Design and Analysis of Algorithms 2008
(Home work 1)

August 28, 2008

- Due on Thursday, September 11, before 8 a.m.
- You have a total of 7 late days in the whole term.
- Please give precise arguments for all statements that you write.
- Please do not hesitate to contact me if you do not understand the problems.
- Each problem in this homework bear 5 points.
- Collaboration is encouraged, but you should not copy solutions, but write down your own answers. If copying is detected that may immediately lead to a grade less than 7. (**This would be followed strictly**)
- Credits would be given to partial solutions also.
- The answers should be typed or written clearly and a hard copy is to be submitted.

1. Consider the following algorithm MIN as discussed in class:

   Algorithm MIN(L)
   1. $min \leftarrow L[1]$
   2. **for** $i = 1$ to n
   3.   **if** $L[i] < min$,
   4. $min \leftarrow L[i]$
   5.   **end if**
   6. **end for**
   7. **return** min

   We want to determine the average number of times the assignment in line 4 is executed. Assume that the array L contains n elements and is a random permutation of the set $S = \{1, 2, 3, \ldots, n\}$.

(a) If a number x is randomly selected from the set S. What is the probability that x is the minimum of S.

(b) When line 4 of the above program is executed, what is the relationship between $L[i]$ and $L[j]$ for $1 \leq j \leq i$?

(c) For each i in the range $1 \leq i \leq n$, what is the probability that line 4 is executed.

2. Write the pseudo-code of an algorithm to find both the maximum and minimum of an array of n numbers. Do not assume anything about n, and the number of comparisons should be around $\frac{3}{2}n$.

3. Consider the harmonic number

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

(a) Express the sum $\sum_{k=1}^{n} 1/(2k-1)$ in terms of harmonic numbers.

(b) Show that $\ln(n) < H_n < \ln(n) + 1$.

(c) Group the terms of the harmonic number as below

$$H_n = 1 + \underbrace{\frac{1}{2} + \frac{1}{3}} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}} + \cdots$$

Note that in the above grouping the first group contains 2^0 terms, the second group contains 2^1 terms and so on. So, the k-th group contains 2^{k-1} terms. Now, consider that $1/n$ is in the k-th group, show that $H_n > k/2$.

4. Consider the problem of evaluating a polynomial at a point. Given n integer coefficients $a_0, a_1, \ldots, a_{n-1}$ and and integer x, we want to compute $\sum_{i=0}^{n-1} a_i x^i$. Write the pseudo-code of an algorithm to do this. How many additions and multiplications are required to do this. Try to make your algorithm as efficient as possible.

5. Write a pseudo-code of the linear search problem as discussed in class. Discuss the number of comparisons required for searching an element in the best case, the worst case and the average case.

6. Let A be an array of n distinct numbers. If $i < j$ and $A[i] > A[j]$, then the pair (i, j) is called an *inversion* of A.

(a) List the inversions of the array $< 2, 3, 8, 6, 1 >$.

(b) Which array containing the elements from the set $S = \{1, 2, 3, \ldots, n\}$ contains the maximum number of inversions. How many does it have?

(c) What is the relationship between the running time of *insertion sort* and the number of inversions present in the array being sorted? Justify your answer.