Functions

Definition 1 *If S and T are nonempty sets then a function from S to T is a subset, F, of $S \times T$ such that for every $s \in S$ there is an unique $t \in T$ such that the ordered pair $(s,t) \in F$.*

The above definition precisely describes a function. But we would prefer to think a function as a rule which associates any element of S to some element in t. The rule being: *associate $s \in S$ with $t \in T$ if and only if $(s,t) \in F$.* We shall call t as the *image* of s under the function F.

We denote a function τ from S to T by the notation $\tau : S \to T$. If t is an image of s under τ then we shall usually write $\tau(s) = t$ [1].

Example 1. Let S be a set. Define $I : S \to S$ as $I(s) = s$ for all $s \in S$. I is called the identity function of S.

Example 2. Let $\mathbb{Q}$ be the set of rational numbers and let T be $\mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers. Given $s \in \mathbb{Q}$, we can write $s = m/n$ where $m, n \in \mathbb{Z}$ such that they have no common factors. Define $\tau : \mathbb{Q} \to T$ as $\tau(s) = (m, n)$.

Example 3. Let S and T be sets; define $\tau : S \times T \to S$ by $\tau(a, b) = a$. This τ is called a projection of $S \times T$ on S. We can similarly define the projection of $S \times T$ on T.

Example 4. Define $\tau : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as $\tau(a, b) = a + b$. This is an example of a *binary operation* on the set $\mathbb{Z}$. For a general set S, given a function $\tau : S \times S \to S$, we could use it to define a product $*$ in S by declaring $a * b = c$ if $\tau(a, b) = c$.

Example 5. Let $S = \{x_1, x_2, x_3\}$, define $\tau : S \to S$ by $\tau(x_1) = x_2$, $\tau(x_2) = x_3$ and $\tau(x_3) = x_1$

Example 6. Let $\mathbb{Z}$ be the set of integers and $B = \{0, 1\}$. Define $\tau : \mathbb{Z} \to B$ as $\tau(x) = 1$ if x is even and $\tau(x) = 0$ if x is odd.

We shall have the opportunity to see many more examples as we proceed. But for the time being let us proceed with our discussion.

Definition 2 *Give $\tau : S \to T$, the inverse image of $t \in T$ with respect to τ is the set $\{s : \tau(s) = t\}$.*

For example in Example 6 the inverse image of 1 is the set of all even numbers. It can be so that for some element in T the inverse image with respect to a function τ is empty. As in Example 2 the inverse image of $(4, 2)$ is the empty set.

Definition 3 *The function $\tau : S \to T$ is called* onto T *if for any $t \in T$, there exists an $s \in S$ such that $\tau(s) = t$. An onto function is called a* surjection.

Definition 4 *The function $\tau : S \to T$ is called* one-to-one *if whenever $s_1 \neq s_2$, then $\tau(s_1) \neq \tau(s_2)$. A* one-to-one *function is called an* injection.

Definition 5 *A function which is both one-to-one and onto is called a* bijection.

[1] Other notations are also in use like $t = Fs$ or $sF = t$, the reader should be cautious about this while following other texts

Definition 6 *The two functions σ and τ from S to T are called equal if for all $s \in S$, $\sigma(s) = \tau(s)$*

Now let us suppose that there are two functions $\sigma : S \to T$ and $\tau : T \to U$. We now want to combine these two functions σ and τ to yield another function from S to U. The obvious way to do this is to first apply the function σ to obtain an element in T and then again apply τ to obtain an element in U. This operation is called *composition of functions* which is formally defined as follows:

Definition 7 *If $\sigma : S \to T$ and $\tau : T \to U$ then the composition of σ and τ is the function $\tau \circ \sigma : S \to T$ defined as $\tau \circ \sigma(s) = \tau(\sigma(s))$ for every $s \in S$.*

Next we illustrate the composition operator with an example

Example 7. Let $S = \{x_1, x_2, x_3\}$ and let $T = S$. Let $\sigma : S \to S$ be defined by

$$\sigma(x_1) = x_2$$
$$\sigma(x_2) = x_3$$
$$\sigma(x_3) = x_1$$

and $\tau : S \to S$ by

$$\tau(x_1) = x_1$$
$$\tau(x_2) = x_3$$
$$\tau(x_3) = x_2$$

Thus we have

$$\tau \circ \sigma(x_1) = \tau(x_2) = x_3$$
$$\tau \circ \sigma(x_2) = \tau(x_3) = x_2$$
$$\tau \circ \sigma(x_3) = \tau(x_1) = x_1$$

Lemma 1. *(Associative Law) If $\sigma : S \to T$, $\tau : T \to U$ and $\mu : U \to V$, then $\mu \circ (\tau \circ \sigma) = (\mu \circ \tau) \circ \sigma$*

Proof. First note that $\tau \circ \sigma$ makes sense and takes S to U, also $\mu \circ (\tau \circ \sigma)$ makes sense and takes S to V. Similarly $(\mu \circ \tau) \circ \sigma$ is meaningful and takes S to V. Thus we can talk of the equality or inequality of $\mu \circ (\tau \circ \sigma)$ and $(\mu \circ \tau) \circ \sigma$.

To show the equality, we must show that for any $s \in S$,

$$\mu \circ (\tau \circ \sigma)(s) = (\mu \circ \tau) \circ \sigma(s)$$

Now, from the definition of composition of functions we have

$$\begin{aligned} \mu \circ (\tau \circ \sigma)(s) &= \mu(\tau \circ \sigma(s)) \\ &= \mu(\tau(\sigma(s))) \end{aligned}$$

Similarly,

$$\begin{aligned} (\mu \circ \tau) \circ \sigma(s) &= (\mu \circ \tau)(\sigma(s)) \\ &= \mu(\tau(\sigma(s))) \end{aligned}$$

Thus we have $\mu \circ (\tau \circ \sigma)(s) = (\mu \circ \tau) \circ \sigma(s)$ for all $s \in S$.

Now we shall prove two more important properties of functions:

Lemma 2. *Let $\sigma : S \to T$ and $\tau : T \to U$, then*

1. *$\tau \circ \sigma$ is a surjection if each of σ and τ are surjections.*
2. *$\sigma \circ \tau$ is an injection if each of σ and τ are injections*

Proof. 1. By hypothesis σ and τ are surjections, i.e., for every $t \in T$ there exists a $s \in S$ s.t. $\sigma(s) = t$ and for every $u \in U$ there exist a $t_1 \in T$ s.t. $\tau(t_1) = u$. So given any $u \in U$, we have a $s \in S$, s.t., $u = \tau(\sigma(s))$. So, $\tau \circ \sigma$ is a surjection.

2. Suppose $s_1, s_2 \in S$ and $s_1 \neq s_2$. As, σ is an injection, so $\sigma(s_1) \neq \sigma(s_2)$. And, τ being an injection $\tau(\sigma(s_1)) \neq \tau(\sigma(s_2))$. Which implies that for $s_1 \neq s_2$, $\sigma \circ \tau(s_1) \neq \sigma \circ \tau(s_2)$, which means that $\sigma \circ \tau$ is an injection.

Suppose $\sigma : S \to T$ is a bijection, i.e., it is both one-to-one and onto. For such a function we can define a function $\sigma^{-1} : T \to S$ by $\sigma^{-1}(t) = s$ if and only if $\sigma(s) = t$. We call σ^{-1} the inverse of σ. It is easy to verify that $\sigma^{-1} \circ \sigma$ is a function from S onto S, and similarly $\sigma \circ \sigma^{-1}$ is a function from T onto T. Now let $s \in S$, then $\sigma(s) = t$, for some t in T. Now, by definition $\sigma^{-1}(t) = s$, so

$$\sigma^{-1} \circ \sigma(s) = \sigma^{-1}(\sigma(s)) = \sigma^{-1}(t) = s.$$

We have shown that $\sigma^{-1} \circ \sigma$ is an identity function from S onto S. By a similar computation it can be shown that $\sigma \circ \sigma^{-1}$ is an identity function from T onto T.

Conversely, if $\sigma : S \to T$ is such that there exists a $\mu : T \to S$ with the property that $\mu \circ \sigma$ and $\sigma \circ \mu$ are identity mappings on S and T respectively then σ is a bijection. We formalize this in the next lemma:

Lemma 3. *The function $\sigma : S \to T$ is a bijection if and only if there exists a function $\mu : T \to S$ such that $\mu \circ \sigma$ and $\sigma \circ \mu$ are identity functions on S and T respectively.*

Proof. We have already shown that if σ is a bijection then there exist a function σ^{-1} such that $\sigma^{-1} \circ \sigma$ and $\sigma \circ \sigma^{-1}$ are identity functions on S and T respectively.

To prove the other way, we assume there exists a $\mu : T \to S$ with the property that $\mu \circ \sigma$ and $\sigma \circ \mu$ are identity mappings on S and T respectively. So for a given $t \in T$, $\sigma \circ \mu(t) = \sigma(\mu(t)) = t$, so for any $t \in T$, t is the image of $\mu(t) \in S$ under the function σ. This shows that σ is onto. Further, if $\sigma(s_1) = \sigma(s_2)$ then we have

$$s_1 = \mu \circ \sigma(s_1) = \mu(\sigma(s_1)) = \mu(\sigma(s_2)) = \mu \circ \sigma(s_2) = s_2,$$

as $\mu \circ \sigma$ is a identity function on S. Thus we have shown that σ is one-to-one. Thus σ is a bijection.