Solving Linear Homogeneous Recurrence Relations with Constant Coefficients: The Method of Characteristic Roots

In class we studied the method of characteristic roots to solve a linear homogeneous recurrence relation with constant coefficients. This handout is to supplement the material that we saw in class[1].

1 Linear Homogeneous Recurrence Relations With Constant Coefficients

The Fibonacci recurrence is defined by $f_0 = 0$, $f_1 = 1$ and

$$f_n = f_{n-1} + f_{n-2}, n \geq 2. \tag{1}$$

In the class we saw some algorithms to find f_n for any given n, in this handout we would see a general method to solve certain kind of recurrences, this method would enable us to find f_n analytically.

The Fibonacci recurrence falls under a general class of recurrence relations which are called *linear homogeneous recurrence relations with constant coefficients*. The general form of such recurrence is

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_p a_{n-p}, n \geq p, \tag{2}$$

where $c_1, c_2, \ldots c_p$ are constants and $c_p \neq 0$. Such a recurrence is called linear as all terms a_k occur to the first power and it is called homogeneous as there is no term which does not involve some a_k, $n - p \leq k \leq n - 1$. Since the coefficients c_i, $1 \leq i \leq p$ are are constants hence the recurrence in eq. (2) is a linear homogeneous recurrence relations with constant coefficients. It is easy to see that the Fibonacci recurrence as described in (1), also falls under this general category.

2 The Method of Characteristic Roots

The recurrence in eq. (2) has a unique solution when the values of the first p terms $a_0, a_1, \ldots a_{p-1}$ are specified, these are called the initial conditions. Given $a_0, a_1, \ldots a_{p-1}$, we van use the recurrence in (2) to find a_p, and then using the values of $a_1, \ldots a_p$ we can find the value of a_{p+1} and so on.

For having a general solution of the recurrence in (2), we replace a_k by x^k in (2) and solve for x. Making the substitution we obtain

$$x^n - c_1 x^{n-1} - c_2 x^{n-2} - \cdots - c_p x^{n-p} = 0. \tag{3}$$

Dividing both sides of eq. (3) by x^{n-p} we obtain

$$x^p - c_1 x^{p-1} - c_2 x^{p-2} - \cdots - c_p = 0. \tag{4}$$

[1] The discussion closely follows the text:*Applied Combinatorics*, by Fred Roberts, Prentice Hall 1984

Equation (4) is called the characteristic equation of the recurrence (2). It is a p degree polynomial equation and so would have p roots, let us call these p roots as $\alpha_1, \alpha_2, \ldots, \alpha_p$. Some of these roots may be repeated roots (i.e., all of $\alpha_1, \alpha_2, \ldots, \alpha_p$ may not be distinct), also it may be so that some of the roots are complex numbers. These roots are called the *characteristic roots.* For example consider the Fibonacci recurrence in (1), the characteristic equation of this recurrence would be $x^2 - x - 1 = 0$ and the characteristic roots would be $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$.

Now, if α is a characteristic root of the recurrence in 2, and if we take $a_n = \alpha^n$, it follows that the sequence (a_n) satisfies the recurrence. Thus corresponding to each characteristic root we have a sequence which would be the solution to the recurrence. But for the case of the Fibonacci recurrence we notice that neither $f_n = (\frac{1+\sqrt{5}}{2})^n$ nor $f_n = (\frac{1-\sqrt{5}}{2})^n$ satisfies the initial conditions $f_0 = 0$ and $f_1 = 1$.

Now we see that if the sequences (a'_n) and (a''_n) both satisfies the recurrence (2) then a sequence (a'''_n) where $a'''_n = \lambda_1 a''_n + \lambda_2 a'_n$ and λ_1 and λ_2 constants should also satisfy (2). To see this, observe that

$$a'_n = c_1 a'_{n-1} + c_2 a'_{n-2} + \cdots + c_p a'_{n-p} \tag{5}$$

and

$$a''_n = c_1 a''_{n-1} + c_2 a''_{n-2} + \cdots + c_p a''_{n-p} \tag{6}$$

Multiplying the above equations by λ_1 and λ_2 respectively and adding gives us

$$\begin{aligned} a'''_n &= \lambda_1(c_1 a'_{n-1} + c_2 a'_{n-2} + \cdots + c_p a'_{n-p}) + \lambda_2(c_1 a''_{n-1} + c_2 a''_{n-2} + \cdots + c_p a''_{n-p}) \\ &= c_1(\lambda_1 a'_{n-1} + \lambda_2 a''_{n-1}) + c_2(\lambda_1 a'_{n-2} + \lambda_2 a''_{n-2}) + \cdots c_p(\lambda_1 a'_{n-p} + \lambda_2 a''_{n-p}) \\ &= c_1 a'''_{n-1} + c_2 a'''_{n-2} + \cdots + c_p a'''_{n-p}. \end{aligned}$$

Thus (a'''_n) satisfies (2).

In general, suppose that $\alpha_1, \alpha_2, \ldots, \alpha_p$ are the characteristic roots of recurrence the recurrence in (2). Then our reasoning shows that if $\lambda_1, \lambda_2, \ldots, \alpha_p$ are constants, and if

$$a_n = \lambda_1 \alpha_1^n + \lambda_2 \alpha_2^n + \cdots + \lambda_p \alpha_p^n,$$

then a_n satisfies (2). Additionally, it turns out that *every* solution of (2) can be expressed in this form *provided the roots* $\alpha_1, \alpha_2, \ldots, \alpha_p$ *are distinct.* Thus we can summarize these facts in the following theorem.

Theorem 1. *Suppose a liner homogeneous recurrence with constant coefficients as in (2) has characteristic roots* $\alpha_1, \alpha_2, \ldots, \alpha_p$. *Then if* $\lambda_1, \lambda_2, \ldots, \lambda_p$ *are constants, every expression of the form*

$$a_n = \lambda_1 \alpha_1^n + \lambda_2 \alpha_2^n + \cdots + \lambda_p \alpha_p^n \tag{7}$$

is a solution to the recurrence. Moreover if the characteristic roots are distinct, then every solution to the recurrence have the form of eq. (7) for some constants $\lambda_1, \lambda_2, \ldots, \lambda_p$. *We call the expression in (7) as the general solution.*

Using this result we can now solve the Fibonacci recurrence. We already found out that the characteristic equation for the Fibonacci recurrence is $x^2 - x - 1 = 0$, and thus the characteristic roots are $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$. Thus according to Theorem the general solution for the Fibonacci recurrence would be

$$f_n = \lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{8}$$

For some constants λ_1 and λ_2. Thus solving the Fibonacci recurrence now reduces to finding the values of the constants λ_1 and λ_2 such that the general solution conforms with the given initial conditions $f_0 = 0$ and $f_0 = 1$. Substituting the initial conditions in the general solution we obtain

$$f_0 = \lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$$

$$f_1 = \lambda_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + \lambda_2 \left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

which translates to the following system of linear equations

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0 \\ (1+\sqrt{5})\lambda_1 + (1-\sqrt{5})\lambda_2 &= 2 \end{aligned}$$

This system of equations have an unique solution $\lambda_1 = \frac{1}{\sqrt{5}}$ and $\lambda_2 = -\frac{1}{\sqrt{5}}$. Thus, the unique solution to the Fibonacci recurrence is

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

2.1 The Case of Multiple Roots

Consider the recurrence

$$a_n = 6a_{n-1} - 9a_{n-2}, \tag{9}$$

with $a_0 = 1$, $a_1 = 2$. Its characteristic equation is $x^2 - 6x + 9 = 0$, or $(x-3)^2 = 0$. The two characteristic roots are 3 and 3, i.e., 3 is a multiple root. Hence the second part of Theorem 2 does not apply. Though it is still true that 3^n is a solution of (9), and it is also true that $\lambda_1(3)^n + \lambda_2(3)^n$ is always a solution, it is not true that every solution of (9) is of the form $\lambda_1(3)^n + \lambda_2(3)^n$. In particular there is no solution of the form $\lambda_1(3)^n + \lambda_2(3)^n$ for (9) which satisfies the given initial conditions. For the initial conditions would give us the equations

$$\begin{aligned} \lambda_1 + \lambda_2 &= 1 \\ 3\lambda_1 + 3\lambda_2 &= 2. \end{aligned}$$

There are no λ_1 and λ_2 which satisfies these two equations simultaneously.

Suppose α is a characteristic root with multiplicity u; i.e., α appears as a root of the characteristic equation u many times. Then it turns out that not only $a_n = \alpha^n$ satisfy the recurrence but so do $\alpha = n\alpha^n$, $a_n = n^2\alpha^n, \ldots,$ and $n^{u-1}\alpha^n$. So for our example of the recurrence in eq. (9) as we have the characteristic root $\alpha = 3$ with multiplicity $u = 2$, so both $a_n = 3^n$ and $a_n = n3^n$ are its solution. So for some constants λ_1 and λ_2, $a_n = \lambda_1 3^n + \lambda_2 n 3^n$ is a solution to (9). Using the initial conditions $a_0 = 1$ and $a_1 = 2$, we get the equations

$$\begin{aligned} \lambda_1 &= 1 \\ 3\lambda_1 + 3\lambda_2 &= 2 \end{aligned}$$

These equations have the unique solution $\lambda_1 = 1$, $\lambda_2 = -\frac{1}{3}$. Hence $a_n = 3^n - \frac{1}{3}n3^n$ is a solution to the recurrence (9) with initial conditions $a_0 = 1$ and $a_1 = 2$. It follows that this must be the unique solution.

This procedure generalizes as follows. Suppose that a recurrence (2) has characteristic roots $\alpha_1, \alpha_2, \ldots, \alpha_q$, with α_i having multiplicity u_i. This means that

$$\sum_{i=1}^{q} u_i = p.$$

Then,

$$\alpha_1^n, n\alpha_1^n, \ldots, n^{u_1-1}\alpha_1^n, \alpha_2^n, n\alpha_2^n, \ldots, n^{u_2-1}\alpha_2^n, \ldots, \alpha_q^n, n\alpha_q^n, \ldots, n^{u_q-1}\alpha_q^n$$

must all be solutions of the recurrence. Let us call these solutions as the *basic solutions*. There are p of these basic solutions in all. Let us denote them by $b_1, b_2, \ldots, b_p$. Then for any constants $\lambda_1, \lambda_2, \ldots \lambda_p$,

$$a_n = \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_p b_p$$

is also a solution of the recurrence and every solution of it is of this form. Thus we can summarize as

Theorem 2. *Suppose that a linear homogeneous recurrence with constant coefficient as in (2) has basic solutions $b_1, b_2, \ldots, b_p$. Then the general solution is given by*

$$a_n = \lambda_1 b_1 + \lambda_2 b_2 + \ldots + \lambda_p b_p \tag{10}$$

for some constants $\lambda_1, \lambda_2, \ldots, \lambda_p$.

To end this discussion let us consider solving a recurrence using the results we stated. Consider the recurrence

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}, \tag{11}$$

$a_0 = 1$, $a_1 = 2$, $a_2 = 0$. Then the characteristic equation would be

$$x^3 - 7x^2 + 16x - 12 = 0$$

The characteristic roots would be 2,2,3. Thus the general solution would be of the form

$$\lambda_1 2^n + \lambda_2 n 2^n + \lambda_3 3^n$$

Using the initial conditions solve for $\lambda_1, \lambda_2, \lambda_3$ and thus obtain an unique solution to (11) (EXERCISE!).