Life's Still Lifes

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Abstract

The de Bruijn diagram describing those decompositions of the neighborhoods of a one dimensional cellular automaton which conform to predetermined requirements of periodicity and translational symmetry shows how to construct extended configurations satisfying the same requirements. Similar diagrams, formed by stages, describe higher dimensional automata, although they become more laborious to compute with increasing neighborhood size. The procedure is illustrated by computing some still lifes for Conway's game of *Life*, a widely known two dimensional cellular automaton.

1 Introduction

Public attention was drawn to cellular automata by Martin Gardner's monthly column *Mathematical Recreations*, a regular feature of *Scientific American* for many years. The October, 1970, issue[2] featured the game of *Life*, which had been invented about that time by the British mathematician John Horton Conway. Sufficient interest was aroused by the game for it to be followed up in several later columns, and to support a newsletter[8] for nearly three years. Gardner's columns have now been collected into one of the compilations that are regularly published by W. H. Freeman and Company[3], while Conway's own version of the game is available in the recent Academic Press book[1] *Winning Ways*.

However, there had been much previous interest in cellular automata, beginning at least with the work[5] of Warren McCulloch and Walter Pitts on neural nets, later including John von Neumann's investigations[6] into self reproduction and automatic factories. Interest still continues, a recent example being Stephen Wolfram's examination[10] of one dimensional automata from the point of view of chaos in complex systems theory.

One of the fundamental expectations in the theory of automata is that the automaton will eventually settle down into a fixed cycle of states, which will then characterize its long term behavior. Some modification of this principle must be expected for infinite automata, nevertheless the search for states of low period constituted an important part of the activity inspired by the announcement of *Life*. Someone with a crystallographer's frame of mind might well have undertaken a classification of all such states, beginning with those of period 1, which Conway called "still lifes."

This article describes how such a classification can be obtained.

2 Cellular automata

Mathematically, an automaton consists of a set of states, together with a set of mappings of the state set into itself. Each mapping is identified with a signal, which is supposed to cause a change of state. Signals can therefore be considered as inputs to the automaton, which in turn could be considered as a neural net, an electronic circuit, or some other structure. In that case, outputs might also considered, and altogether the groundwork has been laid for some kind of fundamental theory of computation, or at least of computing devices. Much of the theory of automata procedes in that direction.

Cellular automata are those for which a large number of similar automata—the cells—are connected together in some regular pattern, and for which the signals are the information which each cell has concerning some of its neighbors, most likely including selfawareness. From time to time the cells change their state, according to this knowledge. McCulloch and Pitts would have the connectivity of the cells modelling some physiological system, but lacking definite structures to follow, the tendency has been to use crystallographic lattices of low dimension. Von Neumann worked with two dimensions, which was also the arena for Conway's game.

Life presupposed binary cells occupying a two dimensional square lattice, the neighborhood of each cell consisting of itself, four lateral neighbors, and four diagonal neighbors; a total of nine cells altogether. Many other combinations are possible, but Conway chose one of them, as well as a particular rule of transition, for his game after discarding many alternatives. Adopting his picturesque ecological metaphor, binary cells are either dead or alive; in each generation,

- new cells are born to three live "parents"
- old cells survive if they have two or three live neighbors
- all other cells either die or remain dead

There are 2^9, or 512, different combinations of dead and live neighbors. Each combination can evolve in its own way, giving the enormous number of 2^{2^9} different rules, or games, which Conway could have chosen; nevertheless that one choice has lived up to his expectations of finding an interesting game. Part of the choice consisted in selecting a symmetric rule; it is reasonable to suppose that a square lattice would evolve similarly if it were rotated or reflected, as well as if any configuration were shifted to on side by a given distance. Beyond this, the rule depends on *numbers* of live neighbors, not on particular groupings.

Whatever the reasons for choosing one rule in preference to another, the analysis which follows is applicable to all cellular automata; so *Life* just happens to be a particularly interesting special case. Consequently its results are available for comparison and checking against other rules.

3 Still life

For any given rule of evolution, some cells will retain their existing state, while others change; generally there is no correlation between the two alternatives, giving the automaton a different appearance from generation to generation. Still, it could happen that there are particular combinations which remain immobile. One such, deliberately included in Conway's choice of a rule, is that if a cell and all its neighbors are dead, it remains dead. Automata which follow this requirement are said to have a quiescent state; live cells cannot appear spontaneously, but only near other live cells.

A quiescent state is not usually considered to be a still life; the latter term is reserved for collections of live cells for which cells neither die nor are born with succeeding generations.

It is curious that the simplest possible approach, if managed properly, suffices to enumerate the still lifes for an automaton. To begin with, neighborhoods can be classified as "good" or "bad" according to how their cell evolves. For the moment, a good neighborhood is one whose cell remains fixed; a cell that changes state is in a bad neighborhood. Obviously we are only interested in good neighborhoods.

The next step checks the neighborhoods of the neighbors; but not all pairs of neighborhoods overlap consistently. Only good neighborhoods that overlap well need be considered. The direct approach procedes along some path, fitting good neighborhoods together until an inconsistency results. By backtracking and considering alternative neighborhoods, it might be possible to generate a whole region which is unaffected by evolution. Trying again and again until all the possibilities are exhausted would eventually produce a complete list of static regions.

The whole plane can never be covered by this process; but it would be possible to stop when a quiescent border was reached, or even if the region began to repeat itself after a certain distance. So, at the very least, it should be possible to find all the still lifes covering a fixed area. Of course, the quantity of computation required grows exponentially with the area to be covered.

Giving first priority to the compatibility of overlapping neighborhoods, later rejecting those whose evolution is not satisfactory, places the computation on a firmer foundation. In either case, there is a simple diagram from which the compatibility of neighborhoods can be ascertained.

4 De Bruijn diagram

The representation of overlapping sequences and the establishment of some of their properties is facilitated by using a diagram which is often called the de Bruijn diagram, or its associated connectivity matrix. The concepts involved have had a fairly long history[7]; sometimes the diagram is given other names.

Basically, a diagram is prepared whose nodes represent short segments taken from a sequence; an example would be a string of three binary numbers, eight nodes corresponding to all the possible sequences. Links are drawn in the diagram according to the ways that the first member of the sequence can be discarded, and a new final member appended; it is natural to label the links by the longer segments, including the discarded and adjoined elements.

In this binary example, 0 could be discarded from the sequence 011; then 0 adjoined to produce 110 or else 1 to produce 111. Accordingly 011 would be linked to each of the nodes 110 and 111, but no others. The first link would be labelled 0110, the second 0111.

In the case of *Life*, and for automata in general, we are interested in dissecting the neighborhood of a cell into two overlapping pieces, each of which overlaps an appropriate partner among adjoining neighborhoods. In the following sample,

a	b	c	d	e
f	g	h	i	j
k	l	m	n	o

the cells g, h, and i have the respective neighborhoods and partial neighborhoods

cell	neighborhood	left half	right half
g	$a\ b\ c$ $f\ g\ h$ $k\ l\ m$	$a\ b$ $f\ g$ $k\ l$	$b\ c$ $g\ h$ $l\ m$
h	$b\ c\ d$ $g\ h\ i$ $l\ m\ n$	$b\ c$ $g\ h$ $l\ m$	$c\ d$ $h\ i$ $m\ n$
i	$c\ d\ e$ $h\ i\ j$ $m\ n\ o$	$c\ d$ $h\ i$ $m\ n$	$d\ e$ $i\ j$ $n\ o$

The de Bruijn diagram for *Life* and other automata based on the same neighborhood has 64 nodes, due to six binary cells forming each overlapping half. Eight links emanate from each node, since three binary cells are discarded and three added to advance from one neighborhood to the next. Links and neighborhoods correspond exactly; there are 512 altogether. Octal notation readily labels the nodes; if

$$\begin{aligned}\alpha &= 4a + 2f + k \\ \beta &= 4b + 2g + l \\ \gamma &= 4c + 2h + m,\end{aligned}$$

then the link $\alpha\beta\gamma$ joins the node $\alpha\beta$ to the node $\beta\gamma$.

The large number of nodes and links would make such a diagram laid out on a small sheet of paper overly crowded; a better representation would be the connectivity matrix of the diagram, or even a simple listing in which each line contained its own node followed by a list of the nodes to which it was linked.

The choice of a de Bruijn diagram whose links are the neighborhoods in Conway's *Life* means that any path through the diagram represents a possible row of cells in the automaton, surrounded by their respective neighborhoods. The nodes are partial neighborhoods; the lack of a link between a particular pair shows that they cannot be overlapped to form a complete neighborhood.

Insofar as the links represent neighborhoods, they can be considered to reflect the properties of their neighborhoods as well. By dropping the links corresponding to bad neighborhoods, any remaining paths through the diagram can only represent a good row, which in the present context would be a row of a still life. In other words, there exists a diagram from which all the still life rows can be read off just by following paths through the diagram.

Building up still rows is only the first stage of construction; a new second stage de Bruijn diagram governs the overlapping of rows to cover the plane.

5 First stage

The maximal first stage de Bruijn diagram has 64 nodes connected by 512 links. The nodes are representable by a pair of octal numbers or equivalently, by a single number modulo 64. In that case, it is easy to describe the connectivity matrix M_{ij}, whose elements are defined by

$$M_{ij} = \begin{cases} 1 & i \to j \\ 0 & \text{otherwise} \end{cases}$$

This matrix is shown in greater detail in Figure 1.

$$M_{ij} = 1 \begin{cases} j = 8i & \mod 64 \\ j = 8i+1 & \mod 64 \\ j = 8i+2 & \mod 64 \\ j = 8i+3 & \mod 64 \\ j = 8i+4 & \mod 64 \\ j = 8i+5 & \mod 64 \\ j = 8i+6 & \mod 64 \\ j = 8i+7 & \mod 64 \end{cases}$$

$$= 0 \qquad \text{otherwise}$$

Figure 1: the full first stage 64×64 de Bruijn matrix

By inspection, $Trace(M) = 8$ and M^2 is a matrix solidly filled with 1's. Therefore $Trace(M^2) = 64$ and M satisfies the minimal equation

$$M^3 = 8M^2,$$

from which its characteristic equation can be obtained. It is evident from calculating traces of its powers that there are many loops of all possible lengths in the diagram. There are also numerous Hamiltonian loops, these latter passing through all 64 different nodes, giving the longest possible consecutive sequences of neighborhoods that can be formed without repeating one of them.

Dropping links from the complete de Bruijn diagram will break some loops, maybe even isolate certain nodes completely. Normally such artifacts would be discarded; if a node had no exit links, it would mean that there was a partial neighborhood for which no right border existed so that the central cell would remain constant in the next generation. Such partial neighborhoods are barriers to extending a region of still lifes, therefore inappropriate to an infinite plane.

Such barriers would terminate any recursion reaching them in the direct approach to still lifes; the advantage of using de Bruijn diagrams is just that their occurrence is clearly located in the context of the finite number of distinct sequences of neighborhoods which exist.

Although emphasis has been placed on still lifes, it should be noted that the evolved cell can be compared to any of the other cells in the neighborhood from which it arose, not just the original cell. The simplest comparisons are with the shifted cells; by symmetry the interesting ones would be the longitudinal neighbor in the direction of progressive overlapping, the transversal neighbor perpendicular to the direction of extension, and the diagonal neighbor. Using the symbol (x, y, t) to designate a pattern in which a cell matches the one x cells to the right and y cells up after t generations, the fundamental neighborhood is large enough to detect the patterns (0,0,1), (0,1,1), (1,0,1), and (1,1,1). The first of these are the still lifes, the others might be called gliders. Strictly, Conway's glider corresponds to the pattern (1,1,4), his space ships to (0,2,4) or (2,0,4).

In principle any Boolean combination of the evolved cell with the cells of its neighborhood could form the basis of a pattern. For example, a pattern could consist of configurations which vanished after a single generation.

Figure 2: the de Bruijn matrix for *Life*'s still lifes

Figure 2 shows the connection matrix of the first stage de Bruijn diagram for still lifes, from which its resemblance to the complete diagram can be judged. A better understanding can be formed by inspecting several of the lower powers of the matrix, as shown in Figure 3 on the next page.

The block structure of M is evident by the power M^3; the fact that M^2 is still

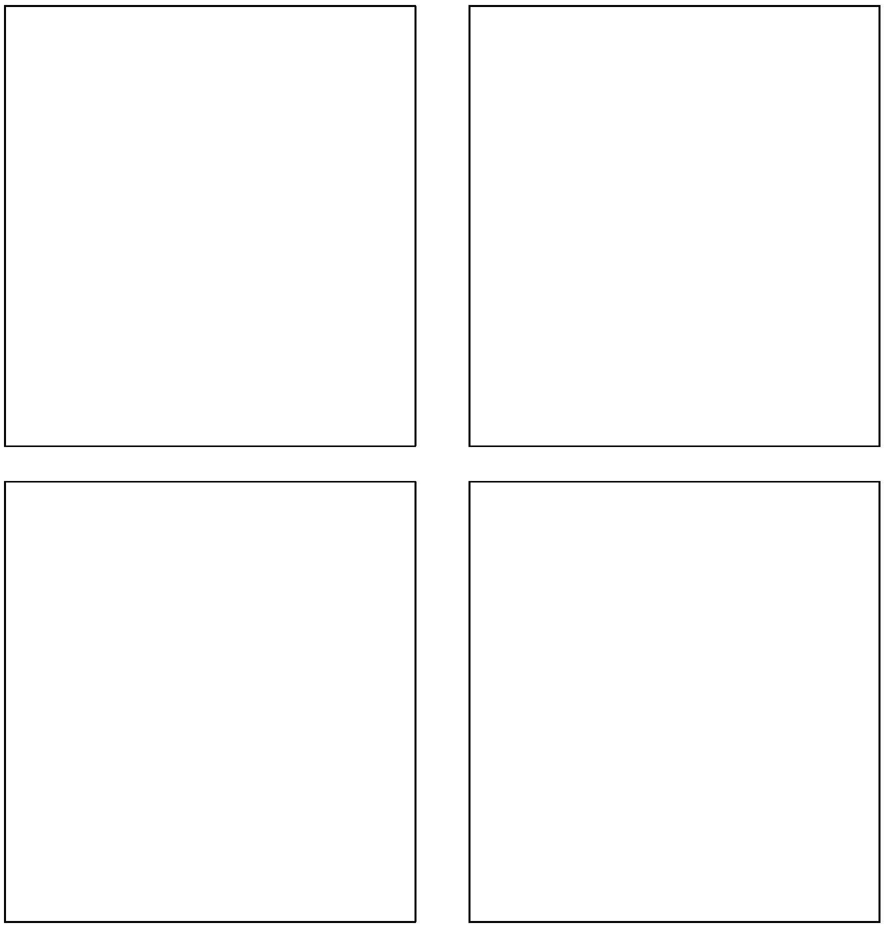

Figure 3: second through fifth powers of the de Bruijn matrix

fairly sparse but M^3 is not indicates the presence of an important component of period 3. This situation has been strongly evident to those who have examined large empirical collections of still lifes, among which the length three is a magic number. In terms of the properties of positive matrices, an explanation would be that the second largest eigenvalue was triply degenerate (in absolute value).

Although there is probably no real "explanation" of why 3 is magic, those who have worked with *Life* have noticed that the still lifes for a 3×3 torus are just those configurations with four live cells, an easy requirement to satisfy. The same rule works for a general $3 \times N$ torus, although an additional family of still lifes is also possible. Apparently breaking the periodicity in the cross direction still leaves ample opportunity to form still lifes. It would be interesting to know whether this phenomonon persists for other rules or for other lattices.

The ergodic set of the diagram has 57 nodes, the remaining seven never appearing in any still life. This is not surprising, since it is foreordained that the live central cell in neighborhood built either by extending the nodes 73, 76, or 77 to the right, or the nodes 37, 67, or 77 to the left will die. The node 57 cannot be extended to the right, nor 75 to the left, but each can be extended in the other direction. They, too, have to be removed from the diagram.

Usually the exclusion of nodes is more subtle, requiring much higher powers of the connectivity matrix to establish the pattern clearly.

Although relatively few nodes are lacking from the full de Bruijn diagram, about half the links are missing. In general such numbers are explained by the fact that there are two classes of neighborhoods, the good and the bad, and that they occur in roughly equal numbers. Thus there is a 50% probability of dropping any given link from the diagram.

Eight links must be dropped to leave a node without exit links; since $(\frac{1}{2})^8 = \frac{1}{256}$, there is about 0.4% chance of finding one such node, which means that not too many bare nodes will be found among 64. The fact that seven are missing has to be considered as a fluctuation, due to some special characteristic of Conway's game. Experience with other rules tends to confirm this evaluation.

Similarly rough estimates apply to the chance of finding loops; we expect 8^N loops of length N (with repeated subloops included) but cutting any link ruins the loop, so that there is one chance in 2^N of finding an intact loop of that length. Thus the expectation is that the number of loops will quadruple with any increment in length. The dominant eigenvalue of the de Bruijn matrix gives the most accurate estimate of this factor, which is generally close to 4.0.

The following table presents the number of loops originating with the quiescent state, the total number of loops, and the maximum number possible. Bear in mind that the trace of the connection matrix counts a loop once for each node which it contains, but not multiple traverses of portions of a loop, so the crude data does not reflect the usual number of cycles as they are commonly counted.

width	$(0, 0)$ element	Trace	maximum Trace
1	1	3	8
2	1	23	64
3	4	156	512
4	24	499	$4,096$
5	103	$2,613$	$32,768$
6	455	$12,320$	$262,144$
7	$1,114$	$57,235$	$2,097,152$
8	$10,708$	$279,523$	...
9	$51,006$	...	...

This data shows that the number of loops is multiplied by approximately four for each increment in length, and that there is a strong component of period 3; as confirmed by experimental tallies of the number of still lifes.

6 Second stage

The first stage de Bruijn diagram shows which rows of cells can form a still life (or other pattern). The second stage selects a width, then determines all possible rows of that length. A new de Bruijn diagram, whose links are these rows, describes strips of fixed width but arbitrary height embodying the desired pattern.

It is natural to ask why a fixed width is required. Amongst other representations, the admissible rows can be defined by regular expressions. From the expressions defining the link sequences (three rows high) can be extracted those defining the node sequences (two rows high); compatible rows would be described by intersections of regular expressions. It would seem that one has an instance of Post's correspondence problem, in trying to relate the two classes of regular expressions. Keeping the width fixed and finite eliminates this uncertainty from the calculation.

The simplest criterion for a row is that it be periodic, which means that it would be formed from loops of the first stage diagram of a given length. Other boundary conditions could be imposed, but they would seem to be rather artificial unless the context determining them were included. One possibility, somewhat contrary to the spirit of the generality being described, would be to build up a collection of still lifes with given boundaries. Matching such strips to get even wider strips would reveal some, but not all, of the still lifes having the the composite width.

A very important exception to the reccommendation to eschew such constructions is when the quiescent state forms the boundary. If the de Bruijn diagram contains a link connecting the quiescent state to itself, isolated patterns can be formed. If in turn the strips themselves are bounded by the quiescent state, there will be free standing figures which can exist in complete isolation from any others.

If "0" is the quiescent state, we need to concentrate on the (0,0) element of the connection matrix. In any event, suppose that the following illustration represents two consecutive rows of a strip of width 6,

a	b	c	d	e	f
g	h	i	j	k	l
m	n	o	p	q	r
s	t	u	v	w	x

This time the cells are the rows $ghijkl$ and $mnopqr$ with the respective neighborhoods and partial neighborhoods

cell	neighborhood	upper half	lower half
$ghijkl$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|}\hline a & b & c & d & e & f \\ \hline g & h & i & j & k & l \\ \hline m & n & o & p & q & r \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|}\hline a & b & c & d & e & f \\ \hline g & h & i & j & k & l \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|}\hline g & h & i & j & k & l \\ \hline m & n & o & p & q & r \\ \hline\end{array}$
$mnopqr$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|}\hline g & h & i & j & k & l \\ \hline m & n & o & p & q & r \\ \hline s & t & u & v & w & x \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|}\hline g & h & i & j & k & l \\ \hline m & n & o & p & q & r \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|c\|c\|}\hline m & n & o & p & q & r \\ \hline s & t & u & v & w & x \\ \hline\end{array}$

It is typical of the second stage that there are many broken loops and even isolated nodes, in contrast to the first stage where they are relatively infrequent. The same informal probabilistic arguments given for the first stage explain why.

The full diagram for a strip of width N would have 2^{2N} nodes with 2^{3N} links, thus 2^N links per node. But instead of a single neighborhood, half of whose links might be bad, there are N cells, whose "halves" are compounded, reducing the number of good links for the whole row by a factor of $\frac{1}{2}^N$; the two effects compensate, leaving us with an estimated single link per node. Fluctuations can just as readily leave a node without a link as provide it with a pair of links, so that the surviving core of loops—the ergodic set—will still have a certain amount of variety.

One dreads to think of what would happen if this reasoning remained valid for a third stage, as in a three dimensional cellular automaton.

Although the simplest cases are almost trivial, they are also easy to understand, which will help to fix our ideas and understand the general case.

Figure 4: second stage de Bruijn diagram for width 1

Choosing width 1 means working with constant rows; such a system is a one dimensional linear cellular automaton evolving according to Wolfram's Rule 22. Our table predicts three still life neighborhoods out of eight possible; the pertinent de Bruijn diagram is shown in Figure 4. There is just one still life, discounting the quiescent state; it consists of alternating rows of live and dead cells.

Although this is the simplest case, it already shows some structure, namely that the de Bruijn diagram can consist of two disconnected pieces. It also shows how links and nodes can be dropped from the full de Bruijn diagram. Showing the actual diagram as a subset of the full diagram would be more instructive if a larger diagram were used, but showing complete diagrams for greater widths is too cumbersome for the printed page.

Figure 5: second stage de Bruijn diagram for width 2

Width 2 shows slightly more variety. It is equivalent to working with a (4,1) cellular automaton (4 states, first neighbors).

The unrefined de Bruijn diagram has 11 nodes with 23 links, which reduces to 7 nodes with 19 links and finally to the cyclic core with 7 nodes and 9 links shown in Figure 5. Inspection shows that the de Bruijn diagram decomposes into three disjoint pieces. The first contains just the quiescent state linked to itself; of course we can ascribe any periodicity to the quiescent state that we wish, making it a common feature of all diagrams.

The second is really double the configuration of width 1, in which rows of live cells alternate with rows of dead cells. There is just one new configuration, the third piece, a sort of grillework in which the minimum length of each tier of vertical bars is two cells. Both these latter configurations have to extend indefinitely, since there is no transition bringing either of them to the quiescent state.

	00	01	02	04	03	05	06	07	10	20	40	30	50	60	70	17	27	47	71	72	74	33	35	36	53	55	56	63	65	66
00	1	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
01	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
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07	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.
10	1	1	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
20	1	1	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
40	1	1	1	1	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
30	.	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
50	.	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
60	.	.	.	.	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
70	.	1	1	1	1	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
17	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
27	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
47	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
71	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
72	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
74	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
33	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
35	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
36	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
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56	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
63	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
65	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
66	.	.	.	.	.	.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Figure 6: second stage de Bruijn matrix for width 3 - connected component of 00

In contrast to narrower strips, width 3 is striking both for the complexity of its still lifes and the simplicity of the rule generating some of them. The de Bruijn diagram is too complicated to show; however its two disconnected pieces can be exhibited in matrix form.

The first, consisting of 30 nodes, is the least regular, but is the connected component of the quiescent state. If pairs of octal numbers are used to index the nodes, each each member of the pair translates directly into a three-cell cross section of the strip. The pertinent connectivity matrix is shown in Figure 6.

In all diagrams and connectivity matrices a certain amount of symmetry must be present, because the rule of evolution remains the same even though the row from which the diagram is derived is longitudinally shifted or even reflected. The symmetry can manifest in various forms according to whether the still life has the same symmetry as the row, or whether there are several equivalent patterns arising from one another via the symmetry operations. For longer rows whose lengths have various distinct divisors, intermediate cases of partial symmetry can arise.

	11	12	14	21	22	24	41	42	44	13	15	16	23	25	26	43	45	46	31	32	34	51	52	54	61	62	64
11	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
12	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
14	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.
21	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
22	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
24	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.
41	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
42	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
44	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.
13	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.
15	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.
16	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1
23	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.
25	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.
26	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1
43	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.
45	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1	.	.	.
46	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1	1	1
31	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
32	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
34	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
51	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
52	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
54	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
61	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
62	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
64	.	.	.	.	.	.	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Figure 7: second stage de Bruijn matrix for width 3 - connected component of 11

The second component, consisting of 27 nodes, is disconnected from the quiescent state, but is extremely regular. To begin with, the rule of formation is extremely elegant; given any neighborhood containing exactly four live cells, three cells are to be chosen for the new margin so that the overlapping neighborhood also has four live cells; a construction clearly unique to *Life*. Its connectivity matrix constitutes Figure 7.

Close inspection shows that sections of odd parity—with a single live cell—alternate with sections of even parity—having two live cells—in the ratio of two odd sections to one even section; thus the importance of the height being a multiple of three. Moreover it is pairs of sections which count—the half neighborhoods—so the sequence even-odd, odd-odd, and odd-even must be rigorously followed. Consequently the de Bruijn matrix is imprimitive of degree 3, explaining the magical properties of the number 3.

Further inspection shows that the matrix is the tensor product of a cyclic shift matrix with a full de Bruijn matrix for three states and two stages, which is a consequence of the transversal symmetry. In other words, it doesn't matter which section of a given parity is chosen, but from there on further choices must all be consistent.

Figure 8: typical still lifes for a strip of width 3

To better visualize the still lifes of width three, Figure 8 shows typical samples, one for the component of 00, the other for the component of 11. The former contains three fundamental motifs; horizontal bars which must have castellations whenever they are not surrounded by similar bars; rows of blocks, and extended snakes, using the standard terminology of *Life*. The motifs may follow in any sequence, and with any phase relative to one another.

Going on to widths of 4 or wider, more and more complicated diagrams are encountered. However, a new phenomonon becomes visible with width 4, which is that the quiescent state may have nontrivial connected components in both the first stage and the second stage. Since the quiescent state is always self-connected, an arbitrary number of additional quiescent states may always be inserted wherever a single one was encountered. Consequently the live cells from any configuration so found continue to form a still life when moved to an environment which is not periodic.

Furthermore, with fewer diagrams, visual presentations for slightly wider strips are still possible.

Figure 9: freestanding strip of width 4

Strips of width 4 show the first nontrivial component of the quiescent state; the second order diagram for this component has 24 nodes, as shown in Figure 9. This is the smallest diagram in which completely freestanding figures occur; they are separable from one another or others by arbitrarily long quiescent intervals, either horizontally or vertically.

Of course, the full de Bruijn diagram for width 4 could also be exhibited; but it isn't since the intricacy of the diagrams increases exponentially with width, making their explicit representation increasingly cumbersome and less visual. Eventually the only feasible representation consists of a table, whose entries list the possible successors of each node in turn.

7 Comments

It is disconcerting to see the degree to which the detailed labor of hundreds of people, and untold hours of computer time, has been reduced to an algorithm. Even though the necessary computations can be realized quite quickly, it is no less impressive to foresee the stupendous amount of time which would be required to obtain certain additional results which initially seem quite simple. For instance, it is out of the question to use the same approach directly to calculate any structures of period 2, much less gliders and space ships which require period 4.

Much of the activity reported in Wainwright's newsletter[8] involved tracking the evolution of diverse small figures, a byproduct of which was a gradually increasing catalog of small still lifes. Many of them grouped themselves into families, whose general structure could be readily perceived, and many people seem to have become quite skilled at designing still lifes and other predictable patterns. Moreover, after having reviewed large collections of still lifes, one develops an eye for flaws and a feeling for what constitutes a proper still life. Which implies that there must be some pattern present which can be recognized.

Golomb's book[4] on shift register sequences disseminated the use of de Bruijn diagrams to characterize long sequences of overlapping symbols, although an application to automata does not seem to have been published until Wolfram's article[9] in 1984. Nowadays they can be seen as a tool for quickly obtaining the personalities of arbitrary automata, subject to the limitations imposed by exponential growth with respect to any of the parameters involved—dimension of the automaton, length of the period, number of states, and so on. Perhaps an even more elaborate theory of higher dimensional de Bruijn diagrams will eventually result.

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