What Has
and
What Hasn't
Been Done With
Cellular Automata

Harold V. McIntosh
Departamento de Aplicación de Microcomputadoras,
Instituto de Ciencias, Universidad Autónoma de Puebla,
Apartado postal 461, 72000 Puebla, Puebla, México.
E-mail:mcintosh@servidor.unam.mx

November 10, 1990

Abstract

Research on the subject of cellular automata is surveyed, with the intention of distinguishing between what *has* and what *has not* been accomplished during the course of its history.

1   Introduction

The subject of cellular automata has passed through several stages of evolution during the fifty or so years during which it may be thought to have existed. It is interesting to see what has been accomplished during this time, and not only to speculate as to what remains to be done but also to wonder to what degree the subject has applications, and how successful it will be in meeting the expectations which may be held for it.

If an automaton is supposed to be a mechanism capable of performing intricate tasks or exhibiting complex behavior, the concept is surely as old as antiquity; but it was not until the mastery of electricity at the end of the last century that really delicate and subtle constructs could be imagined. Far beyond that, it required an advanced degree of electronics, and even of microelectronics in the form of reliable vacuum tubes, transistors, and nowadays integrated circuits, before truly elaborate automata could become a reality.

For this reason, the innovative ideas of Warren McCulloch and Walter Pitts[50] and the designs of John von Neumann[53] are more properly considered to be the foundations of automata theory than, say, Charles Babbage's[3] proposal for his difference engine or the more advanced analytical engine which made use of ideas taken from the Jacquard loom. As it is, there is a wide variety of sources from which to choose, whatever one's point of view. Little of this will be taken into further account here, but for those interested there are excellent references which can be consulted[60, 2, 30, 26].

McCulloch and Pitts' investigations concerned the possibility of constructing a model of mental processes. While derived from their knowledge of the physiology of nervous systems, their model was intended mostly to reflect characteristics such as the connectivity of the neural net or the interactions in Boolean terms of signals propagating through it. Neither the strict form of the signals, nor the material composition of the neurons were considered to be primary issues.

In similar fashion, von Neumann thought of different models, some of them quite physical, through which his plans for automatic construction could be realized. Nevertheless, as is quite the custom among mathematicians, the abstract approach proved to be the formulation which seemed best to capture the central ideas.

From such a beginning voluminous results have followed, both in the form of mathematical theorems and in the design of specific circuits, all of which have been duly recorded in the scientific literature. Cellular automata achieved substantial popular notoriety when John Conway took up the subject[16] and Martin Gardner reported one of his most interesting discoveries in *Scientific American* [28]. The evangelical efforts of Stephen Wolfram[70] have given a whole new dimension to the topic in recent years.

2 What has been done

When certain subjects are mentioned, they are immediately associated with a certain body of results; others are more tentative or may even be in the process of revision. There is probably a certain consensus as to what the theory of cellular automata comprises, overlapping somewhat with automata theory in general, and touching upon the theory of probabilistic automata. The following synopsis is a mixture between a catalog of topics and an account of the historical development.

2.1 The concept of an automaton

Whether or not the ideas of McCulloch and Pitts have any bearing on neurophysiology, there is no doubt that their proposals stirred up a considerable amount of interest, some of which can fairly be seen as having resulted in the

present theory of automata. Indeed, von Neumann made quite direct and extensive reference to their ideas in the process of designing the EDVAC as well as in his work on self reproducing automata.

Another direct outgrowth of their work was origin of the theory of regular expressions, which was formulated by Stephen Kleene[45] as a rigorous logical and algebraic foundation for their ideas. In turn, regular expressions have been found to lie at the foundation of the theory of formal languages; indeed they constitute just about the simplest of all languages, because the others always include them. This is a subject which can be pursued in great detail, but which is not strictly a part of the theory of *cellular* automata. Nevertheless the circle closes when it is found that regular expressions are invaluable for describing paths through graphs, one of the preferred representations by which the evolution of cellular automata may be described.

All of these early ideas have undergone extensive development since they were first proposed. To start with, von Neumann was deeply involved with a critical phase in the development of computers, which coincided with his interest in cellular automata. As a mathematician he had an interest in symbolic logic and the theory of computation which permeated his approach to computer design. Setting out to equip a computer with the organs required to realize a certain philosophy of computation is a rather different proposition from uncovering all the capabilities of a complex circuit which one has designed.

The logical issues which preoccupied von Neumann would not have concerned all designers, nor did he necessarily cast computer design into a mold which would not otherwise have shaped it. Nevertheless decisions ranging from the choice of the binary number system to the systematic use of stored programs evolved during that formative era. Previously computers had not been complex enough to make the issues important, although Ada Lovelace touched on some of the points in her description of Babbage's Analytical Engine; subsequently computers worked well enough and were sufficiently well understood that after 1956 or thereabouts they tended to be copied rather than redesigned as they went into commercial production.

Another direction which the early ideas took was the growth of the general theory of automata as well as the study of formal languages. In fact the McCulloch-Pitts "neurons" were only one example of digital circuitry, as distinguished from analog circuitry; naturally the use of such circuits required a theory to explain them. Boolean algebra served for combinatorial circuits; the incorporation of memory elements or time delay elements extended the theory to take into account sequential circuits, the kind that could be used in computing machines.

General automata differ from cellular automata in two important respects. Ordinary automata are concerned with input and output, that is, the transformation of signals through the intervention of the automaton. Moreover, design problems are typically concerned with constructing the precise automaton which will produce a given transformation, ascertaining the possible equivalence of two

automata in their signal handling abilities, finding the simplest automaton for a given purpose, and so on. The second characteristic is that they are typically small, in any event finite, being intended for use in actual construction.

In contrast cellular automata are required to form lattices, infinite in principle if not in practice, and do not work with signals. Rather, the "signal" which activates each cell is its knowledge of its own state and the states of a certain number of its neighbors at any given moment. Were the lattice not infinite, there would be no discussion of Turing machines, universal computation, or the like. On the other hand, just because of its large scale regularity, a cellular automaton lends itself readily to implementation in terms of microelectronics.

In any event, it could hardly be expected that it would be possible to discuss cellular automata without first taking a certain amount of the general theory into account.

2.2 The evolution of finite systems into cycles

It is not surprising that the reasons for studying automata change from time to time, and that the amount of detail required keeps increasing. Basically a cellular automaton is a lattice formed from cells, each of which has k states, conveniently numbered from 0 to $kkkkkkkkkkkkkk - 111111111111111$. As a consequence of forming part of the lattice, each cell may be associated with several nearby cells, which form its neighborhood. The geometrical shape of the neighborhood may vary, but typically only the closest neighbors up to some radius r are included. Neighborhoods are supposed to have the same form for all cells, and are therefore translates of one another. Consequently there is a single function φ to specify the state of a cell at the next generation in terms of the states of its neighbors (generally including itself) in the present generation.

Although φ relates cells to neighborhoods, its systematic application throughout the lattice defines a mapping Φ from the configuration—assignment of states to lattice sites—of the automaton in one generation to that of the next. Of course, the central problem of the theory of cellular automata is obtaining a satisfactory description of Φ.

If an automaton, such as von Neumann's universal constructor, had been carefully constructed to serve a special purpose, the greatest interest in its operation would naturally lie in whether it performed according to specification or not. On the other hand, if an arbitrary automaton were presented for examination, the most natural reaction would be to observe the course of a typical evolution, that is, the result of repeated iteration of Φ. Different concepts of "typical" would be likely to produce somewhat different results. The initial configuration might be a simple pattern, such as a group of cells in one state with the remainder in another. Alternatively, some short sequence of cell states might be repeated indefinitely, giving a periodic initial configuration. If still greater variety were desired, the states of the individual cells might be assigned with the help of a random number generator.

4

The best way to reach a compromise between finite and infinite automata would be to combine the second alternative with the third. But then one would soon find that mappings of finite sets into themselves have one fundamental property which turns up over and over again in practically any context whatsoever. Their iterations must eventually become periodic. Evolution being the pertinent mapping for cellular automata, the inescapable conclusion is that the sequence of configurations through which the automaton evolves must eventually become cyclic, oftentimes even static. Thus one is advised to begin the analysis of an automaton by ascertaining all of its periods, both temporal and spatial.

Knowing whether an automaton is large enough to be considered infinite, distinguishing between periodic and aperiodic extensions, and resolving other philosophical dilemmas have a practical importance. Recognizing an entity such as the square root of two so that the diagonal of the unit square can have a length only requires that that such a number can be represented to an arbitrary accuracy. Likewise, although a Turing machine must never be denied additional tape, a generous allotment might suffice for all the calculations required by some project. Thus it is convenient to work with finite structures in such a way that their scale does not have to be stated explicitly, making them implicitly infinite. It then remains to identify any additional structure which may have inadvertently been introduced.

Describing the asymptotic behavior of infinite, but not periodic, configurations within an automaton is a relatively recent development, much of whose motivation arises from trying to identify the computational powers of automata. At the same time, it has been realized that the transient portion of their evolution, particularly when it persists for an exceptionally long time, is not entirely uninteresting, and that its statistical properties warrant investigation. Also, there is interest in locating some or all of the cycles without the necessity of performing all the trial evolutions, the possibility of which is part of the theory of computational complexity.

Whatever the degree of sophistication of the presentation, it is not likely that one is going to find a discussion of a cellular automaton which omits all reference to its periodicity, from its static configurations onward.

2.3   Self-reproducing automata

Crystal growth does not qualify as a reproductive process, whilst the real process involving DNA molecules and a whole chemical environment remains far from being fully understood; nevertheless one is skeptical that it will be found to contain some complex computing mechanism. Certainly comprehension of biological reproduction has advanced considerably since those days when chromosomes were known but genes were merely conjectured on the basis of empirical observations, which was still the case when cellular automata came into being.

Questions of reproduction, be it sexual, asexual, or self reproduction, tend to be rather philosophical in nature, along with questions of consciousness, the borderline between the living and the nonliving, and similar issues. A robot is easy enough to imagine, not to mention the possibility of its having a certain dexterity and the ability to follow instructions of a sort. Never mind that functional examples may be in short supply, but it was just this gap between fanciful speculation and actual practice which led von Neumann to settle instead for self patterning wallpaper. The suggestion came from Stanislaw Ulam, who had been experimenting extensively with functional transformations and means of representing them.

By working with a medium which could support a variant of Alan Turing's hypothetical computer, the constructor which von Neumann sought could avoid the monotony of reproduction through crystal growth by calculating the nature of its intended construct, opening quite a few possibilities of variability, adaptation, and evolution. Presumably the use of a full Turing machine would leave the constructor with the utmost versatility possible in choosing its products, or the strategies for carrying out its constructions. Consequently it would appear that whatever such a model lacked in realism might be compensated by the possibility of establishing theorems similar to those which Turing proved for general computations, namely the existence of a universal machine and of undecidable calculations.

The resulting object was cumbersome, containing twenty nine state cells arranged on a two dimensional square lattice, and a finished universal constructor that would occupy an area thousands of squares on a side if it were ever to be "built." Given that the construction originally contrived to prove a point is not necessarily the simplest or most elegant route to the same conclusion, it is not surprising that there are other alternatives, such as one with eight states found by Edgar Codd[15]. Nor is it surprising that von Neumann himself found that as he worked out the details of more and more of his proposed machine, he encountered ideas that would simplify the construction of the parts of the machine which he had already designed. Earlier this trap had thoroughly ensnared Babbage, who repeatedly laid aside plans for one machine in favor of another, still more elegant one, while gradually running out of financing.

The requirements which von Neumann laid on his machine may have been much more severe than necessary. If only the emulation of a Turing machine is required, Alvy Ray Smith III[65] has shown a fairly straightforward way to do so; if only reproduction is required, Christopher Langton[46],[47] has shown some simple variants of Codd's automaton which will grow, neither using a computer to guide the expansion, nor even their own description. Langton does argue, however, that a description of sorts is inherent in the form of the simplest reproducing unit of the organism.

The search for automata with elegant computational properties still goes on, and automata continue to be judged either by the power of the computation that they are able to carry on, or the power of the computer required to understand

6

their behavior.

2.4   The Garden of Eden

The evolution of an automaton is nicely represented by a transition diagram or graph; every configuration is represented by a node, the nodes are linked according to the rule of evolution. Chains formed from the links describe long term evolution. The deterministic character of evolution implies that there is exactly one outgoing link for each node. In the forward direction, chains must close into loops; they could also continue indefinitely when the number of configurations is infinite.

In the backward direction the additional possibility exists that a configuration has no ancestor. For a finite diagram that is a necessary alternative if any of the chains are confluent. The reason is simply that after counting exactly one link per node in the outgoing direction, a node with two incoming links must deprive some other node of a compensating incoming link.

Simple counting does not work for infinite diagrams, but the fact that the neighborhoods defining the evolution have to overlap to some degree allowed Edward F. Moore[51] to show that a similar conclusion nevertheless follows, namely that whenever some configurations have multiple ancestors, there must be others which have none; the term Garden of Eden seemed appropriate to describe the situation.

The result is an interesting limitation on the kind of constructions which an automaton can perform. But a universal constructor is not expected to construct *everything*, and according to this theorem, clearly cannot. Generally it is considered adequate that it can construct objects at least as complicated as a Turing machine, or slightly more so, so as to be able to make copies of itself.

There are straightforward procedures for ascertaining the Garden of Eden configurations of a one dimensional automaton, as well as the actual ancestors for the remaining configurations. Jen[43] has shown one such construction, based on a symbolic de Bruijn diagram. It is typical that the corresponding calculations are undecidable for arbitrary two dimensional automata or beyond, generally because of conflicts arising from the Post correspondence principle. Nevertheless two Garden of Eden configurations for *Life* have been published [29, page 248],[9, page 829], and a third cited[29, page 248], all of them were evidently encountered after a long series of exhaustive eliminations.

Garden of Eden configurations represent one end of the evolutionary tree, its leaves, which lie at the opposite extreme from the cycles, which form its roots. Discovering them requires more than simply following out trial evolutions, but their inclusion nevertheless forms an interesting part of the description of any given automaton.

2.5 Conway's *Life*

For some time the theory of cellular automata consisted of the inevitability of evolution into cycles, the existence of pristine initial configurations, and a respectful reverence for the complexity of von Neumann's detailed plans and the philosophical implications of his efforts. In the practical direction, a variety of efforts with image processing and photographic interpretation were understood to be related as much to cellular automata as to Fourier or Walsh transforms[58], and some thought was given to designing circuit arrays as automata.

One way or another. the lore of automata theory has gradually spread around. Studying cellular automata has become a popular computer game in addition to being a learned academic subject in several stages. Perhaps the initial step resulted from John Conway having decided to review the area just at the time that computers acquired versatile visual display equipment.

By taking the approach of first defining the automaton, then examining its capabilities, Conway[28],[29] encountered a two-dimensional binary cellular automaton of incredible elegance and beauty. Christened *Life*, it was eventually shown to have the same powers of universal computation and self-reproduction as von Neumann's automaton[9]. Even so, much work and several years were required to acquire familiarity with *Life*; many of the results were reported as they were being discovered in a quarterly newsletter published for nearly three years by Robert T. Wainwright[68]. Later some of the information was published more formally in a book written by William Poundstone[57]. The substance of all the notices in Martin Gardner's columns was collected in one of his reprint volumes[29].

From the outset Wainwright undertook to catalog the *Life* configurations that were being discovered, classifying them roughly along the lines of the general theory. Conway knew about some travelling configurations called gliders, but was apparently unprepared for the discovery of first "glider guns" and then "puffer trains" which between them permitted the assembly of incredibly complex artifacts. Nevertheless he was able to use them to achieve von Neumann's results, although in terms of an equally arduous construction spreading over a fantastic area if it were ever to be realized in practice. The number of contributors to Wainwright's newsletter was considerable, but the contributions of William Gosper and several associates at MIT's Artificial Intelligence Laboratory were fairly outstanding. Some of their adventures have been recounted in Stephen Levy's book, *The Hackers*[48].

Conway's choice of a particular rule and lattice resulted from careful experimentation, giving *Life* a relatively distinguished setting. Poundstone[57] reports experiments with an alternative rule, Packard and Wolfram[55] surveyed numerous two dimensional automata; in both cases reporting that their automata fail to meet Conway's criteria. It would appear to be an interesting challenge, either to account for the singular nature of *Life* or to encounter additional specimens.

Besides offering a new and interesting automaton to the existing repertoire,

Conway's contribution was to begin with specific automata with the intention of ascertaining their potential behavior, rather than beginning with the application and searching for the automaton which would fulfil the requirement.

2.6 Wolfram's classes

As computer power steadily increases, there have been ever increasing opportunities to perform experimental mathematics, both by professionals and by amateurs. An excellent example of this tendency has been all the recent work on fractals, nonlinear differential equations, dynamical systems, and similar topics. Many extremely classical results, such as the results of Pierre Fatou and Gaston Julia concerning functional iteration, some very esoteric topological results such as Stephen Smale's strange attractors, or even such advanced theorems relating to differential equations such as Kolmogoroff, Arnol'd and Moser's results in celestial mechanics, acquire an entire new perspective when their consequences can be followed through a numerical calculation.

A rather interesting confrontation of this nature took place when Stephen Wolfram began to experiment with the evolution of one dimensional cellular automata, and felt that he saw a certain amount of analogy between the phenomenological characteristics of the evolution of the automata and some classifications of limit sets which had been found relevant in nonlinear dynamics. As a result of numerous observations, he proposed a system of four classes, which reflected the same number of different kinds of evolution which he had been observing. Roughly speaking, cellular automata seem to settle down to a constant field (Class I), isolated periodic structures (Class II), uniformly chaotic fields (Class III), or isolated structures showing complicated internal behavior (Class IV).

In addition to a series of computer experiments in one and two dimensions, with varying numbers of states per cell and sizes of neighborhoods, Wolfram, occasionally working with some coauthors, has analyzed the mathematical aspects of the rules of evolution of their automata. Thus[72] discusses the relation of automata to formal language theory, including the observation that the evolute of a configuration described by a regular expression is still a regular expression, but may fail to remain so in the limit. Lyman P. Hurd has taken up the explicit question of the limiting behavior in his thesis[39] and related publications[38],[40].

Wolfram also discusses some measures of complexity, such as the size of the automata associated with the regular expressions.

Martin, Odlyzko, and Wolfram[49] have worked out the evolution of automata for which φ is defined by a linear combination of the states of the cells of the neighborhood, assuming them to be elements of the corresponding finite field; an example would be evolution according to the rule of parity, in which each cell transforms into the sum of the three neighbors, *modulo* 2. Although such automata may not be entirely typical, the fact that quite complete and

exact results can be obtained makes for useful comparisons.

Wolfram has edited a collection of most of his own papers and a good assortment of others, including several useful appendices containing a wide variety of data[73].

Notwithstanding the fact that assigning an automaton to one of Wolfram's classes is an undecidable proposition[17], his investigations represent the first time that a systematic and exhaustive study had been made of large classes of one dimensional linear cellular automata, rather than selecting a single automaton for concentrated attention. At the same time the classification has a strong visual appeal and has been widely adopted by other investigators, along with a few notational details such as his numbering scheme.

2.7   Probability measures

For various reasons, the statistical properties of the evolution of cellular automata have attracted attention. That there are statistical properties to be investigated is immediately apparent whenever there are graphical means available to display the evolution, such as can be done readily with most computers nowadays. Typically, if an initial configuration is randomly chosen, it quickly develops a characteristic texture which persists until the inevitable cycles make their appearance and dominate the evolution; thus there are usually two inherent time scales and three textures associated with the evolution of any given automaton.

The first phase can be orderly or random, in either event reflecting the details of the initial pattern or random number generator used to produce it. Its duration gives some idea of the incompatibility between the evolutionary rule and the initial arrangement; for random patterns the time is typically quite short, but for quiescent automata with a recognizable "velocity of light," a reasonable propagation time must be allowed for quiescent regions to feel the influence of their neighbors.

One would like to think that there might be a "self consistent probability" associated with any particular evolutionary rule, but calculations based on the simple rules applicable to independent probabilities produce indifferent results, traditionally disclaimed by invoking unaccounted correlations. Evidently an appeal must be made to more elaborate statistical concepts if better results are required; the crude theory usually produces results which match observations at the level of 10% to 20% error, which is often considered adequate.

Otherwise it is necessary to investigate measures which commute with the evolution or are otherwise related to it. Related concepts such as entropy are also relevant, and can also be studied. One of the first studies to be made was that of Schulman and Seiden[62], who tried to explain their measurements of Conway's *Life*. Later Dresden and Wong[24] made corrections for correlations, but the most comprehensive studies have been made by Wilbur, Lippman and Shamma[69], and then by Gutowitz, Victor and Knight[34].

There are actually three or four levels at which self consistent probabilities can be approached. All of them involve comparing the probability that a cell lies in a certain state with the probability that the cells in the neighborhood are in just the states in which they are found. Self consistency requires that the two probabilities agree, but the methods differ in the way that the probabilities are estimated.

The simplest comparison is to ask how many of the possible neighborhoods can evolve into the state in question; the answer may range from 'none' through 'an equal share' to 'all of them'. The uniform fields resulting from the extremes produce no surprise, a state with scarcely any ancestors would not be expected to be very common, while a democratic assortment of rules would lead one to expect a representative mix of states. In general terms these expectations are seem to be confirmed by experience and are thus well founded.

It is a general statistical principle that the variance in averages decreases inversely as the square root of the sample size; as applied to the evolution of automata it is possible to obtain very stable frequencies for the occurrence of the different states by collecting them from an automaton with a very large number of states – say a thousand or more. When this is done, discrepancies of tens of percent are found with respect to estimates by rule of thumb, such as counting transition rules.

The next level of self consistency consists in estimating the probability of a neighborhood by combining the probabilities of its cells according to the traditional rules, while assuming the validity of the traditional assumptions of independence. Called mean field theory, it provides polynomial equations to be solved for the self consistent probabilities. Indeed, the simpler estimate is just the probability that would be assigned the standard distribution – the one in which all states have the same probability – and thus could be the starting point for an iterative solution of the mean field equations.

Empirically mean field theory provides better frequencies, but not all automata perform equally well and discrepancies still range on the order of 10% or so, a useful but hardly a precision result. So the next level concentrates on taking correlations and other factors into account. The block structure theory of Gutowitz *et.al.* is based on Bayesian extension; empirical studies have shown that by taking into account probabilities of blocks of six, eight, ten or more cells, frequencies can be matched to two or more figures. The self consistency equations involve rational fractions rather than polynomials, and can also be solved by iteration.

The final level would be to obtain the invariant measure of the automaton, but having to describe it numerically leaves one on the level of local structure theory with very long blocks. Whatever the level of detail at which they are attempted, it would seem that the probabilistic approach is a significant alternative, and a complement to, the point of view of formal language theory.

11

2.8   De Bruijn diagrams

Whether working with probabilities, measures, or simple evolution, the overlapping which occurs between different neighborhoods comprises the greatest technical obstacle to computations; unless this problem is resolved adequately, further progress is almost impossible. Fortunately, for one dimensional automata, a diagrammatic technique which lies at the heart of shift register theory[31] saves the situation; the diagrams are called de Bruijn diagrams, but they are just simple graphs showing the possible ways in which different neighborhoods can overlap[59]. Erica Jen has shown how many properties of automata can be extracted from such diagrams, especially the static and periodic configurations on a cylinder of fixed circumference[44].

In principle, such a diagram could be extended to automata of higher dimensions, but a problem arises from selecting partial neighborhoods that will join to form full neighborhoods in all directions. The straightforward approach of building up strips of successively higher dimension runs afoul of Post's correspondence principle when arbitrary intermediate strips have to be matched to form the strips of the next higher dimension. If only periodic solutions are required, the problem is still soluble, but again the conflict between large systems and unbounded systems arises, tending to leave the generic properties of aperiodic systems undecidable.

At least in one dimension, there is nothing difficult about a de Bruijn diagram; as applied to cellular automata it is simply a graph in which partial neighborhoods are the nodes, with links connecting those which may overlap to form a full neighborhood. Given this correspondence between links and full neighborhoods, each link is also associated with the evolved cell belonging to the neighborhood. Consequently characteristics of the evolution can be used to select subgraphs of the de Bruijn diagram; for example, there is a subgraph composed of the neighborhoods whose central cell never changes. Global properties of the automaton can be read off in terms of the chains to be found in such a subdiagram; for the present example, the chains determine all the static configurations.

The advantage of using a de Bruijn diagram is that many problems concerning automata are thereby transformed into known problems regarding the tracing of paths through a graph. For instance, no loop can be longer than the total number of nodes in the graph without repeating some segment; but then there must exist still other loops in which the repeated segment is traversed an arbitrary number of times. For example, a binary automaton depending upon nearest neighbors has eight distinct neighborhoods, representable as eight links connecting four nodes, it follows that no static configuration can be more than four cells long without repeating some two-cell partial neighborhood. Thus the static configurations are rather severely constrained.

Sometimes the de Bruijn diagram reveals information about localized aspects of a configuration. For example if an acceptable path terminates at a node in

which *all* the outgoing links are acceptable, it need continue no further. Likewise if all the incoming links are acceptable, the path may begin just as though it had been part of a loop. Thus semi infinite structures may be located, or even finite ones if both ends have such universal terminations. This leads to the phenomenon of membranes and macrocells which Wolfram noticed during the course of his investigations. That is, an automaton may have patches which are isolated from one another by static regions, whose evolutions proceed quite independently.

The converse process is also possible, to define the rule of evolution of an automaton by postulating that the de Bruijn diagram have prescribed properties; for example that the unit cell (0101) must be a static configuration. To the extent that the requirements are not contradictory, and all the possibilities are covered, an automaton may practically be designed to order.

Enumerating the paths through a graph is a classical task, to which many papers have been devoted, but which has a particularly elegant solution in terms of regular expressions. Conway's book on regular algebra[16] expounds the technique; a later article of Backhouse and Carré[4] gives a very thorough presentation.

Probabilistic versions of both the de Bruijn diagrams and the evolutionary diagrams exist, being useful for studying correlations between blocks of cells, or in the numerical calculation of self consistent block probabilities. All the standard theorems regarding positive matrices and stochastic matrices apply. All told, the introduction of graphical techniques to automata studies is very profitable.

3   What has not been done

It is hard to know whether there are likely to be any surprising new developments in automata theory, in order to be able to predict them. To judge from past experience, each technological episode which has produced substantial new computing power has brought with it some new development in automata theory. The industrial revolution allowed Babbage to envision a mechanism which transcended dolls having the style of a Swiss music box operated by cams, levers, and gear wheels; even so, inadequacies of technology and project management kept a functioning machine from becoming a reality.

A century later, the first completely electronic computer inspired von Neumann to speculate about the possibilities of automatic factories, leaving a huge gap between his mathematical symbolism and the ability to design specific cellular automata, much less their translation into performing mechanisms. Greatly improved computers, interactively operable, permitted following up his original designs and simplifying them. Still further advances, in the form of symbolic programming languages and visual displays, were waiting when Conway's discoveries attracted attention. Finally, Wolfram's forays into one dimensional

automata would seem to have coincided with the advent of a generation of widely accessible computers, endowed with ample memory and rapid operation.

When circuits become available, for which large numbers of cells with large numbers of states can rapidly evolve in parallel according to programmable rules, there is no doubt that new insights will be found, very likely through empirical observation. In the meantime it is always possible to say that more details could be filled in with respect to any of the facets of the theory which have already been studied. So the problem is to take some feature which seems to show promise, to see which of its aspects make it interesting, anticipate what results are expected, and try to find the best way to go about establishing them.

3.1   Convergence of block probabilities

Kolmogoroff's theorem asserts that a family of consistent measures converges to a limit measure. Gutowitz *et.al.*[34] use this result to assure a meaningful construction of a system of equations for self consistent neighborhood probabilities, wherein extrapolation via Bayes' theorem yields probabilities for the counterimages of individual neighborhoods. They incidentally remark that the Bayesian extension yields maximum entropy; in other words, the least additional information about the extension.

However, each different length of neighborhood, or order, possesses its own set of equations defining self-consistency, making the measure deduced vary from one order to the next. It would be desirable to have an estimate of how much these measures can differ from one another, to know whether the sequence of measures deduced from successively longer neighborhoods can be demonstrated to converge to a unique limit, and at what point a desired degree of accuracy has been reached.

The approach of Gutowitz *et.al.*was to select rules for study, keep using longer blocks until the results stabilized, and apply statical tests for goodness of fit between the result and empirical observations. In its context, this is an entirely valid methodology. But each rule needs its own analysis; some rules seem to be much more amenable to the procedure than others.

As an idea of the hazards which are to be encountered, consider the Garden of Eden states. They can be described in terms of excluded sequences, which necessarily have probability zero. Nevertheless a zero probability cannot be obtained via Bayesian extrapolation from the probabilities of shorter blocks not from the Garden of Eden, so that explicit block probabilities at least as long as the shortest excluded sequences are required. However, there is not necessarily a "longest shortest excluded block," so that there may be no finite approximation at all which is logically correct. In practice, the excluded sequences may already be assigned such small probabilities by local field theories of low order that their nonvanishing would never be detectable.

In point of fact, each periodic configuration of the automaton gives rise to a discrete measure concentrated on the orbit of that particular configuration, so

14

that without further restrictions, the determination of a measure is not at all unique. Likewise there are rules of evolution based on the linear algebra of finite fields for which periodicities of one sort or another are so scrupulously observed that the measures also retain a thoroughly discrete structure.

No wonder that it would be nice to have a more comprehensive understanding of invariant measures and their relationships to one another. Especially with respect to establishing error bounds.

3.2   Automata which are necessarily infinite

The very most exotic applications of the theory of computation require computers which are infinite in extent, at least in principle. Otherwise one is dealing with a finite state machine which can presumably always be evaluated by an exhaustive enumeration. In practice, a large enough extent is sufficiently close to infinite, and supposedly any quirks of infinite machines can be approximated up to a point by making the finite machine large enough. Nevertheless, there are certain theoretical considerations which are practically equivalent to choosing a definition of infinity, for which finite approximations will definitely not suffice. In any event, think of how much more tractable analysis becomes when real numbers can be used freely rather than rational approximations.

To a certain extent, the problem which arises is not with the limiting behavior of an automaton as with describing that behavior. Some limits are complicated and have a complicated description, but others have an exceedingly simple description with respect to other terms of reference. The recognition of balanced parentheses is the classical example; their description by regular expressions is infinite, substantially a listing of cases, while their description by a "context free language" is very short. No matter that the automaton required in either case is the same, it is the difference in the length of the descriptions required by the two schemes which counts.

Then again, it is likely that the majority of limit sets will be horrendously complicated, no matter what representation is chosen for them. Wolfram repeatedly emphasizes this point.

3.3   Computational requirements

Numerous variants on the theme of cellular automata have been proposed, such as stochastic evolutionary rules, varying the rule from one cell to another or from time to time, or even permitting such variations in the definitions of the neighborhoods. Unfortunately the quantity of computation required to simulate an automaton increases exponentially, and usually with a rather large exponent, with the size of the parameters involved. Thus any increase in the number of parameters will only aggravate a situation which is already rather difficult. If a significant practical application of one of these variants were to be discovered, no doubt the means would be found to compute its properties. In the meantime

the regular version of cellular automata has enough unsolved problems to keep one occupied.

There are two visible tendencies. One is the approach of Conway, or of those who already have an automaton and want to find out what it does. The other is the original approach of von Neumann, to go ahead and construct the automaton which one requires, regardless of the cost in cells and states. Given that a Turing machine can be embedded in cellular automata in a fairly standard way, there will always be *some* automaton which will realize a given calculation; typically such a straightforward realization will be neither aesthetic nor efficient.

In either event enormous computer power is required. In the former, the de Bruijn diagrams are enormous; for two dimensional binary automata, it is hardly even possible to compute the period 2 states for *Life*, for instance. For one dimensional binary nearest neighbor automata, the calculation of cycles of length 34 or periods of length 15 strain present microcomputer capacity. Yet computer experiments reveal numerous configurations which are even bigger or repeat with longer periods. So, although much surer, systematic computation does not yet extend to revealing many fairly common experimental results.

Self consistent probabilities likewise require exponentially growing facilities, as the number of nonlinear equations to be solved increases with order. In the second case these concerns may be set aside, but it is still necessary to manage an extremely large table of transitions and to ensure its own internal consistency, especially in the face of backtracking from changed decisions as the rule is gradually elaborated.

3.4 Practical applications

So far there has been a shortage of physical, biological, social, mathematical, or whatever sort of systems, of independent interest, which have been shown to follow a rule of evolution which would qualify them as cellular automata. There is a result from the theory of symbolic dynamics that cellular automata define the continuous mappings with respect to a particular topology, but it is not known whether this has led to the detailed examination of any particular automata.

Some examples have been proposed; Preston and Duff[58] describe image processing applications, Wolfram's reprint collection[73] includes applications to nonlinear equations describing chemical reactions. It has often been proposed that the discretization of partial differential equations will render them into cellular automata. However, one has to wonder whether this approach gives any new insight or produces useful methods of solution. That is, the utility of the typical theorems of cellular automata theory such as the evolution into cycles, or the existence of Garden of Eden configurations, has to be contrasted with the existence theorems, stability criteria, and so on of traditional differential equation theory.

There is no doubt that if an actual cellular automaton were built mimicing

16

Laplace's equation, for example, that it would be enormously useful. Still, each cell would have to have enough states to represent a real number to a significant accuracy and would have to have a capacity for at least addition and shifting. The number of such cells would have to be large enough to interpolate a plane area to a reasonable degree of accuracy. So, the cost of the design of an appropriate integrated circuit, and subsequent volume of sales, would have to be balanced, on the one hand, against the cost and utility of designing some other comparably complex circuit, and on the other against the ease of performing the same simulation in a contemporary single or parallel computer.

4 Conclusion

With any body of knowledge, there is a set of definitions and axioms, followed by a series of results—the theorems. One would like to know that the basic assumptions lead to a well defined structure which then exhibits certain remarkable characteristics which set it apart from other structures.

In the case of cellular automata, the definitions lead first to the association of "cells" with "neighborhoods" via a function φ defining cellular evolution. Then the association is repeated uniformly throughout some lattice, leading to the function Φ mapping configurations from one to another, and thus the evolution of the lattice. The theory of cellular automata is essentially the theory of the interrelation of these two functions, and especially the behavior of Φ under iteration.

In the forward direction, the theorems relate to the limit sets of the evolution, including the rate at which limits are approached and the nature of the limiting configurations, often from the point of view of formal language theory. "Crystallographic" limits, cyclic or shifting in space and periodic in time, are of a particular practical interest; de Bruijn diagrams lead directly to their determination in the great majority of cases. Theoretically and philosophically important are the aperiodic limits, particularly those whose linguistic description is especially complex. In this latter category may be included the universal constructors of von Neumann, the emulators of Turing machines, and the like.

In the backward direction, the overlapping of neighborhoods allowed Moore to prove the Garden of Eden theorem, which does not apply to arbitrary mappings.

The existence of a probability measure μ compatible with Φ permits the calculation of equilibrium frequencies and entropies for the lattice and so is an important adjunct to Φ itself; consequently the determination of either μ or reasonable approximations thereto play an important role in theory of cellular automata.

Finally, the existence of an extensive collection of well studied examples is a delightful part of the theory, particularly because many of them have proved to have a considerable entertainment value.

17

References

[1] Jürgen Albert and Karel Culik II, "A Simple Universal Cellular Automaton and its One-Way and Totalistic Version," *Complex Systems* **1** 1-16 (1987).

[2] Stan Augarten, *Bit by Bit*, Ticknor and Fields, New York, 1984 (ISBN 0-89919-302-1).

[3] Charles Babbage, *Charles Babbage and his Calculating Engines*, (Philip and Emily Morrison, eds.), Dover Publications, Inc., New York, 1961.

[4] R. C. Backhouse and B. A. Carré, "Regular Algebra Applied to Path-finding Problems," *Journal of the Institute of Mathematics and its Applications*, **15** 161-186 (1975).

[5] Carter Bays, "The game of three dimensional life" (11/20/86) unpublished (available as a supplement to A. K. Dewdney's February 1987 column).

[6] Carter Bays, "Candidates for the Game of Life in Three Dimensions," *Complex Systems* **1** 373-400 (1987).

[7] Carter Bays, "Patterns for Simple Cellular Automata in a Universe of Dense-Packed Spheres," *Complex Systems* **1** 853-875 (1987).

[8] Carter Bays, "Classification of Semitotalistic Cellular Automata in Three Dimensions," *Complex Systems* **1** 373-400 (1987).

[9] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning Ways for your Mathematical Plays*, Academic Press, 1982 (ISBN 0-12-091152-3) vol. 2, chapter 25.

[10] Abraham Berman and Robert J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979 (ISBN 0-12-092250-9).

[11] Janusz A. Brzozowski, "Derivatives of regular expressions," *Journal of the Association for Computing Machinery* **11** 481-494 (1964).

[12] David J. Buckingham, "Some facts of life," *Byte* vol. 3, no. 12 (December, 1978), pp. 54-67.

[13] Arthur W. Burks (editor), *Essays on Cellular Automata*, University of Illinois Press, Urbana, 1970.

[14] Gerardo Cisneros and Harold V. McIntosh, *Notas sobre los lenguajes REC y Convert* (Spanish), Universidad Autonoma de Puebla, 1986.

[15] E. F. Codd, *Cellular Automata*, Academic Press, New York, 1968.

[16] J. H. Conway, *Regular Algebra and Finite Machines*, Chapman and Hall, Ltd., London, 1971 (ISBN 412-10620-5).

[17] Karel Culick II and Sheng Yu, "Undecidability of CA Classification Schemes," *Complex Systems* **2** 177-190 (1988).

[18] Karel Culick II, Jan Pachl, and Sheng Yu, "On the Limit Sets of Cellular Automata," *SIAM Journal on Computing* **18** 831-842 (1989).

[19] Predrag Cvitanovic, "Invariant measurement of strange sets in terms of cycles," (preprint).

[20] A. K. Dewdney, "Computer Recreations - Building computers in one dimension sheds light on irreducibly complicated phenomena," *Scientific American*, May 1985, pp. 10-16.

[21] A. K. Dewdney, "Computer Recreations - The game Life acquires some successors in three dimensions," *Scientific American*, February 1987, pp 8-13.

[22] A. K. Dewdney, *The Armchair Universe*, W. H. Freeman and Company, New York, 1988. (ISBN 0-7167-1939-8 pbk)

[23] A. K. Dewdney, "Computer Recreations - A cellular universe of debris, droplets, defects, and demons," *Scientific American*, August 1989, pp 88-91.

[24] M. Dresden and D. Wong, "Life Games and Statistical Models," *Proceedings of the National Academy of Sciences (U.S.A.)* **72** 956-960 (1975).

[25] Manfred Eigen, "Abstract: The Hypercycle: A Principle of Natural Self-Organization," *International Journal of Quantum Chemistry*, **Quantum Biology Symposium 5** 219 (1978).

[26] Glen Fleck (Ed.), *A Computer Perspective*, Harvard University Press, Cambridge, 1973 (ISBN 674-15625-0).

[27] F.R. Gantmacher, *The Theory of Matrices*, volume 2, Chelsea Publishing Company, New York, 1959.

[28] Martin Gardner, "Mathematical Games - The fantastic combinations of John Conway's new solitaire game 'life,' " *Scientific American*, October 1970, pp. 120-123.

[29] Martin Gardner, *Wheels, Life, and Other Mathematical Amusements*, W. H. Freeman and Company, New York, 1983. (ISBN 0-7167-1589-9 pbk)

[30] Herman H. Goldstine, *The Computer from Pascal to von Neumann*, Princeton University Press, Princeton, 1972 (ISBN 0-691-08104-2).

[31] Solomon W. Golomb, *Shift Register Sequences*, Holden-Day, Inc., San Francisco, 1967.

[32] Howard A. Gutowitz, "Local Structure Theory for Cellular Automata," *Thesis* Rockefeller University, 1987.

[33] Howard A. Gutowitz and Jonathan D. Victor, "Local structure theory in more than one dimension," *Complex Systems* **1** 57-68 (1987).

[34] Howard A. Gutowitz, Jonathan D. Victor, and Bruce W. Knight, "Local structure theory for cellular automata," *Physica* **28D** 18-48 (1987).

[35] Howard A. Gutowitz and Jonathan D. Victor, "Local structure theory: calculation on hexagonal arrays, and interaction of rule and lattice," *Journal of Statistical Physics* **54** 495-514 (1989).

[36] Fred Hapgood, "Let there be Life," *Omni*, vol. 9, no. 7 (April 1987) pp. 40-46, 116-117.

[37] Brian Hayes, "Computer Recreations - The cellular automaton offers a model of the world and a world unto itself," *Scientific American*, March 1984, pp. 10-16.

[38] Lyman P Hurd, "Formal Language Characterizations of Cellular Automaton Limit Sets," *Complex Systems* **1** 69-80 (1987).

[39] Lyman P. Hurd, "The Application of Formal Language Theory to the Dynamical Behavior of Cellular Automata," *Thesis*, Princeton University, 1988.

[40] Lyman P Hurd, "The Non-wandering Set of a CA Map," *Complex Systems* **2** 549-554 (1988).

[41] Erica Jen, "Global Properties of Cellular Automata," *Journal of Statistical Physics* **43** 219-242 (1986).

[42] Erica Jen, "Invariant Strings and Pattern-Recognizing Properties of One-Dimensional Cellular Automata," *Journal of Statistical Physics* **43** 243-265 (1986).

[43] Erica Jen, "Scaling of preimages in cellular automata," *Complex Systems* **1** 1045-1062 (1987).

[44] Erica Jen, "Cylindrical cellular automata," *Communications in Mathematical Physics* **118** 569-590 (1988).

[45] S. C. Kleene, "Representation of events in nerve nets and finite automata," *in* Claude E. Shannon and John McCarthy (editors), *Automata Studies*, Annals of Mathematics Studies #34, Princeton University Press, Princeton, 1956. (ISBN 0-691-07916-1), pp. 3-41.

[46] Christopher G. Langton, "Self-Reproduction in Cellular Automata," *Physica* **10D** 135-144 (1984).

[47] Christopher G. Langton, "Studying artificial life with cellular automata," *Physica* **22D** 120-149 (1986).

[48] Stephen Levy, *Hackers: Heroes of the Computer Revolution*, Anchor Press/ Doubleday, Garden City, New York, 1984 (ISBN 0-385-19195-2), chapter 7.

[49] O. Martin, A. Odlyzko, and S. Wolfram, "Algebraic aspects of cellular automata," *Communications in Mathematical Physics* **93** 219-258 (1984).

[50] W. S. McCulloch and W. Pitts, "A logical calculus of the ideas immanent in nervous activity," *Bulletin of Mathematical Biophysics* **5** 115-133 (1943).

[51] Edward F. Moore, "Machine models of self-reproduction," in A. Burks (ed), *Essays on Cellular Automata*, University of Illinois Press, Urbana, 1970. pp. 186-203.

[52] Scott Morris, "Games - The game of Life," *Omni*, vol. 7, no. 1 (October, 1984), pp. 188-189.

[53] John von Neumann, *Theory of Self-reproducing Automata* (edited and completed by A. W. Burks), University of Illinois Press, 1966.

[54] Mark D. Niemiec, "Life Algorithms," *Byte* vol. 4, no 1 (January, 1979) pp. 90-97.

[55] Norman H. Packard and Stephen Wolfram, "Two-Dimensional Cellular Automata," *Journal of Statistical Physics* **38** 901-946 (1985).

[56] Kenneth E. Perry, "Abstract mathematical art," *Byte*, vol. 11, no. 13 (December 1986), pp. 181-192.

[57] William Poundstone, *The Recursive Universe*, William Morrow and Company, New York, 1985 (ISBN 0-688-03975-8).

[58] Kendall Preston, Jr., and Michael J. B. Duff, *Modern Cellular Automata*, Plenum Press, New York, 1984 (ISBN 0-306-41737-5).

[59] Anthony Ralston, "De Bruijn Sequences—A Model Example of the Interaction of Discrete Mathematics and Computer Science," *Mathematics Magazine* **55** 131-143 (1982).

[60] Brian Randell (Ed.), *The Origins of Digital Computers: Selected Papers*, Springer Verlag, Berlin, 1973 (ISBN 3-540-06169-X).

[61] D. R. Rutherford, *Substitutional Analysis*, Edinburgh, at the University Press (1948).

[62] L.S. Schulman and P.E. Seiden, "Statistical mechanics of a dynamical system based on Conway's game of Life," *Journal of Statistical Physics* **19** 293-314 (1978).

[63] Eugene Seneta, *Non-Negative Matrices*, John Wiley and Sons, New York, 1973 (ISBN 0-470-77605-6).

[64] Claude E. Shannon and John McCarthy (editors), *Automata Studies*, Annals of Mathematics Studies #34, Princeton University Press, Princeton, 1956. (ISBN 0-691-07916-1).

[65] Alvy Ray Smith III, "Simple computation-universal cellular spaces," *Journal of the Association for Computing Machinery* **18** 339-353 (1971).

[66] Stanislaw Ulam, "On some mathematical problems connected with patterns of growth of figures," in A. Burks (ed.) *Essays on Cellular Automata*, University of Illinois Press, 1970.

[67] Richard S. Varga, *Matrix Iterative Analysis*, Prentice Hall, Inc, Englewood Cliffs, New Jersey, (1962).

[68] Robert T. Wainwright (editor), *Lifeline*, a quarterly newsletter with 11 issues published between March 1971 and September 1973.

[69] W. John Wilbur, David J. Lipman, and Shihab A. Shamma, "On the prediction of local patterns in cellular automata," *Physica* **19D** 397-410 (1986).

[70] Stephen Wolfram, "Statistical mechanics of cellular automata," *Reviews of Modern Physics* **55** 601-644 (1983).

[71] Stephen Wolfram, "Universality and complexity in cellular automata," *Physica* **10D** 1-35 (1984).

[72] Stephen Wolfram, "Computation theory of cellular automata," *Communications in Mathematical Physics* **96** 15-57 (1984).

[73] Stephen Wolfram (Ed.), *Theory and Applications of Cellular Automata*, World Scientific Press, Singapore, 1986 (ISBN 9971-50-124-4 pbk).

[74] Takeo Yaku, "The constructibility of a configuration in a cellular automaton," *Journal of Computer and System Sciences* **7** 481-496 (1973).