

Fundamentals of Algebra for Computer Science (Home work 2)

November 4, 2006

- Due on Wednesday, November 8, before 4 p.m.
- Late submissions will not be accepted.
- Please give precise arguments for all statements that you write.
- To disprove a fact it is enough to give a counter example.
- Please do not hesitate to contact me if you do not understand the problems.
- Each problem in this homework bear 5 points.

1. An **Automorphism** of a group G is an isomorphism of G onto itself.
 - (a) Let G be a group and \mathcal{A} denote the set of all automorphisms of G . Show that \mathcal{A} is a group.
 - (b) Let G be a group. For $h \in G$ define $f_h : G \rightarrow G$ by $f_h(g) = hgh^{-1}$. Prove that f_h is an automorphism of G .
 - (c) Let G be an abelian group. Prove that $f(g) = g^{-1}$ is an automorphism of G .
 - (d) Let g be a generator of a cyclic group G and $f : G \rightarrow G$ be an automorphism. Show that $f(g)$ is also a generator of G .
2. **Direct products:** Let A, B be groups, then $A \times B = \{(a, b) : a \in A, b \in B\}$ is also a group under the operation defined as $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$.
 - (a) Let $f_1 : G \rightarrow A$ and $f_2 : G \rightarrow B$ be two group homomorphisms. Let $f : G \rightarrow A \times B$ be defined by $f(g) = (f_1(g), f_2(g))$. Show that f is a group homomorphism.
 - (b) Show that:

$$\ker(f) = \ker(f_1) \cap \ker(f_2)$$

3. Cyclic groups

- (a) Prove that the subgroup of a cyclic group is always cyclic.
 - (b) Let G be a cyclic group of (finite) order n with a generator g . Show that G is isomorphic to \mathbb{Z}_n , by showing that the map $f : G \rightarrow \mathbb{Z}_n$ defined by $f(g^i) = i \pmod n$ is such an isomorphism.
 - (c) Let G be a cyclic group of infinite order, with a generator g . Show that G is isomorphic to \mathbb{Z} , by showing that the map $f : G \rightarrow \mathbb{Z}$ defined by $f(g^i) = i$ is such an isomorphism.
4. Let G be a finite group and H a subgroup of G . For A, B subgroups of G , define A to be conjugate to B relative to H , if $B = x^{-1}Ax$ for some $x \in H$. Prove
- (a) This defines an equivalence relation on the set of subgroups of G .
 - (b) The number of subgroups of G conjugate to A relative to H equals the index of $N(A) \cap H$ in H , where $N(A) = \{x \in G \mid xAx^{-1} = A\}$