Design and Analysis of Algorithms 2009 (Extra Credit Problems)

December 3, 2009

- This is a set of bonus problems. It is not mandatory and not submitting them will not hamper your grade. But if you answer then correctly you can upgrade your score by at most 1 point (in the 10 point scale).
- Due on December 20 midnight.
- Please give precise arguments for all statements that you write.
- Credits would be given to partial solutions also.
- The answers should be typed and mailed to debrup@cs.cinvestav.mx with the subject DAA09-ECP.

1. We saw in one of our homework that adding two Toeplitz matrices of size $n \times n$ can be done in $\mathcal{O}(n)$ time by using a special representation of the matrix. Show that a $n \times n$ Toeplitz matrix can be multiplied with a $(n \times 1)$ vector in $\mathcal{O}(n \lg n)$ time.

2. Consider the following game: the casino has placed a sequence $S[1, \dots, n]$ of dollar bills, of various denominations, in a line on the table. The i-th bill is $S[i]$ dollars. Two players take turns; during your turn, you can pick up and keep either the first or the last bill remaining in the line. The goal is to collect as much money as possible. Design two algorithms: (1) a $O(n^2)$ time algorithm that, given $S[1, ..., n]$, pre-computes some information for use by the second algorithm; (2) an O(1) time algorithm that, given a configuration that can occur in the resulting game and the pre-computed information, identifies the best next move.

3. Suppose we have a directed graph $G = (V; E)$ describing a computer network, where vertices correspond to hosts or routers and edges correspond to network links. Also assume that for each network link $(u, v) \in E$, we are given a measure $r(u, v)$ of the reliability of this link: specifically, $r(u, v) =$ the probability that a packet sent across the link (u, v) will not be lost while it is transiting that link. You may assume that these probabilities are independent. So, if we have a path $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$, the probability that a packet sent along this path makes it from v_0 to v_k successfully

is given by $r(v_0, v_1) \times r(v_1, v_2) \times \cdots r(v_{k-1}, v_k)$. Given the graph G, the reliability measure $r(.,.)$, and vertices $s, t \in V$, your job is to find a path from s to t of maximum reliability. Design an efficient algorithm to solve this problem. **Hint:** This is almost a shortest-path problem, but we are maximizing instead of minimizing, and we have products instead of sums. But can we modify the problem slightly to use a shortest-path algorithm?

4. Each clause in a 2CNF formula looks like $(x \vee y)$ which can be written as $\bar{x} \rightarrow y$. Reduce the problem of 2CNF-SAT to a problem involving directed graphs which can be solved in polynomial time, thus showing that 2CNF-SAT is in the class $\mathcal{P}$.

5. An independent set of a graph $G = (V, E)$ is a subset $V' \subseteq V$ of vertices such that each edge in E is incident on at most one vertex in V'. The independent set problem is to find a maximum size independent set in G.

   (a) Formulate a related decision problem for the independent set problem and prove that it is NP complete.

   (b) Suppose you are given a subroutine ti solve the decision problem you defined above. Give an algorithm to find the independent set of maximum size. The running time of your algorithm should be a polynomial in $|V|$ and $|E|$. Assume the calls to the subroutine subroutine takes single steps.