Complexity Theory 2010
(Home work 1)

June 4, 2010

- Due on Monday, June 21, before 10 a.m.
- Late home works will not be accepted.
- Please give precise arguments for all statements that you write.
- Please do not hesitate to contact me if you do not understand the problems.
- Collaboration is discouraged, but not prohibited. It is recommended that you try to solve the problems on your own. You can discuss the questions with your colleagues but you should not copy solutions. Always write down your own answers. If copying is detected that may immediately lead to a grade less than 7. (**This would be followed strictly**)
- Credits would be given to partial solutions also.
- The answers should be typed or written clearly and a hard copy is to be submitted.

1. **Warm up problems**. [60 points.]

   (a) If $L_1, L_2 \in \mathbf{NP}$, is $L_1 \cup L_2 \in \mathbf{NP}$?

   (b) If $L_1, L_2 \in \mathbf{NP}$, is $L_1 \cap L_2 \in \mathbf{NP}$?

   (c) If $L_1 \leq_p L_2$, is $L_2 \leq_p L_1$?

   (d) Prove that $P \subseteq \mathbf{NP} \cup \mathbf{coNP}$?

   (e) Prove that if $\mathbf{P} = \mathbf{NP}$, then $\mathbf{NP} = \mathbf{coNP}$.

   (f) For languages L_1, L_2, let us define $L_1 \oplus L_2$ as

   $$L_1 \oplus L_2 = \{x : x \text{ is in exactly one of } L_1, L_2\} = (L_1 \setminus L_2) \cup (L_2 \setminus L_1).$$

   Suppose $L_1, L_2 \in \mathbf{NP} \cap \mathbf{coNP}$, then show that $L_1 \oplus L_2 \in \mathbf{NP} \cap \mathbf{coNP}$

2. Let $\mathsf{HALT} = \{(\alpha, x) : M_\alpha \text{ halts on input } x\}$. Recall that M_α denotes the Turing machine represented by the string α. Show that HALT is **NP** hard. Is HALT, **NP** complete? [20 points].

3. Given an undirected graph $G = (V, E)$, a coloring of G means to assign colors to the vertices of G such that no adjacent vertices gets the same color. Thus, a graph coloring is a function $c : V \leftarrow \mathsf{COLORS}$, where COLORS is a finite set whose elements are called colors, the function c should have the property that for every edge $(u, v) \in E$, $c(u) \neq c(v)$. G is called k colorable if it has a coloring with k colors. We define the languages $\mathsf{2COL}$ and $\mathsf{3COL}$ as follows:

$$\mathsf{2COL} = \{G : G \text{ is two colorable}\}$$

$$\mathsf{3COL} = \{G : G \text{ is three colorable}\}$$

   (a) Show that $\mathsf{2COL} \in \mathbf{P}$ [10 points]

   (b) Show that $\mathsf{3COL}$ is **NP** complete. [30 points]

4. In this problem, we analyze a reduction from $\mathsf{3SAT}$ to the following language:

$$\mathsf{MAX2SAT} = \{(\phi, k) \mid \phi \text{ is a 2-CNF formula, and there is an assignment that satisfies at least } k \text{ clauses}\}.$$

Our reduction is the following: Given a $\mathsf{3SAT}$ instance ϕ, we will output a $\mathsf{MAX2SAT}$ instance (ϕ', k), where ϕ' is a 2-CNF formula. To construct ϕ', we do the following: for each clause $(x \vee y \vee z)$ in ϕ, add the following 10 clauses to ϕ' (where w is a new variable for each clause):

$$(x), (y), (z), (\bar{x} \vee \bar{y}), (\bar{y} \vee \bar{z}), (\bar{z} \vee \bar{x}), (w), (x \vee \bar{w}), (y \vee \bar{w}), (z \vee \bar{w})$$

Find a value of k such that $(\phi', k) \in \mathsf{MAX2SAT}$ if and only if $\phi \in \mathsf{3SAT}$. Prove the correctness of the reduction. [30 points.]