying, we obtain:

$$
\begin{aligned}
&f_1(\mathbf{x}) + f_2(\mathbf{x}) + f_3(\mathbf{x}) + \cdots + f_{M-1}(\mathbf{x}) + f_M(\mathbf{x}) = \\
=&\frac{|x_1|}{|x_1| + |x_2| + \cdots + |x_M|} + \frac{|x_2|}{|x_1| + |x_2| + \cdots + |x_M|} + \cdots + \\
&\frac{|x_M|}{|x_1| + |x_2| + \cdots + |x_M|} = \frac{|x_1| + |x_2| + \cdots + |x_M|}{|x_1| + |x_2| + \cdots + |x_M|} = 1
\end{aligned}
$$

Hence, from i) and ii), we have that given an $\mathbf{x} \in [0,1]^M$ such that $\mathbf{x} \neq \mathbf{0}$, $F(\mathbf{x})$ is a mixture.■

Appendix A.4. Any solution is a Pareto optimal solution

Now, we show that any $\mathbf{x} \in \Omega$[1] is Pareto optimal solution of its corresponding MOP. By definition of the formulated MOPs, we know that $\mathbf{z} = (f_1(\mathbf{x}), \ldots, f_M(\mathbf{x}))$ is a mixture for any $\mathbf{x} \in \Omega$, i.e., $z_i \geq 0$ ($i = 0, \ldots, M$) and $\sum_{i=1}^{M} z_i = 1$ (see the proofs in the above sections).

Let us assume that there is an objective vector $\mathbf{y}$ such that $\mathbf{y}$ dominates $\mathbf{z}$. Notice that $\mathbf{y}$ is a mixture. Then, from the definition of Pareto dominance, we have that there exists $i \in \{1, \ldots, M\}$ such that $y_i < z_i$ and $y_j \leq z_j$ for all $j \in \{1, \ldots, M\}$ with $i \neq j$. Therefore $\sum_{j=1, j\neq i}^{M} y_j \leq \sum_{j=1, j\neq i}^{M} z_j$. Since $y_i < z_i$, $\sum_{j=1, j\neq i}^{M} y_j + y_i < \sum_{j=1, j\neq i}^{M} z_j + z_i$. Therefore, $\sum_{i=1}^{M} y_i < 1$. Hence, $\mathbf{y}$ is not a mixture.■

[1] $\Omega = [0,1]^{M-1}$ for MOP1 and MOP2, and $\Omega = [0,1]^M \setminus \{\mathbf{0}\}$ for MOP3