Equivalence Relations

A subset R of the set $A \times A$ is called a relation on A. A relation of specific interest to us is an *equivalence relation.*

A subset R of $A \times A$ is called an equivalence relation on A if

– $(a, a) \in R$ for all $a \in A$
– $(a, b) \in R$ implies $(b, a) \in R$
– $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

Instead of talking of subsets of $A \times A$ we can conveniently talk of a binary relation on elements of the set A, i.e., when $(a, b) \in R$ we denote it by $a \sim b$ and call it as a related to b. With this notation we can restate the definition of a equivalence relation as below

Definition 1 *The binary relation $\sim$ on A is said to be a equivalence relation on A, if for all a, b and c in A,*

– *$a \sim a$ [Reflexivity]*
– *$a \sim b$ implies $b \sim a$ [Symmetry]*
– *$a \sim b$ and $b \sim c$ implies $a \sim c$ [Transitivity]*

Example 1. Let S be a set and define $a \sim b$, for $a, b \in S$, if and only if $a = b$. This clearly defines a equivalence relation on S. In fact, an equivalence relation is generalization of equality, measuring equality up to some property.

Example 2. Let S be the set of all triangles in a plane. Two triangles are defined to be equivalent if they are similar (i.e., have corresponding angles equal). This defines a equivalence relation on S.

Example 3. Let S be the set of points in a plane. Two points a and b are defined to be equivalent if they are equidistant from the origin. This defines an equivalence relation on S.

Example 4. Let S be the set of all integers. Given $a, b \in S$, define $a \sim b$ if $a - b$ is an even integer. We verify that this is an equivalence relation of S.

1. Since $a - a = 0$ is even, so $a \sim a$
2. if $a \sim b$ then $(a - b)$ is even, then $b - a = -(a - b)$ is also even, so $b \sim a$
3. If $a \sim b$ and $b \sim c$ then $a - b$ and $b - c$ are even, whence $a - c = (a - b) + (b - c)$ is also even, thus $a \sim c$

Definition 2 *If A is a set and if $\sim$ is an equivalence relation on A, then the* equivalence class *of $a \in A$ is the set $\{x \in A : a \sim x\}$. We write it as $cl(a)$*

Now let us see what are the equivalence classes in the examples that we just described. In Example 1, the equivalence class of a consists only of a. In Example 2 $cl(a)$ consists of all triangles which are similar to a. In Example 3, $cl(a)$ consists of all points in the plane which lie on a circle whose center is the origin and which passes through a. In Example 4, $cl(a)$ consists of all integers of the form $a + 2m$, where $m = 0, \pm 1, \pm 2, \ldots$.

Now we are ready to prove an important theorem regarding equivalence relations.

Theorem 1. *Distinct equivalence classes of an equivalence relation on A provide us with a decomposition of A as an union of mutually disjoint subsets. Conversely, given a decomposition of A as an union of mutually disjoint, nonempty subsets, we can define an equivalence relation on A for which these subsets are the distinct equivalence classes.*

Proof. Let $\sim$ be a equivalence relation on A. For $a \in A$ let $cl(a)$ be the equivalence class of a. As $a \sim a$, thus, for all $a \in A$, $a \in cl(a)$. So, $\cup_{a \in A} cl(a) = A$. So we have proved that the union of the equivalence classes in A gives A.

Now, we need to show that for two distinct elements $a, b \in A$ either $cl(a) = cl(b)$ or $cl(a)$ and $cl(b)$ are disjoint. To show this let us assume that $cl(a)$ and $cl(b)$ have a non-empty intersection, and let $x \in cl(a) \cap cl(b)$. So, we have $x \in cl(a)$ and $x \in cl(b)$. Thus, by definition of a equivalence class we have $a \sim x$ and $b \sim x$. And $b \sim x$ implies $x \sim b$. Also, $a \sim x$ and $x \sim b$ together imply $a \sim b$. Now if $y \in cl(a)$ then $y \sim a$, also as $a \sim b$, so $y \sim b$, which means $y \in cl(b)$. Thus $y \in cl(b)$. So we conclude that $cl(a) \subseteq cl(b)$. This argument is symmetric and we can by the same argument conclude that $cl(b) \subseteq cl(a)$. Thus $cl(a) = cl(b)$. Thus we have proved that if $cl(a)$ and $cl(b)$ have a nonempty intersection then they must be equal.

To prove the other part of the theorem, we assume that A_α, $\alpha \in I$ be a decomposition of A such that $\cup_{\alpha \in I} A_\alpha = A$ and $A_\alpha \cap A_\beta = \phi$ for all $\alpha, \beta \in I$ s.t. $\alpha \neq \beta$. Now, we need to define an equivalence relation on A using this decomposition of A. For $a, b \in A$ we define $a \sim b$ iff a and b belongs to the same subset A_α. What is left is to prove that $\sim$ defined in the above manner is indeed an equivalence relation. We leave this as an exercise.