Discrete Mathematics 2015
(Home work 3)

October 09, 2015

- Due on Wednesday, October 19, before 10 a.m. (You can submit in class)
- Late home works will not be accepted.
- Please give precise arguments for all statements that you write.
- Please do not hesitate to contact me if you do not understand the problems.
- Collaboration is discouraged, but not prohibited. It is recommended that you try to solve the problems on your own. You can discuss the questions with your colleagues but you should not copy solutions. Always write down your own answers. If copying is detected that may immediately lead to a grade less than 7. (**This would be followed strictly**)
- Credits would be given to partial solutions also.
- The answers should be typed or written clearly and a hard copy is to be submitted.

1. [**20 points**] Prove or disprove the following:

    (a) Every partially ordered set has a minimum (smallest) element.

    (b) For a linearly ordered set a minimal element is also the minimum element.

    (c) If a partially ordered set has a single minimal element, then it is also the minimum element.

    (d) Let (X, R) be a partially ordered set. An element $x \in X$ is maximal in (X, R) iff x is maximal in R^{-1}.

2. [**15 points**] Let $X_n = \{1, 2, 3, \ldots, n\}$ and $\mathcal{B}_n = (2^{X_n}, \subseteq)$. Note, for a set X, 2^X denotes the power set of X.

    (a) Prove that $\mathcal{B}_n$ is a partial order.

(b) Construct a linear order $\sqsubseteq$ on X_4 which extends $\subseteq$, ie for any $A, B \in 2^X$, if $A \subseteq B$, then $A \sqsubseteq B$.

(c) Prove that $\omega(\mathcal{B}_n) = n + 1$.

3. **[10 points]** Prove that the set $\mathbb{R}^2$ of all ordered pairs of real numbers has the same cardinality as $\mathbb{R}$.

4. **[20 points]** We discussed in class that the set of rational numbers $\mathbb{Q}$ is countable. We argued this by arranging the rational numbers in a infinite matrix and enumerating them in a strange fashion, which we termed as enumerating by the diagonal method. This, though correct, has the problem that there are repetitions of the rational numbers in the matrix which makes it a bit in-elegant. Here we discuss another method to enumerate the rationals which leads to no repetitions.

   We build an infinite binary tree as follows:

   - Label the root as $\frac{1}{1}$.
   - If, the label of a node is $\frac{i}{j}$, then its left child has the label $\frac{i}{i+j}$ and the right child has the label $\frac{i+j}{j}$.

   Now we enumerate the the nodes of the tree level-wise from left to right. The list would start as

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \ldots$$

   Now, prove the following facts about this construction and enumeration:

   (a) All fractions appearing as a label of a node in the tree is reduced, i.e., if $\frac{r}{s}$ appears as a label then r and s are relatively prime.

   (b) Every reduced fraction $\frac{r}{s}$ appears in the tree.

   (c) Every reduced fraction appears exactly once.

   (d) The denominator of the n-th fraction in the list equals the numerator of the $(n + 1)$-st fraction.