Discrete Mathematics 2015
(Home work 1)

September 11, 2013

- Due on Wednesday, September 23, before 10 a.m.(You can submit in the class)
- Late home works will not be accepted.
- Please give precise arguments for all statements that you write.
- Please do not hesitate to contact me if you do not understand the problems.
- Collaboration is discouraged, but not prohibited. It is recommended that you try to solve the problems on your own. You can discuss the questions with your colleagues but you should not copy solutions. Always write down your own answers. If copying is detected that may immediately lead to a grade less than 7. (**This would be followed strictly**)
- Credits would be given to partial solutions also.
- The answers should be typed or written clearly and a hard copy is to be submitted.

1. [**5 × 5 = 25 points**] Prove the following by mathematical induction:
   (a) $\forall n \in \mathbb{N}, 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
   (b) $\forall n \in \mathbb{N}, x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots x^2 + x + 1)$.
   (c) $\forall n \in \mathbb{N}, 3 | n^3 + 2n$.
   (d) $\forall n \in \mathbb{N}, \sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$.
   (e) For all $n \in \mathbb{N}$, $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer.
2. [**5 × 6 = 30 points**] Prove the following. In all the exercises of this question f_n denotes the n-th Fibonacci number, i.e., $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.
   (a) $f_n \geq 2^{\frac{n-1}{2}}$, for all $n \geq 1$.

(b) $2 | f_{3n}$, for all $n \geq 1$.

(c) $f_n^2 - f_{n+1}f_{n-1} = (-1)^{n-1}$, $n \geq 1$

(d) $\sum_{i=0}^{n} f_i = f_{n+2} - 1$

(e) $\sum_{i=1}^{n} f_n^2 = f_n f_{n+1}$

(f) $\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$

3. [**5 points**] Prove that the ten's digit of any power of three is even. [For example the tens digits of $3^1(= 3)$, $3^2(= 9)$, $3^3(= 27)$, $3^4(= 81)$, $3^5(= 243)$, $3^6(= 729)$; are 0, 0, 2, 8, 4 and 2 respectively; and they are all even.]

4. [**10 points**] Let there be n persons in a room and each person shakes hand with all other persons. Prove by induction that the total number of handshakes that takes place is $\frac{n(n-1)}{2}$.

5. [**10 points**] Let us draw n lines in a plane in such a way that no two are parallel and no three intersect in a common point. Prove that the plane is divided into $n(n+1)/2+1$ regions.

6. [**10 points**] Show that all numbers of the form $10017, 100117, 1001117, 10011117, \ldots$ are divisible by 53.
(**Hint:** Let $u_i = 7 + \left(\sum_{j=1}^{i} 10^j\right) + 10^{i+3}$, $i \geq 1$; then u_i represents the i-th number in the given sequence)

7. [**10 points**] Let there be n circles drawn on a plane such that two circles intersect each other in two points and no three circles pass through the same point. Let R_n denote the number of regions into which the plane is divided when n such circles are drawn. We are interested in finding R_n for all $n \geq 1$.

   (a) For $n \geq 1$, what is the relation between R_n and R_{n+1}? Explain in one sentence why this relation is true.

   (b) Try to guess (or find) a closed form solution for R_n, i.e., express R_n only in terms of n.

   (c) Prove that your guess is correct.