A novel approach to select the best portfolio considering the preferences of the decision maker

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Abstract

The challenges of Portfolio Optimization have led to an increasing interest from the multi-objective evolutionary algorithms research community; however, little attention has been paid to the particular preferences of the investor in order to select the most preferred portfolio from a set of mathematically equivalent alternatives in presence of many criteria. The main goal of this work is thus modeling the preferences of the investor in order to find the most satisfactory portfolio from the investor's perspective when many objective functions are considered. Here, the investor's behavior facing risk, the estimations of the portfolios' future returns, and the risk of not attaining those returns are all represented by means of probabilistic confidence intervals. The imperfect knowledge related to the subjectivity of the investor is modeled on the basis of Interval Theory and the outranking method. The proposed approach aggregates the many criteria on the basis of the investor's particular system of preferences producing a selective pressure towards the most preferred portfolio while the investor's cognitive effort in the final selection is reduced.

An illustrative example in the context of stock portfolio optimization is provided, where several investors interested in many criteria are simulated. The considered criteria are confidence intervals around the portfolios' expected returns, and indicators from the so-called fundamental and technical analyses. Our approach is compared, using real historical data, with an outstanding multi-objective evolutionary algorithm, MOEA/D, and some well-known benchmarks in Modern Portfolio Theory and Finance Theory, namely, the Mean-Variance approach and the Dow Jones Industrial Average index. The results show an evident superiority of the proposed approach in both the context of the underlying criteria (confidence

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intervals and financial indicators) and the context of the actual returns. Thus, we conclude that the proposed approach was able to find satisfactory portfolios in the context of the experiments.

Keywords: Evolutionary multi-objective optimization; portfolio optimization; preferences modeling; uncertainty management.

1 Introduction

In modern society, many objectives are commonly contemplated when allocating resources, that is, generating portfolios (see Steuer *et al*., 2008; Zopounidis and Doumpos, 2013a and 2013b; and Aouni *et al*, 2014). Some of the objectives most commonly mentioned in the related literature are:

- Maximization of the portfolio's return (e.g., Markowitz, 1952; Solares *et al*., 2018).
- Maximization of social responsibility and ethical considerations (e.g.., Gonzalez *et al*., 2013; Utz *et al*., 2014; and Gupta *et al*., 2014).
- Maximization of liquidity (e.g., Allen and Allen, 2013; Kinlaw *et al*., 2013).
- Maximization of return with respect to some benchmarks (see Steuer *et al*., 2008).
- Maximization of the amount invested in R&D (see Steuer *et al*., 2008).
- Minimization of transaction costs (e.g., Mansini *et al*., 2015, Zhang *et al*., 2012).

From all these objectives, the most outstanding one is maximization of the portfolio's return/profit (Solares *et al*., 2018). This is sometimes the only objective optimized during the allocation of resources; however, given the high complexity involved in the return's forecasting procedure, many criteria (e.g., expected return, risk, so-called fundamental and technical analyses) usually underly such objective. Here, we will address, without loss of generality, the latter situation.

Investors frequently use decision-aiding tools in order to obtain a set of portfolios representing, to a certain extent, the best feasible allocations of resources. But this does not solve the problem; the investor still must choose from among all these portfolios the one that represents the best compromise among the considered criteria. But, as reported by Miller (1956), this is not a trivial task since the cognitive limitations make it very difficult for the investor to consistently select the best compromise in the presence of many criteria. This becomes more complicated when she/he needs to make trade-offs between risk and return. Consequently, a more convenient approach must be followed; the goal of such an approach must be to provide a minimal set of portfolios satisfying the investor's preferences. The main objective of this work is thus to propose an approach able to deal with many criteria in order to create a portfolio that satisfies the preferences of the investor. That is, our approach is intended to find the most preferred portfolio.

Since the groundbreaking work of Markowitz (1952), many authors have presented interesting methods to create portfolios with the goal of maximizing the portfolio return (see e.g., Jorion, 2007; Kolm *et al*., 2014; Fabozzi *et al*., 2007; Greco *et al*., 2013). Beyond the use of Probability Theory, many other types of criteria are considered by real investors (decision makers, DMs). These criteria range, for example, from the financial information of the investment objects to the behavior of their returns through time. There are two main perspectives that consider these criteria in the context of stocks: one, the so-called fundamental analysis, mainly uses ratios to express the real (and probably hidden) value of the companies underlying the stocks (see e.g., Xidonas *et al*., 2009); whereas the other, the so-called technical analysis, principally uses signalizations that indicate the goodness of time to execute transactions of stocks by analyzing their serial returns over time (see e.g., Armano *et al*., 2005). These two types of indicators together with the approximation to the probability distribution of the returns constitute the most mentioned criteria in the literature of portfolio optimization. The necessity of considering all these aspects during the allocation of resources come from the high volatility of the stocks' returns and from the complexity to estimate them. Furthermore, sometimes the DMs face the problem of a too short performance history of some stocks to obtain a reliable approximation to their probability distribution, and/or insufficiency of the available financial information. Hence, solving a many-criteria optimization problem describing all these perspectives can be required. However, published papers considering all these aspects in a multi-criteria optimization problem to select the best portfolio according to the decision maker's perspective are scarce. We believe that the lack of popularity of such an approach is mainly due to its high computational cost, caused by an overwhelming number of points in the optimization's search space. Indeed, considering many (more than three) criteria makes the search space grow to a size that makes the portfolio optimization not solvable by exhaustive methods. Furthermore, the number of alternative solutions in the Pareto front for such problems tend to be also overwhelming, making it very hard for the investor to reach a final decision about what he/she considers the best portfolio.

In order to find the best portfolio, the solution that offers the best compromise among the criteria must be found using the decision maker's particular system of preferences (decision policy). That is, since all the solutions within the Pareto front are mathematically equivalent, the DM should provide additional information for choosing the most preferred one (cf. Hakanen *et al*., 2008). This implies that it is necessary to consider the subjectivity of the decision maker in aspects such as her/his attitude facing risk, the importance that she/he assesses to each criterion, and certain thresholds that dictate when the decision maker considers that a portfolio is at least as good as another. If these aspects are considered during the search process, then a selective pressure is performed and a portfolio that is preferred over other portfolios can be found. Nonetheless, incorporating the DM's subjectivity could be a hard task, mainly due to the imperfect knowledge about the true values of the parameters representing the DM's subjectivity and the cognitive effort required from the DM to reduce this source of imprecision. Such imperfect knowledge must necessarily be taken into account when modeling preferences in decision aiding (Roy *et al*., 2014). However, to the best of our knowledge, there are not published papers that deal with this type of difficulty when the subjectivity of the decision maker is incorporated in the portfolio optimization considering many criteria.

Evolutionary multi-criteria optimization algorithms (whose performance analyzing data have been validated in different fields, e.g., in Refs. Pławiak, 2018a and 2018b) work with a population of solutions and can approximate a set of trade-off alternatives simultaneously. They have been widely accepted as a major tool for addressing the problem of finding "good" portfolios. The main goal of this type of algorithms is finding a set of efficient solutions that approximate the true Pareto front in terms of convergence and diversity. The intervention of the DM is thus not traditionally used in the process. So, rather little interest has been paid in the literature to choosing one of the efficient solutions as the final one in contrast to the interest paid in approximating the whole Pareto front.

In this work, we assume that the situation where the DM is not willing/capable to provide preference information in an interactive way holds and propose an a priori approach based on the outranking method. Unlike other ways of modeling preferences, the outranking method is able to deal with i) ordinal and qualitative information, ii) zones of uncertainty in the investor's mind, iii) intransitive preferences, iv) non-compensatory effects and veto situations, and v) incomparability between solutions. The main argument against the outranking approach is its requirement for many preference parameters and the difficulty of eliciting them. Thus, we use here the recent generalization of the outranking method proposed in (Fernandez et *al*., 2018) that defines the preference parameters as ranges of values instead of defining them as punctual values. So, the DM is now capable to directly provide the parameters values that are most representative of his/her preferences. Therefore, a well-suited portfolio might be found according to the DM's decision policy.

An illustrative example in the context of stock portfolio optimization is provided. The dataset used in the validation consists in the actual monthly returns of the stocks within the Dow Jones Industrial Average (DJIA) index during the period April 2011-March 2016. The results are evaluated in both the context of the original criteria and in the context of the actual returns. With respect to the former, we demonstrate that the portfolios constructed by the approach are satisfactory from the decision maker's perspective. With respect to the latter, comparisons with the actual returns of the DJIA index and other benchmarks show that the performance of the proposed approach's results is evidently better.

The paper is structured as follows. In Section 2, a description of the background used by this work is presented. The main proposal is detailed in Section 3. Section 4 provides the validation process and shows the results obtained. Finally, Section 5 concludes the document.

2 Some background

2.1 Portfolio optimization

Selecting the best portfolios from the investor's perspective is very difficult in practice. Particularly, estimating future returns from time series of historical return data is so challenging, that for some authors it is considered as practically impossible (cf. Merton, 1980; Breen *et al*., 1989; and Brandt and Kang, 2004). One of the most outstanding arguments in this sense is the one stated by Merton (1980), who indicated that "attempting to estimate the expected return on the market is to embark on a fool's errand". Conversely, many authors have found empirical evidence of a positive risk-return trade-off relation supporting the development of methods to construct portfolios (see e.g., Fabozzi *et al.*, 2007; Xidonas *et al.*, 2009; Gorgulho *et al.*, 2011; Greco *et al.*, 2013; Solares *et al.*, 2018). In any case, the high relevance of the stock market at a global level makes the scientific research about the construction of stock portfolios a required activity. Let us now provide a brief background of the so-called stock portfolio problem.

Let $p_0^i, p_1^i, p_2^i, \cdots, p_{T_0}^i$ be the historical prices of the ith stock (investment object) in $T_0 + 1$ periods of time. The T_0 historical rate of returns (only *returns* in the following) of the stock are given by $r_t^i = (p_t^i - p_{t-1}^i)/p_{t-1}^i$; $t = 1, 2, \cdots, T_0$. Here, a portfolio is a vector $x = [x_1, x_2, \dots, x_n]^\top$ in decision space that specifies the proportions of money to invest in n investment objects, such that x_i is the proportion to invest in the ith object. The image of a portfolio in objective space is a real number that states the return produced by the portfolio, $R(x)$. It is widely accepted in the literature that the return produced by the portfolio x can be obtained as (cf. Fabozzi *et al.*, 2007):

$$R(x) = \sum_{i=1}^{n} x_i r_{T_0+1}^i.$$

The issue that investors (Decision Makers) face with the previous definition is that it depends on the future returns of the stocks, $r_{T_0+1}^i$, whose values are unknown. Thus, underlying criteria are used with the purpose of estimating such returns, creating the criteria space. The image of a portfolio in the criteria space is a vector that represents the impact of the portfolio on k criteria. The portfolio problem is then to select the feasible portfolio that maximizes the impact on the criteria. Formally:

$$\underset{x \in \Omega}{\text{maximize}}\left(I(x) = \{I_{1(x)}, I_{2(x)}, \cdots, I_{k(x)}\}\right), \tag{1}$$

where $I_j(x)$ is the impact of portfolio x on criterion j and Ω is the set of feasible portfolios (the set of portfolios that fulfill the decision maker’s constraints).

The approximation to reality reached by the model becomes a crucial aspect in portfolio problems. In this sense, several works have used many criteria to describe the portfolio’s performance in a realistic way (cf. Agarwal, 2017). The need for many criteria originates from the decision maker (DM) being unwilling to accept that the uncertainty of future returns can be fully encompassed by a few criteria; not even through a common way of estimating the return such as the expected value. Hence, besides the commonly used Probability Theory, there are approaches that pursuit to describe the quality of the allocation from other perspectives. For example, the stock evaluation is often carried out through the analysis of financial indicators. The two kinds of financial analyses most mentioned in the literature are the fundamental and technical analyses. The former is often used in the stock selection process as a preliminary step to the creation of portfolios. That is, before allocating a proportion of resources to each stock, investors use this analysis to determine which stocks will be considered in the resource allocation process (see e.g., Xidonas *et al*., 2009). On the other hand, the technical analysis is used to specify goodness of the moment in which the resources will be assigned (Macedo *et al*., 2017; Gorgulho *et al*., 2011).

We now present a brief description of these methodologies.

2.1.1 Describing the probability distribution of the returns

Perhaps the best well-known approaches among the scholars to deal with portfolio optimization rely on Probability Theory. For example, Modern Portfolio Theory bases its principal hypothesis in the assumption that the probability distribution of the returns can be approximated. Through this approximation, not just the portfolio's future return is estimated but also the risk involved in the estimation. Particularly, Markowitz (1952, 1968) proposed the variance and semi-variance as measures of risk. In those works, an important assumption is that the returns follow a Gaussian distribution and/or that the decision maker's utility function is quadratic. Such assumptions, together with the high sensitivity to errors, have led this model to be not widely used in practice (Kolm *et al*., 2014).

Many other authors have also assumed that this approximation can be obtained and seek to represent it using better descriptions. For example, Harvey and Siddique (2000) present evidence that supports the intuition that "if returns have systematic skewness, expected returns should include rewards for accepting this risk". Dittmar (2002) uses similar premises and shows that asset returns are affected by covariance, coskewness, and cokurtosis with the return on aggregate wealth. The paper presented by Saranya and Prasanna (2014) follows these results and uses high statistical moments to extend the classical Modern Portfolio Theory and deal with situations where the returns do not follow a Gaussian distribution nor the decision maker's utility function is quadratic. Greco *et al*. (2013) recently proposed a novel approach where the quantiles of the distribution are used as the underlying criteria and several issues related to the Modern Portfolio Theory are overcome.

All these are interesting ideas to construct portfolios in practice, however they all suffer of at least one of the following limitations: i) it is not possible to explicitly consider the behavior of the DM facing risk; ii) they lack of representativeness of the probability distribution; iii) the quantity of underlying criteria is overwhelming; iv) they do not deal with the increasing of the DM's cognitive difficulty caused by the necessity of considering many criteria; v) they do not allow the incorporation of the DM's system of preferences.

2.1.2 Fundamental analysis

The information provided by the fundamental analysis is mainly used in the literature to select competitive stocks. Although this information may be qualitative, it is often generated in the form of ratios of numerical values taken from the financial statements of the companies. Many works in the literature usually aggregate these indicators in a global evaluation index through a subjective process that may depend on the DM decision policy (see e.g., Xidonas *et al*., 2009). Such aggregation is a problem *per se*.

It is well known that the fundamental analysis can be different for companies with different business activities (Marasović *et al*. 2011). Therefore, the most convenient indicators should be used when the fundamental analysis is exploited (e.g., Xidonas *et al*., 2009). Some fundamental indicators that can be used for trans-business companies are shown in Table 1 (cf. Xidonas *et al*., 2009, Marasović *et al*. 2011).

Table 1. Fundamental indicators that can be used for companies with different business activities.

Indicator	Name	Definition
if_1	Return on assets	Earnings before interest and taxes divided by total assets.
if_2	Return on equity	Net income divided by shareholders equity.
if_3	Earnings Per Share	Net income minus dividends on preferred stocks all divided by average outstanding shares.
if_4	Dividend yield	Annual dividends per share divided by price per share.
if_5	Price on earnings	Market value per share divided by earnings per share

if_6	Price on book	Stock price divided by all total assets minus intangible assets and liabilities.
if_7	Price on sales	Share price divided by revenue per share.
if_8	Price on cash Flow	Share price divided by cash flow per share

2.1.3 *Technical analysis*

The technical analysis studies the market patterns, demand and supply of stocks (Achelis, 2000). It consists of using price data to create rules and exploit them financially by selecting stocks in accordance with them. If the rule associated with a technical indicator shows that the price of a stock is likely to rise, the DM should buy now expecting to sell later at a higher price, thus increasing the return of the portfolio.

Some of the most frequently mentioned technical indicators reported in the literature are (cf. e.g., Armano *et al.*, 2005; Macedo *et al.*, 2017; and Gorgulho *et al.*, 2011): Exponential Moving Average (EMA), Double Crossover (DC), Rate of change (ROC), Relative Strength Index (RSI), Moving average convergence/divergence (MACD), On Balance Volume (OBV), Bollinger Band (BB), and True Strength Index (TSI). Let us now describe the rule associated with each of these indicators.

The EMA is one of the simplest technical indicators, where higher weights are assigned to the most recent data. The EMA for the ith stock in period t, EMA_t^i, is defined as (cf., Macedo *et al.*, 2017):

$$EMA_t^i(ws) = \left[p_t^i - EMA_{t-1}^i(ws)\right] \cdot w + EMA_{t-1}^i(ws),$$

where ws is the length of the sliding window of the exponential moving average, $w = \frac{2}{ws+1}$, and the initial EMA (i.e., when $t = ws$) is calculated as the average of the previous ws periods. A value of $ws = 12$ is commonly used (e.g., Gorgulho *et al.*, 2011). The rule associated with this indicator states that if the price line crosses above the EMA line, then the stock should be supported. Formally (cf. Gorgulho *et al.*, 2011):

$$it_1^i = \begin{cases} 1 & EMA_t^i(12) > p_t^i \wedge EMA_{t-1}^i(12) < p_{t-1}^i, \\ 0 & \text{otherwise.} \end{cases}$$

Where $\wedge$ is the conjunction operator.

The DC uses two moving averages (normally, a short one and a large one) and produces a signal when the shorter crosses above the larger. Normally, a window size of five periods is used for the short line while the large line uses a window size of 20 periods. Hence, the signalization rule for this indicator is as follows:

$$it_2^i = \begin{cases} 1 & EMA_t^i(5) > EMA_t^i(20) \wedge EMA_{t-1}^i(5) < EMA_{t-1}^i(20), \\ 0 & \text{otherwise.} \end{cases}$$

The ROC represents the proportional difference between the current price of the ith stock and the price h periods ago (cf., Armano *et al.*, 2005): $ROC_t^i = \frac{p_t^i - p_{t-h}^i}{p_{t-h}^i}$. Positive values in this indicator are desirable. A value $h = 13$ is commonly accepted (see Gorgulho *et al.*, 2011):

$$it_3^i = \begin{cases} 1 & ROC_t^i(13) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The RSI is a momentum oscillator conceived to measure the relative conditions of the stock in the market with respect to its overbought/oversold condition. The RSI value for the ith stock in period t is defined as (cf. Wilder, 1978; Macedo *et al.*, 2017):

$$RSI_t^i(d) = 1 - \frac{1}{1 + RS_t^i(d)},$$

where $RS_t^i(d) = \frac{\sum_{j=1}^{d} U_{t-j+1}^i}{d} / \frac{\sum_{j=1}^{d} D_{t-j+1}^i}{d}$; $\sum_{j=1}^{d} U_{t-j+1}^j$ is the sum of the positive returns of stock i during d periods before the period t, and $\sum_{j=1}^{d} D_{t-j+1}^i$ is the same sum but with negative returns. It is widely accepted that $d = 14$ (e.g., Wilder, 1978; and Gorgulho *et al.*, 2011). This indicator suggests that the ith stock should be supported in period t if the RSI_t^i value crosses above 30% and the current price is higher than the price of the previous period (see Gorgulho *et al.*, 2011; and Macedo *et al.*, 2017). Formally:

$$it_4^i = \begin{cases} 1 & RSI_t^i(d) > 0.3 \wedge RSI_{t-1}^i(d) < 0.3 \wedge p_t^i > p_{t-1}^i, \\ 0 & \text{otherwise.} \end{cases}$$

The MACD is a combination of EMAs that validates the "convenience" of acquiring a stock through the comparison with a signaling function. The most common configuration of this indicator uses two EMAs with 12 and 26 historical periods, $EMA(12)$ and $EMA(26)$, to create the $MACD(12,26)$ (Armano *et al.*, 2005; Macedo *et al.*, 2017; and Gorgulho *et al.*, 2011). The MACD for the stock i in the period t is defined as $MACD_t^i(12,26) = EMA_t^i(12) - EMA_t^i(26)$. The literature usually traces another moving average that does not depend on the price of the stocks but depends on the $MACD_t^i$ indicator. This new moving average, MM_t^i, is used to create a signal line of momentum about the movement of the prices. The signal line is created as a nine-period EMA of the $MACD_t^i$. The common strategy associated with this indicator states that when the value of $MACD_t^i(12,26)$ crosses above $MM_t^i(9)$, then there is evidence that the stock will increase in price and that it is advisable to invest in it in the current period. Thus, we will consider that this indicator suggests support for the ith stock if (cf., Armano *et al*., 2005; and Gorgulho *et al*., 2011):

$$it_5^i = \begin{cases} 1 & MACD_t^i(12,26) > MM_t^i(9) \wedge MACD_{t-1}^i(12,26) < MM_{t-1}^i(9), \\ 0 & \text{otherwise.} \end{cases}$$

The OBV assumes that a rising volume might precede a rise on the stock's price and is calculated as (Gorgulho *et al*., 2011):

$$OBV_t^i = \begin{cases} OBV_{t-1}^i + vol_t & p_t^i > p_{t-1}^i, \\ OBV_{t-1}^i - vol_t & p_t^i < p_{t-1}^i, \\ OBV_{t-1}^i & \text{otherwise.} \end{cases}$$

Where vol_t is the volume (number of shares/stocks traded) in period t.

The OBV indicates that the ith stock should be supported if the value OBV_t^i is rising simultaneously with price (indicating a clear up trend). Formally:

$$it_6^i = \begin{cases} 1 & OBV_t^i > OBV_{t-1}^i \wedge p_t^i > p_{t-1}^i, \\ 0 & \text{otherwise.} \end{cases}$$

The BB is a strategy of election with strong positive net results (Macedo *et al*., 2017). It is a volatility indicator represented by the bands generated from an l-day price moving average minus 2 standard deviations of price changes over the same l-periods time span:

$$MA_t^i(l) = \frac{\sum_{j=1}^{l} p_{t-j+1}}{l}.$$

$$LB_t^i(l) = MA_t^i(l) - 2\sigma_t^i(l).$$

Where $\sigma_t^i(l)$ is the standard deviation of price changes of stock i for the period t and its previous $l-1$ periods.

The rule associated with the BB states that the ith stock should be supported if in period t its price is simultaneously above $LB_t^i(l)$ and below $MA_t^i(l)$ (to avoid false triggering; cf. Macedo *et al.*, 2017):

$$it_7^i = \begin{cases} 1 & MA_t^i(l) > p_t^i > LB_t^i(l), \\ 0 & \text{otherwise.} \end{cases}$$

The TSI is double smoothed with two moving averages to show the trend and specifying, at the same time, the overbought and oversold conditions (Gorgulho *et al.*, 2011). It can be defined as:

$$TSI_t^i(r,s) = 100 \times \frac{EMA_t^i(S) \text{ of } \left(EMA_t^i(r) \text{ of } diff^i\right)}{EMA_t^i(S) \text{ of } \left(EMA_t^i(r) \text{ of } |diff^i|\right)},$$

Where $diff^i$ is the momentum line which calculates the difference between the current price and the price observed on the previous period, that is $diff^i = p_j^i - p_{j-1}^i$ for a given period j.

The literature often uses an EMA of the TSI as trigger: $Trigger_t^i(m) = EMA_t^i(m)$ of TSI_t^i and an *oversold region*. Such region indicates that the stock's price is lower than it should be, and normally is located in the value -25 of TSI_t^i. The rule associated with the TSI states that the ith stock should be supported if the TSI_t^i crosses above $Trigger_t^i(m)$ on the oversold region, that is:

$$it_8^i = \begin{cases} 1 & TSI_t^i(r,s) > Trigger_t^i(m) \wedge TSI_{t-1}^i(r,s) < Trigger_{t-1}^i(m) \wedge Trigger_t^i(m) \leq -0.25, \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Interval-based outranking approach

Fernandez *et al.* (2018) recently proposed a novel approach called *interval-based outranking* that generalizes the classic outranking method. The interval-based outranking assumes that the implicit parameters used by the DM to make decisions are imperfectly known. Imperfect knowledge is modeled in that approach through the representation of parameters as interval numbers, the principal concept within the so-called interval theory. Let us present here this interval-based outranking method in the context of the portfolio construction.

Suppose a set of portfolios A where each $x \in A$ is evaluated on a family of k criteria $\mathcal{I} = \{g_1, g_2, \cdots, g_k\}$ defined on A; in a way that $g_j(x)$ represents the evaluation of portfolio x from the jth perspective. Without loss of generality, we assume that increasing the performance of $g_j(x)$ improves the performance of portfolio x, for any $j = 1,2, \cdots k$. Some of the parameters used by the interval-based outranking approach are the following. (Note the definition of the parameters as interval numbers.)

- $g_j(x) = [g_j^-(x), g_j^+(x)]$, the performance of portfolio $x \in A$ in the jth perspective;
- $w_j = [w_j^-, w_j^+]$, the weight of criterion g_j;
- $v_j = [v_j^-, v_j^+]$, the veto threshold of criterion g_j; and
- $\lambda = [\lambda^-, \lambda^+]$ reflects a threshold for a sufficient strength of the concordance coalition.

Where $x \in A$ and $g_j \in \mathcal{I}$.

Since the imperfect knowledge on the criterion performances is represented through intervals, no preference and indifference thresholds are used in (Fernandez *et al*., 2018).

Through the previous parameters, the interval-based outranking approach builds a likelihood index between pairs $(x, y) \in A \times A$, $\beta(x, y) \in [0,1]$, of the assertion "x is at least as good as y", xSy. This approach also uses a cutting level, β_0, such that $xSy \Leftrightarrow \beta(x, y) \geq \beta_0$.

The marginal likelihood index, $\alpha_j(x, y)$, on portfolio x being at least as good as portfolio y on criterion g_j is calculated as follows:

$$\alpha_j(x, y) = p\left(g_j(x) \geq g_j(y)\right).$$

Where $p(\cdot)$ is the *possibility function* described by:

$$p(I \geq J) = \begin{cases} 1 & \text{if } \ p_{IJ} > 1, \\ p_{IJ} & \text{if } \ 0 \leq p_{IJ} \leq 1, \\ 0 & \text{if } \ p_{IJ} < 0. \end{cases}$$

Where $I = [i^-, i^+]$ and $J = [j^-, j^+]$ are interval numbers and $p_{IJ} = \frac{i^+ - j^-}{(i^+ - i^-) + (j^+ - j^-)}$.

Furthermore, if $i^{+} = i^{-}$ and $j^{+} = j^{-}$, then $p(I \geq J) = \begin{cases} 1 & \text{if } I \geq J, \\ 0 & \text{otherwise.} \end{cases}$

If we assume the existence of a likelihood threshold δ_j for each criterion g_j, then the set of all criteria for which $\alpha_j \geq \delta_j$ is called concordance coalition with the assertion xSy and is denoted by $C(xS_\delta y)$. This concordance coalition is associated with an index $\delta = \min\{\alpha_j\}$, $j \in C(xS_\delta y)$. δ is the likelihood that all criteria within the concordance coalition are actually in agreement with xSy. (Recall that the performance of the portfolios and the values in the set of preference parameters are imperfectly known, so it is not possible to guarantee a total concordance of the criteria.) Criteria that are not in $C(xS_\delta y)$ compose the discordance coalition, $D(xS_\delta y)$. All this is formalized as

$$g_j \in C(xS_\delta y) \text{ iff } \alpha_j \geq \delta, \text{and}$$

$$D(xS_\delta y) = \mathcal{J} - C(xS_\delta y).$$

The imprecision in the definition of the parameters makes it impossible to guarantee $\sum w_j = 1$, as in the classic outranking model. Any realization of the weights is valid only if that condition is fulfilled. So, we must ensure that it can be fulfilled. The following two feasibility restrictions are established with this purpose.

$$\sum_{j=1}^{k} w_j^{-} \leq 1, \tag{2}$$

$$\sum_{j=1}^{k} w_j^{+} \geq 1. \tag{3}$$

The concordance index of the statement "x is at least as good as y", $c(x, y) = [c^{-}(x, y), c^{+}(x, y)]$, is defined as follows. First, it is intuitive to assume that $c^{-} = \sum_{j \in C(xS_\delta y)} w_j^{-}$ and $c^{+} = \sum_{j \in C(xS_\delta y)} w_j^{+}$. Nevertheless, this is valid only if constraints (2) and (7) are fulfilled. To ensure these constraints fulfill the definition of $c(x, y)$, it is necessary to consider the complete set of criteria, $\mathcal{J}$. By definition, this involves contemplating $C(xS_\delta y)$ and $D(xS_\delta y)$. Thus, considering (2) and (7) in the definition of $c(x, y) = [c^{-}(x, y), c^{+}(x, y)]$, Fernandez et al. (2018) define

$$c^-(x,y) = \sum_{j\in C(xS_\delta y)} w_j^-,$$

only if it is true that

$$\sum_{j\in C(xS_\delta y)} w_j^- + \sum_{j\in D(xS_\delta y)} w_j^- \leq 1\text{, and}$$

$$\sum_{j\in C(xS_\delta y)} w_j^- + \sum_{j\in D(xS_\delta y)} w_j^+ \geq 1.$$

Otherwise, $c^-(x,y)$ is defined as

$$1 - \sum_{j\in D(xS_\delta y)} w_j^+.$$

Similarly,

$$c^+(x,y) = \sum_{j\in C(xS_\delta y)} w_j^+,$$

only if it is true that

$$\sum_{j\in C(xS_\delta y)} w_j^+ + \sum_{j\in D(xS_\delta y)} w_j^- \leq 1\text{, and}$$

$$\sum_{j\in C(xS_\delta y)} w_j^+ + \sum_{j\in D(xS_\delta y)} w_j^+ \geq 1.$$

Otherwise, $c^+(x,y)$ is taken as

$$1 - \sum_{j\in D(xS_\delta y)} w_j^-.$$

Fernandez *et al*. (2018) show that $c^+(x,y) \geq c^-(x,y)$, and that if $C(xS_\delta y) = \emptyset$ then $c(x,y) = [0,0]$ and if $C(xS_\delta y) = \mathcal{J}$ then $c(x,y) = [1,1]$.

Let Δ be the set $\left\{\alpha_j \in \mathbb{R}: p\left(g_j(x) \geq g_j(y)\right) = \alpha_j, j = 1,\cdots,k\right\}$. For each $\delta \in \Delta$ Fernandez *et al*. (2018) state that *x* outranks *y* with marginal likelihood index β_δ and majority strength $\lambda = [\lambda^-, \lambda^+]$ ($\lambda^- > 0.5$), if and only if

i. $p(c(x,y) \geq \lambda) \geq \xi$;

ii. $1 - \max_{g_j \in D(xS_\delta y)} \{p(g_j(y) \geq g_j(x) + v_j)\} \geq \xi$; and

iii. $\beta_\delta = \max\{\xi\}$ fulfilling i and ii.

Where $\delta \geq \xi \in \mathbb{R}$. With the above notation, it is said that x outranks y with likelihood index $\beta(x,y) \in [0,1] = max\{B_\delta\}$ ($\delta \in \Delta$) and majority strength $\lambda = [\lambda^-, \lambda^+]$ ($\lambda^- > 0.5$). If Δ is empty, $\beta(x,y)$ is set to zero. Moreover, it is assumed that the DM uses an implicit likelihood threshold $\beta_0 > 0.5$ such that if $\beta(x,y) \geq \beta_0$ then the assertion "x is at least as good as y" is accepted.

Finally, the concept of dominance is also extended in (Fernandez *et al*., 2018). In that work, dominance is not crisp, but there is a "degree of likelihood", α, of the dominance. Let x and y be two solutions and α a real number; y is α-dominated by x, denoted by $xD(\alpha)y$, if and only if $\min_{1 \leq j \leq k} p\left(I_j(x) \geq I_j(y)\right) = \alpha \geq 0.5$.

2.3 Handling uncertainty through confidence intervals

Recently, Solares *et al*., (2018) presented a proposal in which the portfolios are evaluated by means of significant confidence intervals around the portfolios' expected returns. It is stated in that work that the model can capture the attitude of the DM when facing risk by letting her/him to define the probability with which the confidence intervals contain the expected return. That is, if $E(R(x))$ is a random variable that represents the expected return of portfolio x and $P(\omega)$ is the likelihood that event ω will occur, then $\theta_\gamma(x) = [\alpha, \beta]: P(\alpha \leq E(R(x)) \leq \beta) = \gamma$ is the confidence interval around the expected return and, by letting the DM to select γ, we can have an idea of the DM's conservatism in presence of risk. For example, if we assume that the DM is highly risk-averse, then she/he "would feel more satisfied of making a decision based on intervals with a high probability of containing the actual return". On the other hand, if the DM is lowly risk-averse, then she/he "would prefer to make a decision based on intervals that tend to the expected return". Furthermore, it is assumed that several confidence intervals can be used as criteria to find the best portfolio.

The basic idea is that the rightmost intervals are the preferred ones. Therefore, that work proposes using Interval Theory to address the following problem as a specification of

Problem (1) (see Interval Theory's definition of *possibility function* in Subsection 2.2 that determines when an interval number is not less than another):

$$\underset{x\in\Omega}{\text{maximize}}\left(\theta(x)=\left\{\theta_{\gamma_1}(x),\cdots,\theta_{\gamma_k}(x)\right\}\right). \tag{4}$$

Such proposal is an interesting idea given that i) it allows to work with virtually any kind of probability distribution followed by the returns; ii) the confidence intervals are easily understandable for the DMs; iii) each criterion encompasses multiple points within the probability distribution, hence offering more information to the DM per criterion; iv) it allows to incorporate the DM's behavior in the presence of risk.

However, that approach does not allow to consider other types of criteria (e.g., the financial indicators), and does not intend to reflect the DM's system of preferences.

2.4 Managing preferences in multi-objective Optimization

Recently, the interest in incorporating the DM's preferences during the multi-objective optimization process has increased. Fonseca and Fleming (1993) probably suggested the earliest attempt to incorporate preferences; their proposal was to use MOGA together with goal information as an additional criterion to assign ranks to the members of a population. More recently, the idea of measuring the preference-based distance of the potential solutions with respect to a reference point has gained much interest (see e.g., Li et al., 2017; Wickramasinghe, 2010; and Mohammadi, 2013). Nevertheless, some of them are ad-hoc methodologies and/or treat points outside the preferred region as equally redundant. Particularly, the so-called R-metric (Li et al., 2017) is a very recent and interesting idea.

Three classes of multi-objective optimization methods can be identified according to the role of the DM in the solution process when he/she is available (cf. Hwang and Masud, 1979; and Miettinen, 1999). In *a priori* methods, the DM articulates her/his preference information and hopes before the solution process. The difficulty here is that the DM does not necessarily know the limitations and possibilities of the problem and may have too optimistic or pessimistic hopes. Alternatively, a set of Pareto optimal solutions can be generated first and then the DM is supposed to select the most preferred one among them. Typically, evolutionary multi-objective optimization algorithms do this in an *a posteriori* way. However, it may not be suitable for the portfolio optimization problem addressed in this work given that if there are more than three criteria defined as interval numbers in the problem, it may be difficult for the DM to analyze a large amount of information. On the other hand, generating the set of efficient solutions may be computationally expensive. Furthermore, supplying the DM with a large number of trade-off points provides many irrelevant or even noisy information to the decision-making procedure. Another alternative is that, after each iteration, the DM is provided with one or more efficient solutions that obey the preferences expressed as well as possible and he/she can specify his/her preference information on them in such a way that this information is considered for the next iteration. This seems to be the ideal way to incorporate the DM's preferences into the search process. However, there might be situations where the DM is not willing/capable to get involved in the procedure. Hence, one of the other two ways to incorporate the preferences must be implemented.

Besides using different methods to provide preference information, multi-objective optimization algorithms also differ from each other in the type of information that is utilized in generating new, improved solutions and what is assumed about the behavior of the DM. Perhaps the most intuitive one is the weighting method, which assigns a relative importance to each objective: the larger the weight is, the more important the objective is. Zitzler et al. (2007) used this method combined with the hyper-volume indicator (Zitzler et al., 2003) in order to guide the search based on the DM's preferences expressed by weighting coefficients or a reference point. Deb (2003) developed a modified fitness-sharing mechanism, by using a weighted Euclidean distance, to bias the population distribution. In (Branke and Deb, 2005), Branke and Deb modified the crowding distance calculation in NSGA-II by using a weighted mapping method in order to focus the search on the preferred part of the Pareto front. Another method to use the DM's preferences modifies the original Pareto dominance by classifying objectives into different levels and priorities (e.g., Fonseca and Fleming, 1998; Branke et al., 2001; and Jin and Sendhoff, 2002); thus, creating a ranking of the objectives. A convenient way to perform such ranking is by providing the DM with pairs of objectives and asking her/him to provide a decision about which one is the most important. A relevant limitation with this method is that there might be situations where incomparability exists and creating the whole ranking may become a difficult problem to solve. The third approach combines the classical reference point-based method (Wierzbicki, 1980) with evolutionary multi-objective optimization (e.g., Li et al., 2017; Deb et al., 2006; and Said et al., 2010). In such methodology, the DM supplies for each objective the level that should be achieved according to her/his preferences. The reservation level corresponds to the worst value for which the DM is still satisfied (Bechikh et al., 2015). This method is the most commonly used in the related literature (Bechikh et al., 2015). Another methodology to incorporate the DM's preferences is exploiting the outranking concept (Roy, 1996), which states the credibility index of the statement "solution x is at least as good as solution y" (see e.g. Fernandez et al., 2010, 2011). This is a very convenient way to incorporate the DM's preferences since she/he usually considers more information than just the relative importance of the objectives in order to make decisions. The outranking methods can handle intransitive preferences, incomparability, veto effects, and even qualitative and ordinal information for some criteria. Furthermore, through an adequate way of preference parameters elicitation, the DM's decision policy can be reproduced during the optimization procedure in such a way that the most preferred solution can be found and an arduous work by the DM can be avoided.

3 Our proposal

Here, we present an approach that looks for selecting the best portfolio from the decision maker's perspective. To achieve this, we assume that the DM's implicit decision policy can be represented by means of the interval-based outranking approach (Fernandez *et al.*, 2018), whose parameters allow to encompass the imperfect knowledge in the DM's preferences. Moreover, to represent the DM's conservatism in the presence of risk, we use confidence intervals around the expected returns that also consider both the portfolios' most probable returns and the risk of not attaining those returns (see Subsection 2.3). Additionally, the proposed approach aggregates all the original criteria in only one underlying criterion. Such aggregation is on the basis of the DM's particular system of preferences. Therefore, the search process performs a selective pressure towards the DM's most preferred portfolio.

3.1 Incorporating the DM's preferences

First, we assume that the DM's implicit system of preferences can be represented by the set of parameters $\mathcal{P} = \{w_1, \cdots, w_k, v_1, \cdots, v_k, \lambda, \beta_0\}$ that allows to build an interval-based outranking relation between pairs of portfolios (see Subsection 2.2). The elicitation of these parameters can be achieved in two ways. The first is a direct technique based on interactive communication between the decision analyst (entity that facilitates the decision making) and the DM. Helped by the analyst, the DM is in charge to provide the values of the involved parameters. Alternatively, the indirect procedures, like the so-called preference-disaggregation analysis, use regression-like methods for inferring the set of parameters from a battery of decision examples (Doumpos *et al.*, 2009). Although the preference-disaggregation analysis has recently gained much acceptance in the related literature, it rests on two underlying assumptions: i) the DM feels more comfortable making decisions than explaining the arguments that support them; and ii) a sufficient size set of decision examples from the DM is available or can be obtained. On the other hand, the success of the direct elicitation depends, above all, on the DM's willingness to effectively participate in the elicitation procedure. This kind of approach is widely used in situations involving decisions of strategic character (Doumpos and Zopounidis, 2011). The direct elicitation's principal limitation is that we cannot expect the parameters' values provided by the DM to be completely appropriate, according to the DM's system of preferences (cf. Mousseau and Slowinski, 1998).

3.2 A bi-criteria formulation based on Fuzzy Logic

Following (Fernandez *et al.*, 2010, 2011), a portfolio x is said to be strictly-non-outranked if and only if there is not a portfolio y such that y dominates x, or if y outranks x and x does not outrank y. Formally:

$$x \text{ is non-outranked} \Leftrightarrow \neg\exists y\colon yDx \vee (ySx \wedge \neg xSy)$$

$$\equiv \forall y\colon \neg yDx \wedge (\neg ySx \vee xSy).$$

Using μ to denote "truth degree" and the strict negation operator, we can formulate the previous definition in terms of multivalent logic:

$$\mu(x \text{ is strictly non-outranked}) = \mu\left(\forall y\colon \Big(\big(1-\mu(yDx)\big) \wedge \Big(\big(1-\mu(ySx)\big) \vee \mu(xSy)\Big)\Big)\right).$$

Among different logic approaches, we use the so-called Compensatory Fuzzy Logic (Espin et al., 2006, 2011), which has several desirable properties for rational decision-making (see Espin et al., 2014, 2015, 2016). The compensatory logic operators for conjunction have as limits the minimum operator (Zimmermann, 1996). Other compensatory logic operators are the arithmetic mean and the geometric mean. The latter is considered as the simplest among the quasi-arithmetic means (cf. Espin et al., 2006, 2011). Unlike the minimum operator, the geometric mean satisfies the strict growth axiom of the compensatory fuzzy logic (see Espin et al., 2014). In this work, the conjunction and disjunction operators from the compensatory fuzzy logic based on the geometric mean are taken to obtain a non-outranked degree as follows.

Let U be the universe of portfolios within the Pareto front of Problem (1). For each pair $(x, y) \in A \times A$, $A \subseteq U$, it is possible to obtain through the interval-based outranking approach (see Subsection 2.2): i) a likelihood degree of the assertion "x outranks y", denoted by $\beta(xSy)$; ii) a likelihood degree of the assertion "y outranks x", denoted by $\beta(ySx)$; and iii) a likelihood degree of the assertion "x dominates y", denoted by $xD(\alpha)y$. Now, let us make $\mu(xSy) = \beta(xSy)$, $\mu(ySx) = \beta(ySx)$, $\mu(xDy) = xD(\alpha)y$, $\hat{A} = A - \{x\}$ and $n = card(\hat{A})$; then we define the *non-outranked truth degree* of x in A by means of the compensatory fuzzy logic based on the geometric mean as (cf. Espin *et al.*, 2014):

$$NS_A(x) = \sqrt[n]{NS(x, \hat{A})}.$$

Where

$$NS(x, \hat{A}) = \prod_{y \in \hat{A}} \sqrt{\big(1 - \mu(yDx)\big)\left(1 - \sqrt{\big(1 - (1 - \mu(ySx))\big)\big(1 - \mu(xSy)\big)}\right)}.$$

A high non-outranked degree indicates the lack of arguments to believe that there are better solutions than x. On the other hand, a high non-outranked degree is a necessary condition to be the best compromise, but it is not sufficient. A solution may have a high non-outranked degree and be incomparable with all or many of the solutions in the known Pareto front. Positive arguments are required to affirm the superiority of x over the other optimal solutions under consideration.

In order to enhance the preference information, here we suggest to use the outranking net flow score. This is a very popular measure to rank a set of alternatives on which a fuzzy preference relation is defined (cf. Fodor and Roubens, 1994). If $\beta(xSy)$ is an outranking likelihood on the set A, then the net flow score associated to $x \in A$ is defined as $F_n(x) = \sum_{y \in A - \{x\}} \big(\beta(xSy) - \beta(ySx)\big)$. Note that $F_n(x) > F_n(y)$ is an asymmetric and transitive binary relation on A, indicating to some extent preference of x over y. So, the net flow score may be used to select the most satisfactory solution between x and y when $NS_A(x) = NS_A(y)$. Nonetheless, always that $NS_A(x) > NS_A(y)$, the DM can be confident that portfolio x provides her/him more satisfaction than portfolio y, regardless of the values of $F_n(x)$ and $F_n(y)$. Therefore, a best compromise solution can be found through a lexicographic search.

Taking into account the non-outranking truth degree and the net flow score information, we propose to select the best solutions for Problem (1) as the non-dominated set obtained from:

$$\underset{x \in \Omega}{\text{maximize}} \big(NS_A(x), F_n(x)\big), \tag{5}$$

with preemptive priority favoring $NS_A(x)$.

By incorporating the DM's preferences this way, a selective pressure towards the most preferred portfolio is produced. Furthermore, the decision maker's cognitive effort in the final selection is reduced since he/she no longer considers the k criteria in Problem (1) but just two criteria in Problem (5).

4 An illustrative example: a highly risk-averse investor interested in many criteria

We present in this section a study case where the proposed model is used to create portfolios on the basis of many criteria. We assume that the investor is interested in using confidence intervals (Subsection 2.3), fundamental indicators (Subsection 2.1.2), and technical indicators (Subsection 2.1.3) as underlying criteria with the objective of maximizing the portfolio's return. This assumption reflects that there are some scenarios where the DM is not fully satisfied with the information provided by the statistical analysis nor by the financial analyses. Of course, the assumption made in this illustrative example can be adjusted according to the specific context and requirements from the DM.

4.1 Problem definition

Although the fundamental and technical analyses are widely used by investors in the real world, the combination of both types of analyses is not common in the academic literature (see Section 1). Even less common is the combination of fundamental analysis, technical analysis and decisions on proportions in which the resources should be allocated. Nevertheless, the statistical information might not be available/reliable, and/or the financial information might not be enough to involve the risk caused by volatility. Hence, the DM would consider valuable to perform a portfolio optimization where all the criteria are combined in a multi-criteria optimization problem following her/his own decision policy. To the best of our knowledge, there are no published works in the literature that consider the three analyses in a multi-criteria optimization problem, and that is also capable of representing the DM's attitude facing risk as well as her/his decision policy in the context of portfolio optimization with many criteria.

There are papers in which some of these analyses are used consecutively (e.g., Xidonas *et al*., 2009; and Flotynski, 2016). Typically, the fundamental analysis is performed first in order to select the stocks that will be in the portfolio; the technical analysis is performed secondly to determine the time convenience of investing in each stock; finally, the portfolio creation analysis is carried out afterwards to define the allocation proportions to be assigned. Thus, the value of each financial indicator is evaluated individually for each stock. Since our purpose is to select the approximation to the best portfolio, our solution alternatives are not individual stocks, but portfolios. Hence, we use here a way of evaluating portfolios through financial indicators; such way is on the basis of fuzzy logic. Particularly, Fuzzy Logic is used to define the truth degree of each stock being "good" according to the financial indicators. Let us now present our proposed method to determine if a stock is good, first from the fundamental analysis' viewpoint and later from the technical analysis' viewpoint.

4.1.1 Fundamental analysis as a criterion to evaluate portfolios

In order to determine a comprehensive quality index of a portfolio with respect to the fundamental indicators, we first need to assess each of the stocks within the portfolio. We assume that a stock is good from the fundamental analysis' viewpoint if the following two conditions are fulfilled:

i. In a significant majority of the fundamental indicators considered, the value of each indicator reaches a sufficiently high level.
ii. No indicator has a value significantly lower than certain threshold n_2.

To model the truth degree of condition i, it is only necessary to define what the DM understands for "an important majority" (represented by a relative value δ) and "a sufficiently high level" (represented by a relative number n_1). A piecewise linear function H can be used here, where the independent variable is the proportion p of indicators that reach the level n_1, and fulfills: a) $H = 0$ if p is not greater than 0.5, b) H linearly increases to 1 when p grows from 0.5 to δ, and c) H is 1 for values of p not lower than δ. Thus, H can be defined as follows:

$$H = \begin{cases} 0 & \text{if} \quad p \leq 0.5, \\ (p - 0.5)/(\delta - 0.5) & \text{if} \quad 0.5 < p < \delta, \\ 1 & \text{if} \quad p \geq \delta. \end{cases}$$

The truth degree of condition ii can be modeled by a piecewise linear function f_j^i where the value of the jth fundamental indicator when analyzing the ith stock, $value_j^i$, is the independent variable and f_j^i has the following characteristics: a) f_j^i is zero when the value of the indicator is not greater than the level n_{veto}, b) f_j^i linearly increases to 1 when the value of the indicator moves from n_{veto} to n_2, and c) f_j^i is 1 when the value of the indicator is not less than n_2. Thus, f_j^i can be defined as follows:

$$f_j^i = \begin{cases} 0 & \text{if} \quad value_j^i \leq n_{veto}, \\ (v - n_{veto})/(n_2 - n_{veto}) & \text{if} \quad n_{veto} < value_j^i < n_2, \\ 1 & \text{if} \quad value_j^i \geq n_2. \end{cases}$$

The truth degree which, for every indicator, should have a value greater than or equal to n_2 when evaluating the ith stock, is obtained by the conjunction of the values of all f_j^i. An evident compensation exists among such values. Hence, we propose to use the conjunction of the compensatory fuzzy logic based on the geometric mean (cf. Espin *et al.*, 2006, 2011, 2014) following the reasons exposed in Section 3.2.

Finally, the truth degree of the ith stock being good from the fundamental analysis' viewpoint, F_i, is obtained by the conjunction of the truth values of conditions i and ii. There is no compensation in such conjunction. Hence, we propose to use here the product norm as the conjunction operator.

The aggregation to evaluate the portfolio from this viewpoint then becomes:

$$\bar{F}(x) = \sum_{i=1}^{n} F_i x_i.$$

4.1.2 Technical analysis as a criterion to evaluate portfolios

The evaluation of individual stocks using the technical analysis described in Section 2.1.3 consists in finding the convenience of investing in the stocks. Particularly, if the rule associated to the jth technical indicator states that the ith stock is good, then such indicator takes a value of 1 ($it_j^i = 1$), otherwise its value is 0 ($it_j^i = 0$). Therefore, the aggregation

$$T_j(x) = \frac{\sum_{i=1}^{n} x_i it_j^i}{n},$$

represents the desirable momentum proportion of the stocks supported by the portfolio x from the jth technical indicator perspective. A final aggregation of the technical indicators can be performed to obtain the goodness of portfolio x in the technical analysis' viewpoint:

$$\bar{T}(x) = \frac{\sum_{j=1}^{m} T_j}{m},$$

where m is the number of technical indicators considered.

4.1.3 Multi-criteria optimization problem

Of course, there is some uncertainty involved in the definitions of $\bar{F}(x)$ and $\bar{T}(x)$ originated in the finite-precision arithmetic provided by computers. Hence, we take advantage of Interval Theory and redefine the financial indicators as interval numbers:

$$F(x) = [F(x)^-, F(x)^+]. \quad (6)$$

Where $F(x)^-$ is $F(x)$ rounded down to four digits, and $F(x)^+$ is $F(x)$ rounded up to four digits. The same procedure is followed with the technical indicators to create $\bar{T}(x)$:

$$T(x) = [T(x)^-, T(x)^+]. \quad (7)$$

Both $F(x)$ and $T(x)$ can be seen as quality indexes indicating the convenience of investing in portfolio x.

On the other hand, a highly risk-averse DM can be simulated as the one who requires information about two confidence intervals: one containing the expected return with a 70% of probability, $\theta_{\gamma_{70}}(x)$, and another containing it with the 99% of probability, $\theta_{\gamma_{99}}(x)$ (see Subsection 2.3). Here, three constraints are used: budget, non-negativity, and bounds on individual stocks constraints.

Therefore, we validate the proposal presented in Section 3 through its solutions' performance when solving the following multi-criteria problem on the basis of confidence intervals and financial indicators:

$$\max_{x\in\Omega}\left(\theta_{\gamma_{70}}(x), \theta_{\gamma_{99}}(x), F(x), T(x)\right). \tag{8}$$

Subject to

$\sum x_j = 1 \longrightarrow$ Budget constraint;

$x_j \geq 0 \longrightarrow$ Non-negativity conditions;

$x_j \leq 0.4 \longrightarrow$ Bounds on individual stocks.

Where

x_j is the proportion of resources allocated to the jth stock,

$\theta_{70}(x) = \{[\alpha_{70}, \beta_{70}]: P(\alpha_{70} \leq R(x) \leq \beta_{70}) = 0.70\}$,

$\theta_{99}(x) = \{[\alpha_{99}, \beta_{99}]: P(\alpha_{99} \leq R(x) \leq \beta_{99}) = 0.99\}$,

$R(x)$ is a random variable representing the actual return of portfolio x,

$F(x)$ is the evaluation of portfolio x from the fundamental analysis' viewpoint,

$T(x)$ is the evaluation of portfolio x from the technical analysis' viewpoint, and

$j = 1, \cdots, n$.

Given the exponential increase in the number of solutions required for approximating the entire Pareto front of Problem (8), an incorporation of the DM's preferences is convenient (Ishibuchi *et al.*, 2008). Several authors (e.g., Deb *et al.*, 2006; Deb and Sundar, 2006; and Ishibuchi *et al.*, 2008) argue that it is frequent for evolutionary multi-criteria optimization methodologies to suffer serious difficulties when dealing with four or more criteria. One of these difficulties is the need of a larger number of points to represent a higher-dimensional Pareto optimal front. Such difficulty is worsened when the criteria in the optimization problem are defined as interval numbers. Therefore, finding a preferred and smaller set of Pareto-optimal solutions, instead of the entire frontier, tends to be beneficial for the search process (Deb and Sundar, 2006) and it can be achieved by incorporating preference information (Fleming *et al.*, 2005; Thiele *et al.*, 2007).

4.2 Experimental design

In order to address Problem (8), we use the interval-based outranking approach (Subsection 2.2) to model the DM's preferences and to build the aggregation described in Section 3.2. Since that approach allows the DM to provide imprecise values for its parameters, it is relatively easy to obtain such values directly from the DM. However, in general terms the elicitation of a preference model's parameters comprises some part of arbitrariness, imprecision, and ill-determination (Fernandez *et al.*, 2018). According to Fernandez *et al.* (2018), this is particularly true when "the entity in charge of the decision is a group where its members disagree concerning the parameter values, or when the decision-maker is a mythical or an inaccessible person". Consequently, in the experiments described below we assume that although the values of the interval-based outranking's parameters are directly elicited, there exists some deviation from the most appropriate parameters' values according to the DM's actual system of preferences (see Subsection 3.1 for a further rationalization).

During the experiments, we first generate 20 decision models at random that represent 20 decision makers' decision policies. That is, we create 20 sets of parameters $\mathcal{P}^i = \{w_j^i, v_j^i, \lambda^i, \beta_0^i\}$ $(i = 1, \cdots, 20;\; j = 1, \cdots, 4)$. The values of the parameters to create each $\mathcal{P}^i$ are uniformly randomly taken from ranges of numbers that work as sources. Such sources are shown in Table 2. Recall that β_0^i is the only real number of the interval-based outranking model's parameters, whereas the rest of parameters are defined as interval numbers. Thus, the sources in the rows λ^i, w_j^i and v_j^i are actually used for each of these parameters' bounds. Of course, it is satisfied for the lower bounds of these parameters to be not greater than their respective upper bounds. Particularly, we calculate the weight of criterion g_i as $w_i^- = (1 - \omega_i)\,\widehat{w}, w_i^+ = (1 + \omega_i)\,\widehat{w}$, where ω_i is randomly generated in [0,0.3] and $\widehat{w} = \frac{1}{4}$. In Table 2, the value v_{max}^j is used to represent the maximum impact in the jth criterion of a set of 2000 randomly created portfolios. Constraints (2) and (3) settled by the interval-based outranking are also fulfilled in the creation of each $\mathcal{P}^i$.

Table 2. Sources used to uniformly randomly assign values to the parameters of the interval-based outranking

Parameter	Source
β_0^i	(0.5,0.6)
λ^i	(0.5,0.6)
w_j^i	(0,1)
v_j^i	$(0.3,0.5)v_{max}^j$

Subsequently, we randomly deviate each parameter (bound in the case of the interval numbers) in the simulated decision models between 0.1 and 0.3 to obtain 20 new sets $\mathcal{P}^{i'} = \left\{w_j^{i'}, v_j^{i'}, \lambda^{i'}, \beta_0^{i'}\right\}$ $(i = 1, \cdots, 20,\ j = 1, \cdots, 4)$. These sets simulate the values directly elicited from the DMs and they are used as the actual decision policies in the experiments below. It is plausible to assume that the DM has to be satisfied with $\mathcal{P}^{i'}$. Thus, we assume that she/he requires i) the importance of the confidence intervals to be greater than the importance assigned to the rest of criteria, since she/he is considered to be highly risk-averse; and ii) the order of importance assigned to the criteria in $\mathcal{P}^i$ must be respected in $\mathcal{P}^{i'}$.

For each new set $\mathcal{P}^{i'}$ we obtain the best compromise portfolio in the following way. First, the approximation to the returns' probability distribution in the form of confidence intervals is obtained by Montecarlo simulation; this simulation uses the stocks' historical monthly returns of 36 periods as input and runs 200 statistical points using a pseudo-random numbers generator known as Mersenne Twister (see Matsumoto and Nishimura, 1998 and 2004). Later, the evaluation of the financial indicators for the portfolio is obtained using Eqs. (6) and (7). Once the portfolios' fitness of a subset of candidate solutions has been achieved with respect to Problem (8), each portfolio's fitness is aggregated using the DM's parameters, $\mathcal{P}^{i'}$, and the procedure described in Subsection 3.2. Finally, the set of best compromise solutions to Problem (8) is composed with the nondominated solutions to the underlying Problem (5).

4.3 Dataset

We use the historical monthly returns of the stocks in the DJIA index for the period April 2011-March 2016 to perform a back-testing strategy (cf. Ni and Zhang, 2005); the evaluation of the approach is in the period April 2014-March 2016 (3 years of data are used as the training period for the statistical simulation of returns). That time span is recent, it has several upward, downward, and horizontal market's movements, so it is interesting to analyze it. We use here a sliding time window of 36 months/1 month, similarly to (Lim *et al*., 2014) and (Gorgulho *et al*., 2011) to perform the back-testing. That is, we use three years for training the statistical model (e.g., we obtain metrics of the data set April 2011 to March 2014) and one month for validation (we will use the metrics obtained to create a portfolio in April 2014). The process is then repeated for each period of one month (in a sliding window manner) until the end of the evaluation period. In other words, we select the best stock portfolio of the current month by using the historical metrics of the previous three years, solving Problem (8), and maintaining the portfolio over a one-month investment horizon. Each time we start a new investment horizon, we review the stock portfolio (i.e., we select a new distribution of resources among the stocks) according to the corresponding horizon's valuation.

As done by many authors (e.g., Lwin *et al*., 2017; Almahdi and Yang, 2017; Cesarone *et al*., 2013; Gorgulho *et al*., 2011; Zhu *et al*., 2011), the historical prices used to estimate the returns' probability distribution and to calculate the technical indicators, as well as the returns of the index are obtained from (Yahoo, 2018). And, as done by other works (e.g., Falkenstein, 1996; and Moneta, 2015), the financial data to calculate the fundamental ratios is obtained from (Morningstar, 2018). All the data used, together with all the results obtained, are available for consultation upon request.

4.4 Algorithm

We now present the algorithm used during the assessment of our approach to address Problem (5). Without loss of generality, we assume here that i) the DM's preferences can be represented through the interval-based outranking, ii) the optimization method used to exploit the idea implicit in Problem (5) is Differential Evolution, and iii) the concrete Portfolio Optimization Problem that the decision maker wants to address is the one presented in the illustrative example and formulated in Problem (8). Furthermore, we present the algorithm used to address Problem (5), not the one to assess our approach. Thus, we consider only one of the 20 instances mentioned in Section 4.2 and only one of the 24 periods of time mentioned in Section 4.3.

Since Problem (5) was raised as a lexicographic non-linear optimization problem, we use here differential evolution to address it. Such metaheuristic generally has good performance in non-linear single-objective optimizations (see e.g., Krink and Paterlini, 2011; and Krink et al., 2009). The differential evolution algorithm applied here uses $ps = 100$ individuals as its population size, its stopping criterion is the achievement of $gn = 100$ generations, and it uses an n-dimensional real-valued vector to encode the individuals. Recall that differential evolution requires the settlement of four additional control parameters (cf. Li and Zhang, 2009): the crossover probability, CR; the mutation rate, p_m; the differential weight, F; and the distribution index, η. As done in the mentioned work, here we set these parameters as $CR = 1$, $p_m = 1/ps$, $F = 0.5$, and $\eta = 20$. Algorithm 1 presents the pseudocode of our algorithm.

Require: DM's preferences (described through the interval-based outranking method; that is, $\mathcal{P} = \{w_j^i, v_j^i, \lambda^i, \beta_0^i\}$ $(j = 1, \cdots, 4)$), Problem context (Probability distributions of the stocks returns, Fundamental indicators for each stock, Technical indicators for each stock)

Ensure: Set of portfolios recommended by our approach as the *best portfolios*, ρ_{best}.

1. $i \leftarrow 1$
2. **For** $i \leq ps$ **do**
3. $g \leftarrow 0$
4. $P_g \leftarrow$ CreateInitialPropulation()
5. **For** $g \leq gn$ **do**
6. **For** $j \leq ps$ **do**
7. $H_g^j \leftarrow$ CreateOffspringIndividual(P_g, $selection$, $crossover$, $mutation$)
8. $P_g^j \leftarrow$ SelectBestIndividual(P_g^j, H_g^j)
9. $j \leftarrow j + 1$
10. **End for**
11. $g \leftarrow g + 1$
12. **End for**
13. $\rho_i \leftarrow$ SelectBestSet(P_g)
14. $i \leftarrow i + 1$
15. **End for**
16. $g \leftarrow 0$

$P_g \leftarrow \{\rho_1, \rho_2, \cdots, \rho_L\}$

For $g \leq gn$ **do**

    For $j \leq ps$ **do**

        $H_g^j \leftarrow \text{CreateOffspringIndividual}(P_g, selection, crossover, mutation)$

        $P_g^j \leftarrow \text{SelectBestIndividual}(P_g^j, H_g^j)$

        $j \leftarrow j + 1$

    End for

    $g \leftarrow g + 1$

End for

$\rho_{best} \leftarrow \text{SelectBestSet}(P_g)$

Algorithm 1 first creates an initial population by randomly sampling from Ω. Always that we create an individual, we calculate its fitness in the sense of Problem (8). That is, we estimate its confidence intervals using the probability distributions of the stock returns and its quality indexes according to the financial analyses. Then, for each generation j and for each individual P_g^j of P_g, the individual creates an individual H_g^j by applying the Selection, Crossover and Mutation operators of Differential Evolution. After that, P_g^j and H_g^j are compared on the basis of Problem (5). The best individual is now P_g^j. These steps constitute one generation; we perform $gn = 100$ generations. After performing this number of generations, we obtain the set of individuals ρ_i (likely with cardinality of one) within P_g whose fitness is the best in the sense of Problem (5). All the previous is considered as the ith run. Several runs (up to ps) are performed to obtain a "seed population" of size ps whose individuals are the solutions found in the previous runs. A final run is performed using the seed population as the initial population. The set of best compromise solutions (likely with cardinality of one) to Problem (5) in this final run is presented to the DM as the best portfolios.

The main difference between Algorithm 1 and other approaches is the exploitation of the non-outranked truth degree, NS_x, to select the best portfolio according to the decision maker's preferences. NS_x is used as the representative value that reflects the overall satisfaction of x with respect to a set of portfolios according to the decision maker supplied preference information. Such preference information is in terms of the interval-based outranking method; that is, we consider the weights assigned by the decision maker to each criterion, his/her veto values and his/her thresholds about when a solution is at least as good as another.

4.5 Results

In order to evaluate the performance of our approach, we analyze its recommended solutions to Problem (8) and assess them with respect to some benchmarks in two contexts: the criteria space and the objective space. In the former, we make a comparison in terms of the criteria considered in Problem (8). In the latter, the comparison is in terms of the actual returns of the solutions.

4.5.1 Performance in the criteria space

Here, we compare our portfolios with the ones constructed by MOEA/D, which is a state-of-the-art multi-objective evolutionary algorithm based on decomposition (see Zhang and Li, 2007; and Li and Zhang, 2009). The goal of such comparison is to provide a reference to the capacity of the approach proposed in Section 3 to deal with many criteria and obtain satisfactory solutions.

During the exploitation process of MOEA/D, the individuals are represented as real vectors and three randomly selected individuals are used for the crossover operator. The crossover operator works as follows. Let qG_1, qG_2, qG_3 be the quantity of genes satisfying $x_i > 0$ in parent 1, parent 2 and parent 3, respectively. The idea is that the parents provide similar proportions of genetic material to the offspring. So, the number of genes satisfying $x_i > 0$ in the child solution is up to $qG_C = \frac{qG_1+qG_2+qG_3}{3}$ and each parent gives $\frac{qG_C}{3}$ randomly chosen genes to the offspring solution. The mutation operator simply consists in swapping two randomly chosen genes of the offspring solution. The probability of mutation is $p_m = 0.01$. In preliminary experiments, we found that discarding the infeasible solutions is the most suitable method to obtain solutions with good performance. The Tchebycheff method is used to aggregate the criteria (cf. Zhang and Li, 2007). The dataset described in Subsection 4.2 and the constraints defined in Problem (8) are used here to create the benchmark portfolios.

Both approaches achieve a good approximation to the Pareto front with respect to each other, since they produce a high number of non-dominated solutions (see Subsection 2.2 for the definition of interval-based dominance). From the approximations achieved by the approaches in each of the 24 periods of the dataset, roughly 0.7% of the solutions found by MOEA/D are dominated by at least one of our solutions, and roughly 0.3% of our solutions are dominated by at least one of the solutions from MOEA/D.

It is now interesting to know how "good" the constructed portfolios are from the DM's perspective. Thus, we use the actual system of preferences of the DMs, $\mathcal{P}^i$ $(i = 1, \cdots, 20)$, to compare the solutions built by both approaches. Given that each $\mathcal{P}^i$ already has all the parameters needed by the interval-based outranking approach, the comparison of the solutions is on the basis of such method. Our intention is to find the proportion of times that the strict outranking relation is met between the portfolios created by the proposed approach and the benchmark portfolios, according to the discussion of Subsection 2.2. Just to provide an example, Table 3 shows one (arbitrarily chosen) portfolio from MOEA/D's Pareto front and a portfolio built by the proposed approach for one (also arbitrarily chosen) DM in the period April 2014. As it was specified in Section 2.1, the unit used in Table 3 is the proportion of money to invest in each of the n investment objects. Table 4 shows their respective performances in the criteria, and Table 5 shows the chosen DM's system of preferences.

Table 3. Arbitrarily chosen portfolios built in the period April 2014 using MOEA/D and our approach.

Stock	MOEA/D	Our Approach
American Express Company (AXP)	0	0
Boeing Co. (BA)	0	0
Caterpillar Inc. (CAT)	0	0
Cisco Systems, Inc. (CSCO)	0	0
Chevron Corporation (CVX)	0	0
EI du Pont de Nemours and Co (DWDP)	0	0
Walt Disney Company (DIS)	0	0
General Electric Company (GE)	0	0
Goldman Sachs Group Inc. (GS)	0	0
Home Depot, Inc. (HD)	0.08	0

International Business Machines Corporation (IBM)	0	0.256
Intel Corporation (INTC)	0.34	0.03
Johnson and Johnson (JNJ)	0	0
JPMorgan Chase and Co. (JPM)	0	0
Coca-Cola Company (KO)	0.24	0
McDonald's Corporation (MCD)	0.03	0.349
3M Co. (MMM)	0	0
Merck and Co., Inc. (MRK)	0.30	0
Microsoft Corporation (MSFT)	0	0
Nike Inc. (NKE)	0	0
Pfizer Inc. (PFE)	0	0
Procter and Gamble Co. (PG)	0	0
ATandT Inc. (T)	0	0.365
Travelers Companies Inc. (TRV)	0	0
UnitedHealth Group Inc. (UNH)	0	0
United Technologies Corporation (UTX)	0	0
Visa Inc. (V)	0	0
Verizon Communications Inc. (VZ)	0	0
Wal-Mart Stores Inc. (WMT)	0	0
Exxon Mobil Corporation (XOM)	0	0

Table 4. Evaluation in the criteria space of the portfolios built by the approaches

Criterion	MOEA/D	Our Approach
70 percent confidence interval	[-0.0108, 0.0293]	[-0.0046, 0.0270]
99 percent confidence interval	[-0.0372,0.0469]	[-0.1747, 0.0452]
Fundamental analysis' quality index	[0.3537,0.3538]	[0.8268,0.8269]
Technical analysis' quality index	[0.3414,0.3415]	[0.2244,0.2245]

Table 5. System of preferences of the arbitrarily chosen DM

β_0	λ	w_1	w_2	w_3	w_4	v_1	v_2	v_3	v_4
0.55	[0.51,0.56]	[0.18,0.24]	[0.56,0.78]	[0.05,0.07]	[0.05,0.06]	[0.03,0.04]	[0.10,0.12]	[0.27,0.32]	[0.10,0.13]

Now, let y and x be the portfolios shown in Table 3, which were built using MOEA/D and the approach proposed here, respectively. According to the interval-based outranking method described in Subsection 2.2, the likelihood indexes that the DM (in Table 5) assigned to these portfolios are shown in Table 6.

Table 6. Evaluation of the strinct outranking relation between the solutions shown in Table 3. y: portfolio created by MOEA/D, x: portfolio created by the proposed approach.

Comparison	Value	Strictly outranks
$\beta(x, y)$	0.56	Yes
$\beta(y, x)$	0.0	No

From Table 6 we can deduce that, although there is no dominance between the portfolios, the DM is more satisfied with the portfolio created by the proposed approach than with the benchmark portfolio.

We now perform the same analysis for every simulated DM and for every period in the dataset to have an idea of how satisfied the DM would be on average with the solutions provided by the proposed approach relative to the benchmark. The results are presented in Table 7. This table presents the proportion of times that the solutions of the proposed approach were at least as good as the benchmark's solutions according to the DM's actual system of preferences, $\mathcal{P}^i$, for the 20 simulated DMs, and for the 24 periods of the dataset. In each period, the performance of the solution created by our proposal for each DM is compared to that of each solution in the Pareto front approximated by MOEA/D. In this table, x represents the solutions provided by the proposed approach and y represents the solutions provided by MOEA/D.

Table 7. Comparing the solutions x provided by the proposed approach and the solutions y provided by MOEA/D.

Proportion of times that xSy is met	Proportion of times that ySx is met
0.1242	0.0904

The *paired Wilcoxon test* performed indicated that the difference of these means is considered to be statistically significant.

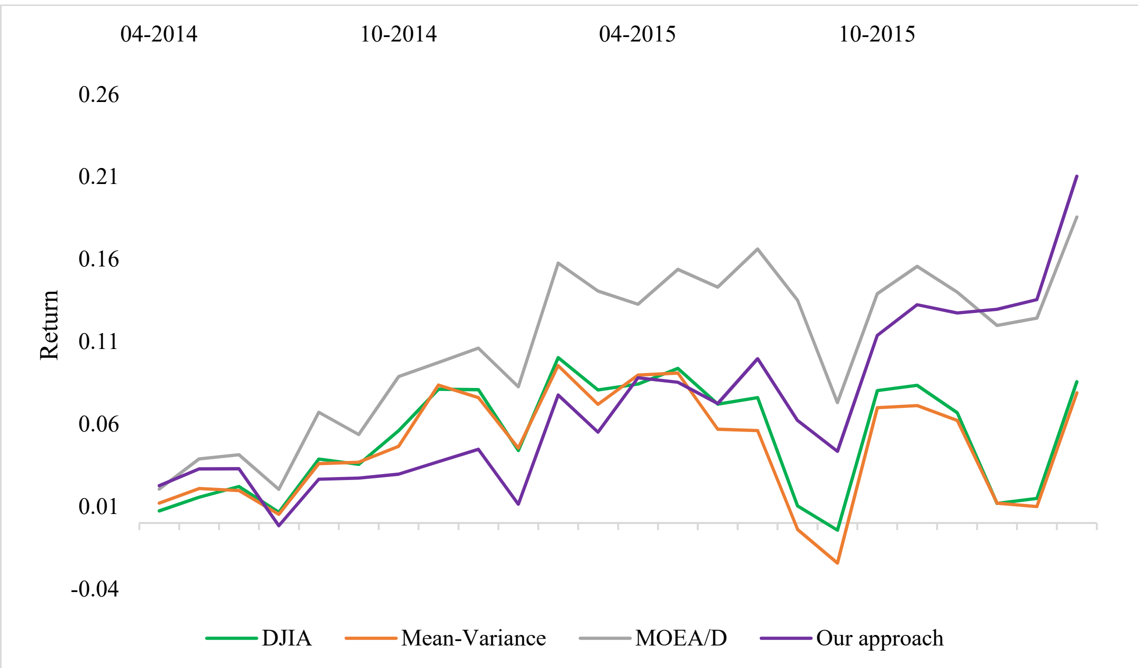
From Table 7 we can state that the proposed approach was able to find more satisfactory solutions than the benchmark algorithm. Of course, this comparison is on the basis of the criteria contemplated in Problem (8). However, we can expect similar results of comparisons on the basis of alternative criteria and defer the validation of such hypothesis for future work.

Finally, it is important to note here that the criteria contemplated in Problem (8) can be easily adjusted to the DM's specific requirements.

4.5.2 Performance in the objective space

In this subsection, we contrast the actual returns of the portfolios constructed by the proposed approach with those of three benchmarks: the Dow Jones Industrial Average (DJIA) index, the mean-variance approach (Markowitz, 1958), and the portfolios built using MOEA/D (Zhang and Li, 2007). The problem addressed by the optimization methods (i.e., the mean-variance approach, MOEA/D and the approach proposed here) is provided in (8). The results are presented in Figure 1. This figure shows the accumulative monthly returns of the portfolios in the 24 periods span between April 2014 and March 2016. In the case of the mean-variance approach and MOEA/D, the accumulative return shown in Figure 1 corresponds to the average returns of the portfolios in their specific Pareto front. For the accumulative return of the DJIA index, we used the monthly returns published in (Yahoo, 2018). In the case of our approach, the accumulative return is formed with the average returns of the portfolios obtained for the 20 DMs.

Figure 1. Actual accumulative returns of the benchmark portfolios and the portfolios built by the proposed approach.



Clearly, the proposed approach outperforms both the mean-variance approach and the DJIA index in these experiments. Specifically, the average return of the portfolios created for the 20 DMs is higher than the return of the DJIA index in 18 of the 24 periods; and it is higher than the average return of the portfolios in the Pareto front found by the mean-variance approach in 14 of the 24 periods. The accumulative return of our approach after the 24 periods was 0.2105, while the accumulative returns of the DJIA index and the Mean-Variance approach were 0.0858 and 0.0791, respectively.

In other comparison, we can see similar average performances between MOEA/D and the proposed approach. Such similarity is originated in both approaches using the same criteria. However, the proposed approach generated the best accumulative return in the whole-time span (the average accumulative returns produced by MOEA/D was 0.1857); this indicates that the results with incorporation of DM's preferences have been, on average, more effective than the optimization without including preferences. It would be interesting to determine the DM's decision policies that generate the best returns. For example, a preliminary analysis suggests that when the simulated DMs fulfill a specific pattern in their systems of preferences the returns tend to grow. Some characteristics of the systems of preferences with such pattern are i) a greater importance assigned to the confidence intervals than to the financial indicators; ii) a greater importance assigned to the interval with the lower probability of containing the actual return; iii) a greater importance assigned to the fundamental analysis' quality index than to the technical analysis' one; and iv) higher relative values for the vetoes assigned to the financial indicators. Given that the main goal of this Section is to provide an illustrative example of the proposed methodology (Section 3), we defer the analysis of the previous and similar assertions for future work.

In the context of the comparison of MOEA/D with our approach for a given period of time, it is important to highlight that the performance of the first is the average return of the portfolios in its Pareto front approximation. Thus, the performance of MOEA/D shown in Figure 1 could be seen as the average performance of several attitudes facing risk, while the performance of our approach is the only portfolio representing a highly risk-averse investor. It is plausible to assume that if the market presents an uptrend (as slightly seen in the considered time span) then the return of the solutions created by MOEA/D should be, in average, better to the solution of our approach.

Finally, the proposed approach has good behavior in the presence of losses, specifically in the periods of August and September 2015 where the steepest fall of the market occurred. This indicates us a satisfactory protection against risk. Moreover, the approach is taking evident advantage of the market upturns, what indicates that it is also capable of finding uptrend opportunities.

5 Conclusions

We presented in this paper a method that aims to build portfolios whose evaluations in the considered criteria satisfy the investor's preferences. The methodology allows the investor to specify as many criteria as she/he requires, by aggregating all the criteria through fuzzy logic in a bi-criterion optimization problem. Such problem incorporates the investor's preferences to perform a guided search to the most preferred portfolio, thus reducing the investor's cognitive difficulty to select the final portfolio. Finally, the uncertainty involved in the investor's preferences as well as in the actual return of the portfolio are both encompassed as ranges of numbers through the so-called interval theory.

An illustrative use case was provided in the context of stock portfolio optimization. The proposed method was assessed considering the investor's own system of preferences and the actual returns of the solutions provided. A comparison with some well-known benchmarks, the Dow Jones Industrial Average index, the mean-variance method, and an approach based on MOEA/D, was performed. We conclude from Tables 6 and 7 that, for the illustrative example, the proposed approach was able to deal with many criteria and, at the same time, find solutions that satisfy the investor's preferences better than the solutions provided by the corresponding benchmark. From Figure 1, we conclude that the proposed approach was able to outperform the benchmarks in the objective space; that is, it found portfolios with greater actual returns. It can also be seen from this figure that our proposal suffered less aggressive falls than the benchmarks (showing good handling of the involved risk) and that it had better exploitation of the rises (showing a good identification of opportunities).

The above remarks indicate us that the proposed approach i) allows the DM to deal with as many criteria as she/he considers necessary to satisfy her/his requirements of information (of course, the method allows the hypothesis used in the illustrative example to be specifically suited to the actual DM's necessities); and ii) uses appropriate elements to construct portfolios that maximize the impact on both the underlying criteria and the final objective in risky and non-risky environments.

Future lines of work are related to i) the analysis of the specific characteristics of the investor's system of preferences that allow to obtain greater returns; ii) the assessment of the proposed approach in different contexts with respect to the number and/or kind of criteria, the nature of investment objects within the portfolios, and the number of elements in the investment objects universe; iii) the consideration of specific useful characteristics of the portfolio problem, such as multi-period optimization and a higher number of constraints. Perhaps the most important limitation of the proposed approach is the setting of parameter values for the interval-based outranking approach. It might be difficult for the investor to provide the most appropriate values even when they can be defined as interval numbers. Thus, it is necessary to infer the preference model's parameters from a set of decision examples in order to fulfill the requirements of the proposed approach without imposing an arduous work to the investor.

We emphasize that although the main proposal was applied in the context of resource allocation, there is a wide range of problems where the proposal can be applied. The general characteristics of such problems are: i) requirement of the preferences of a decision maker to make the final decision, ii) consideration of many criteria, iii) uncertainty in the preferences of the decision maker and/or the impact of the solution alternatives in the criteria.

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