Transitive behavior in reversible one-dimensional cellular automata with a Welch index 1

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Abstract

The problem of knowing and characterizing the transitive behavior of a given cellular automaton is a very interesting topic. This paper provides a matrix representation of the global dynamics in reversible one-dimensional cellular automata with a Welch index 1, i.e. those where the ancestors differ just at one end. We prove that the transitive closure of this matrix shows diverse types of transitive behaviors in these systems. Part of the theorems in this paper are reductions of well-known results in symbolic dynamics. This matrix and its transitive closure were computationally implemented, and some examples are presented.

Keywords: cellular automata, dynamical systems, matrix methods

1 Introduction

Cellular automata are discrete dynamical systems where the dynamics is given by locally interacting parts. Cellular automata have been extensively used as models for natural systems where the local interaction is essential [17] [19], in addition to comprising a very interesting mathematical theory.

The concept of cellular automaton begins in the late 40's with the study of John von Neumann about self-reproductive systems [18]. By the early 70's, this theory acquired a stronger interest due to the work of John H. Conway and the automaton called "Life" [5]. This automaton is characterized by a very simple local behavior but is able to generate a complex global one. Later on, in the 80's, Stephen Wolfram carefully explored the one-dimensional binary cellular automata [19]. His work is notable because it does not study one certain automaton but a sets of automata and the properties they share.

A special type of cellular automaton is one where the system can return to global states that it had already produced. This type of automaton is called reversible. The analysis of reversible cellular automata is important for applications such as simulation of physical and chemical systems [17], [19], or for implementing data coding systems [6], [10].

A very elegant and meticulous paper about reversible automata was developed by Gustav A. Hedlund [7], describing their combinatorial properties as part of his study about the shift dynamical system. Another important paper was made by Jarkko Kari [9]; characterizing the action of a reversible automaton by means of block permutations. But they did not develop any procedure for knowing the dynamical properties of reversible cellular automata. Recent articles have investigated the dynamical properties of cellular automata [11] [4] [3]. In particular Leo Liberti [12] has studied the group properties in the reversible case. However, a simple computable procedure for analyzing the transitive behavior of such systems has not been defined yet.

Based on the two papers described above, and in general concepts of symbolic dynamics, this paper presents a simple matrix representation which is easily implemented in a computer and whose transitive closure allows characterizing the transitive behavior of reversible one-dimensional cellular automata where the ancestors differ just at one end (technically, with a Welch index 1). We prove that the properties of the transitive closure are useful for knowing transitive points, topologically ergodic and topologically mixing sets.

The paper is organized as follows. Section 2 presents the basic concepts and the terminology used in the study of reversible automata. Using block permutations and centered cylinder sets, section 3 provides the matrix representing the global dynamics of reversible automata. Section 4 proves that the transitive closure of this matrix shows transitive configurations, topologically ergodic and topologically mixing subsets. Section 5 contains two examples of the computational implementation of this procedure and the concluding remarks are made in the final section.

2 Definitions

A one-dimensional cellular automaton is a one-dimensional array in which every site or cell has a value from a finite set K of states. The cardinality of K is represented by k. An assignment of states to all cells is a global state or configuration c of the automaton. The set of all the possible configurations is represented by the letter C.

For $n \in \mathbb{Z}^+$, let K^n be the set of sequences of n states, thus the cardinality of K^n is k^n. For some $n \in \mathbb{Z}^+$, every sequence of n cells is a neighborhood of the automaton and n is the neighborhood size. Every neighborhood in a given configuration overlaps with the contiguous ones in $n-1$ elements.

The dynamics of the system is given by an explicit recipe. A mapping from the neighborhood set K^n to the set K of states is defined. It is called the evolution rule φ. From an initial configuration c, the evolution rule is applied to every neighborhood in c, producing a new configuration c'. In this way, the evolution rule φ induces a global mapping Φ between configurations of C.

For $w \in K^m$ with $m \geq n$, $\varphi(w)$ means to apply the evolution rule to each neighborhood in w, which generates a new sequence $w' \in K^{m-n+1}$. Then w is an ancestor of w', having $m - n + 1$ more cells than the sequence it generates.

A cellular automaton is reversible if every configuration has exactly one ancestor. In other words, the evolution rule φ has an inverse rule φ^{-1} (possibly with a different neighborhood radius) and the global mapping Φ is invertible. Before presenting the combinatorial features of reversible automata, we shall explain how an automaton is simulated by another of neighborhood size 2. In this way, we need to study just this case for understanding the rest [1], [8].

Given a cellular automaton of k states and neighborhood size n, take all sequences in K^{2n-2} and their evolutions. Then φ defines a mapping from K^{2n-2} to K^{n-1}. Observe that $K^{2n-2} = K^{n-1} \times K^{n-1}$; take a set S such that $\mid S \mid = k^{n-1}$. Overwrite each element of K^{n-1} for a unique corresponding element in S. Then φ defines a mapping $\varphi' : S^2 \rightarrow S$, which also defines a cellular automaton with neighborhood size 2. In this way, the grouping of states allows simulating any automaton with any neighborhood size by another of neighborhood size 2.

In the case of a reversible automaton, take between the rules φ and φ^{-1} the bigger neighborhood size, and represent both invertible rules with the same neighborhood size. For an evolution rule with smaller neighborhood size, this process is obtained adding redundant states to each neighborhood. Since both rules have the same neighborhood size, the action of the rule φ can be simulated by another rule τ with neighborhood size 2. As φ^{-1} has the same neighborhood size, another inverse rule τ^{-1} also exists with neighborhood size 2. Thus, automata of neighborhood size 2 simulate all the other cases; and throughout this paper, this type of reversible automata will be just analyzed.

We define a matrix representing the evolution rule of this type of automaton. The row indices are the left cells and the column indices are the right cells of the neighborhoods. Each element of the matrix is the evolution of the neighborhood defined by its coordinates.

We describe now the properties of reversible automata. First of all, the results of the paper of Hedlund [7] are presented. His paper includes a combinatorial analysis of the automorphisms of the shift dynamical system, which are reversible one-dimensional cellular automata. Thus, two main properties about finite sequences of states in these systems are:

1. Uniform multiplicity of ancestors: Every finite sequence has the same number of ancestors as all the others. This number is equal to k.
2. Welch indices: If both invertible rules have neighborhood size 2; then for $n \geq 2$, the ancestors of each sequence $w \in K^n$ have L left different cells, a single common central part and R right distinct cells. The values L and R are known as Welch indices fulfilling that $LR = k$.

Thus, the ancestors of any sequence does not proliferate or disappear; and their differences are at the ends (Figure 1).

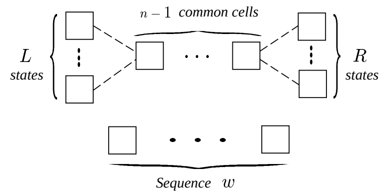

Figure 1: Ancestors of the sequence $w \in K^n$, with $LR = k$.

Another important result in this issue was developed by Jarkko Kari [9], proving that any reversible automaton can be expressed as a combination of two block permutations and a shift, which elegantly explains how the evolution of the automaton preserves the information of the system. We shall explain this result using the properties established by Hedlund.

Take all sequences in K^3 and their evolutions coinciding with K^2. Because both φ and φ^{-1} have neighborhood size 2, then the ancestors of every sequence of 2 cells have a unique central state. With the sequences in K^3 and their evolutions given by φ, we form two sets L_φ and R_φ. The set L_φ consists of every state and its L left ancestor states and the set R_φ consists of every state and its R right ancestor states (Figure 2).

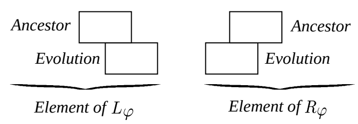

Figure 2: Elements of L_φ and R_φ.

Thus, $\mid L_\varphi \mid = Lk$ and $\mid R_\varphi \mid = Rk$; then:

$$\mid L_\varphi \mid\mid R_\varphi \mid = LRk^2 = k^3 = \mid K^3 \mid \tag{1}$$

In this way, there is a mapping from every sequence in K^3 into a single pair of elements, where the first element belongs to L_φ and the second belongs to R_φ. Define two sets X and Y fulfilling that $\mid L_\varphi \mid = \mid X \mid$ and $\mid R_\varphi \mid = \mid Y \mid$. Then, there are two bijections, one from L_φ into X and another from R_φ into Y. For $x \in X$ and $y \in Y$, every sequence in K^3 has a corresponding pair xy of the product XY.

The preceding procedure is also applied using the inverse evolution rule. We obtain $L_{\varphi^{-1}}$ and $R_{\varphi^{-1}}$ holding that $L_{\varphi^{-1}} = R_\varphi$ and $R_{\varphi^{-1}} = L_\varphi$, because the original ancestor cells are now successor ones. With this $\mid L_{\varphi^{-1}} \mid = \mid Y \mid$ and $\mid R_{\varphi^{-1}} \mid = \mid X \mid$ and there is a bijection from every sequence in K^3 to a unique pair yx of the product YX. Thus, the evolution of a reversible automaton is expressed by two block permutations and a shift between both (Figure 3).

Both Hedlund and Kari carefully explain the combinatorial properties of reversible automata. Nevertheless they do not present a process for detecting and analyzing the dynamical properties of these systems, because Hedlund wanted to understand just the dynamics of the shift system and Kari wished to present how the information in a reversible automaton is preserved. But both papers

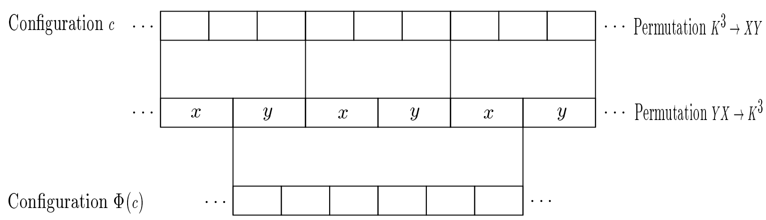

Figure 3: Evolution given by block permutations and a shift.

provide an excellent framework for analyzing the transitive behavior of reversible one-dimensional cellular automata with a Welch index 1.

3 Matrix representing the dynamical behavior

The theory of dynamical systems studies the behavior of some physical system described by the values of a given number of measurements. This theory also analyzes if there is some equivalence between different systems. The concept is also applied to any system which changes its state through time. We shall use the definition of dynamical system given in the book of Brian Marcus and Douglas Lind [13].

Definition 1. *A dynamical system (X, Ψ) consists of a compact space X with a continuous mapping $\Psi : X \to X$. If Ψ is invertible, then the dynamical system (X, Ψ) is invertible.*

A compact space has a finite number of sets covering it. The finiteness of the covering makes easier the analysis. In order to study the dynamical behavior of reversible automata, special attention is made in handling the configurations, so the cells in every configuration will be indexed. Each configuration will have an arbitrary central cell indexed by 0. The cells on the left of the central one will be indexed by negative integers and the cells on the right of the central one will be indexed by positive integers. For $c \in C$ and $i \in \mathbb{Z}$, $c_{[i]}$ represents the cell placed at position i. For $i, j \in \mathbb{Z}$ with $i \leq j$, $c_{[i,j]}$ is the sequence of states of c placed between i, j.

For $i \in \mathbb{N}$ and $w \in K^{2i+1}$, a centered cylinder set $\mathcal{C}_{[w]}$ is the set of configurations such that each configuration $c \in \mathcal{C}_{[w]}$ fulfills that $c_{[-i,i]} = w$. For w_1 and w_2 in K^{2i+1} if $w_1 \neq w_2$, then the centered cylinder sets $\mathcal{C}_{[w_1]}$ and $\mathcal{C}_{[w_2]}$ are disjoint, because each configuration $c \in \mathcal{C}_{[w_1]}$ differs from all configurations in $\mathcal{C}_{[w_2]}$.

For K^{2i+1}, let $\mathfrak{C}_{K^{2i+1}}$ be the family of centered cylinder sets $\mathcal{C}_{[w]}$ such that $w \in K^{2i+1}$; then $\mathfrak{C}_{K^{2i+1}}$ is a finite partition of the configuration set C. Taking (C, Φ) as a dynamical system, the set C is a compact and Hausdorff space by the properties of the family $\mathfrak{C}_{K^{2i+1}}$. The global mapping Φ between configurations induced by φ is continuous [7] and in this case Φ is invertible. Therefore reversible automata are invertible dynamical systems.

In a dynamical system, several types of behaviors are defined according to the transitions between their elements. We shall use the definitions of several transitive behaviors presented in the books of Clark Robinson [16], J. de Vries [2], and Marcus and Lind [13]. For $n \in \mathbb{Z}$ and each set A in the covering of (X, Ψ), a point $x \in (X, \Psi)$ is transitive if $\Psi^n(x) \in A$. In other words, there is a point whose images intersect all sets in the covering. In a reversible automaton, for $i \in \mathbb{N}$ a transitive point is a configuration whose evolution intersects all the centered cylinder sets in $\mathfrak{C}_{K^{2i+1}}$.

Now the transitive behavior between sets of the covering is described. For $n \in \mathbb{Z}$ and each ordered pair A_1, A_2 of sets in the covering, a dynamical system (X, Ψ) is topologically ergodic if $\Psi^n(A_1) \cap A_2 \neq \varnothing$. In this way, the images of every set intersect all the others. For $i \in \mathbb{N}$, a reversible automaton is topologically ergodic if each centered cylinder set in $\mathfrak{C}_{K^{2i+1}}$ intersects all the others by the action of Φ.

Another special transitive property is the following one. For $n_0 \in \mathbb{Z}$ and each ordered pair A_1, A_2 of sets in the covering, a dynamical system (X, Ψ) is topologically mixing if $\Psi^n(A_1) \cap A_2 \neq \varnothing$ for $n \geq n_0$. Thus, the images of every set intersect indefinitely any other by the action of Ψ. For $i \in \mathbb{N}$, a reversible automaton is topologically mixing if every centered cylinder set of $\mathfrak{C}_{K^{2i+1}}$ intersects indefinitely any other after n_0 steps (possibly by the evolution of different configurations in the set).

Using block permutations, we can know the successor centered cylinder sets from a given initial set. Take the family $\mathfrak{C}_{K^3}$, for each sequence $w \in K^3$ concatenate at both sides of w all sequences in K^3. This new sequences are presented by $w_L w w_R$. Every sequence $w_L w w_R$ permutes in three blocks $xy \in XY$. From these three blocks, there are two central blocks $yx \in YX$. Permute these blocks by their corresponding sequences $v_1 v_2$, for $v_i \in K^3$, $i = 1, 2$.

The sequence $v_1 v_2$ permutes in two blocks xy. Take the central block yx and permute this one by its corresponding sequence $u \in K^3$. With this procedure, Figure 4 shows the transition from a centered cylinder set $\mathcal{C}_{[w]}$ into distinct centered cylinder sets $\mathcal{C}_{[u]}$ at the same coordinates.

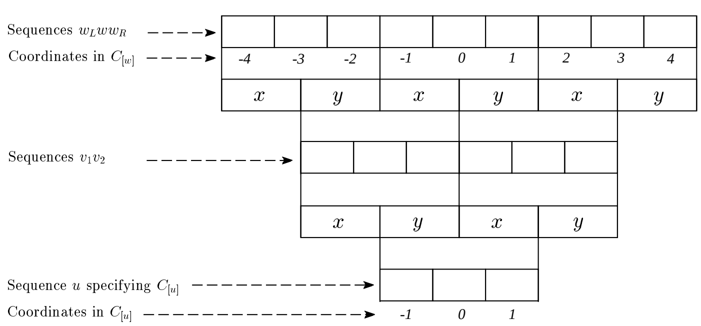

Figure 4: Transitions from the set $\mathcal{C}_{[w]}$ to distinct sets $\mathcal{C}_{[u]}$.

This procedure yields all the possible transitions from a centered cylinder set into all the others in the family $\mathfrak{C}_{K^3}$. These transitions are represented by a matrix M_φ, the indices by rows and columns are the centered cylinder sets. For w_1 and w_2 sequences in K^3, if there is a transition from $\mathcal{C}_{[w_1]}$ into $\mathcal{C}_{[w_2]}$, then the element specified by these sets in M_φ is equal to 1, otherwise is 0. In this way, M_φ is a $0-1$ matrix presenting the transitions of the sets in $\mathfrak{C}_{K^3}$.

In the next section we shall prove that the transitive closure of M_φ shows the properties of the transitive behavior for a given reversible automaton. Theorems 2 and 3 are reductions of well-known results in symbolic dynamics [13].

4 Detecting the transitive properties in reversible automata

We shall use some definitions from the theory of matrices for analyzing the matrix M_φ. Let M be a square nonnegative matrix with elements m_{ij}. The matrix M is essential if none of its rows or columns is zero. For $n \in \mathbb{Z}^+$, let M^n be the nth-power of M. Then, M is irreducible if for each element m_{ij} there is some $n \in \mathbb{Z}^+$ such that $m_{ij} > 0$ in M^n. Finally, M is primitive if there is some $n_0 \in \mathbb{Z}^+$ such that M^n is positive for all $n \geq n_0$.

We shall analyze the transitive behavior of a reversible automaton with a Welch index 1 by means of the matrix M_φ. First of all, a simple but important result is given about the symmetrical behavior of the transitions between centered cylinder sets.

Lemma 1. *For $n \in \mathbb{Z}^+$ and sequences w_1 and w_2 elements of K^3, if $\mathcal{C}_{[w_1]}$ is connected with $\mathcal{C}_{[w_2]}$ in M_φ^n, then there is an image of $\mathcal{C}_{[w_2]}$ intersecting $\mathcal{C}_{[w_1]}$.*

Proof. If some image of $\mathcal{C}_{[w_1]}$ intersects $\mathcal{C}_{[w_2]}$, then there are sequences a, b elements of K^n such that $\varphi^{2n}(aw_1b) = w_2$. Take the configuration $c = \cdots aw_1baw_1baw_1b\cdots$, then $\Phi^{2n}(c) = c'$ where the configuration $c' = \cdots cw_2dcw_2dcw_2d\cdots$ for c, d elements of K^n.

But cw_2d is a finite sequence, so the configuration c' can only generate a finite number of different configurations, and c' has a periodical evolution. Since the automaton is reversible, then the evolution returns to c' but also returns to c. Thus, c is both ancestor and successor of c' and therefore there is an image of $\mathcal{C}_{[w_2]}$ intersecting $\mathcal{C}_{[w_1]}$.

□

The proof of Lemma 1 yields another result about the transitive behavior of the cylinder sets. There is an image of $\mathcal{C}_{[w_1]}$ intersecting it.

Corollary 1. *The transition between centered cylinder sets in M_φ is a reflexive relation.*

The important point now is to obtain the transitive behavior of a given reversible automaton using the matrix M_φ. In other words, for w_1, w_2 and w_3 elements of K^3, if $\mathcal{C}_{[w_1]}$ intersects $\mathcal{C}_{[w_2]}$ and this one also intersects $\mathcal{C}_{[w_3]}$ in M_φ, then some procedure must be performed for knowing if $\mathcal{C}_{[w_1]}$ intersects $\mathcal{C}_{[w_3]}$ through $\mathcal{C}_{[w_2]}$. For a Welch index 1, we shall prove that the transitive closure of M_φ provides this transition.

Theorem 1. *For reversible one-dimensional cellular automata of neighborhood size 2 in both invertible rules and Welch index $L = 1$ in the original rule, if $\mathcal{C}_{[w_1]}$ intersects $\mathcal{C}_{[w_2]}$ and $\mathcal{C}_{[w_2]}$ intersects $\mathcal{C}_{[w_3]}$ in M_φ, then there is a transition from $\mathcal{C}_{[w_1]}$ into $\mathcal{C}_{[w_3]}$ through $\mathcal{C}_{[w_2]}$.*

Proof. For this type of reversible automata, every state has 1 left ancestor and k right ancestors, in other words, $L = 1$ and $R = k$. Take the sequence $w_1 \in K^3$, where some extensions of this sequence evolve into $w_2 \in K^3$ (Figure 5).

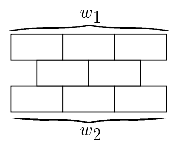

Figure 5: Part of the evolution from w_1 into w_2, the cells presented here have fixed states.

In reversible automata with $L = 1$, the columns of the matrix representing the original evolution rule are permutations of K and the elements of each row are equal to a unique state of K. In Figure 5, we can concatenate k states on the left of each cell as left ancestors, forming k possible states. Analogously, k states may be concatenated on the right of each cell as right ancestors, evolving in a unique state of K. Then, there are k right extensions of w_1 evolving into w_2 (Figure 6).

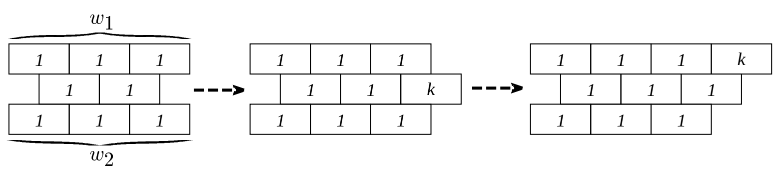

Figure 6: Right extensions of w_1 evolving into the right side of w_2. Here, the value of each cell is the number of states that it can take.

In order to complete the construction in Figure 5, the leftmost cell of the sequence placed between w_1 and w_2 has exactly one left neighbor for evolving into the leftmost cell of w_2. But this left neighbor is evolution from the leftmost cell of w_1 with exactly one left extension. In this way, there is just one left extension of w_1 evolving into w_2 (Figure 7).

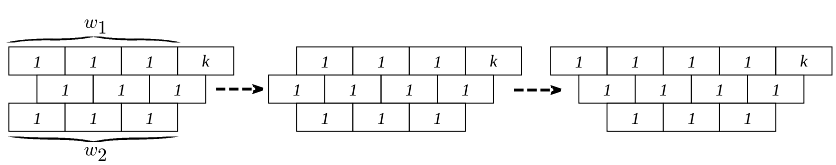

Figure 7: Left extensions of the sequence w_1 which evolve in w_2.

In Figures 6 and 7 each right ancestor is a column in the matrix representing φ. But each column is a permutation of K, therefore every right ancestor can form each state by a suitable choice of its left neighbor. This procedure is used for subsequent transitions in Figure 8.

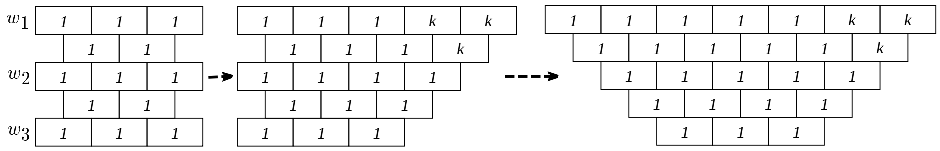

Figure 8: Transition from w_1 into w_3 through w_2.

In this way, there is a transition from w_1 into w_3 through w_2 and therefore from $\mathcal{C}_{[w_1]}$ into $\mathcal{C}_{[w_3]}$.

□

The proof of Theorem 1 implies that the transitive closure of M_φ presents the transitive behavior of the cylinder sets in a reversible automaton.

Corollary 2. *The transitive closure of M_φ gives the transitive behavior of a reversible one-dimensional cellular automaton of neighborhood size 2 for both invertible rules and Welch index $L = 1$ for the original evolution rule. This is analogous for reversible automata with Welch index $R = 1$ for the original evolution rule.*

The transitive closure of M_φ shows how the centered cylinder sets are connected for a large number of steps; using Lemma 1, Theorem 1 and Corollary 1, the next result is obtained.

Corollary 3. *The transitive closure of the matrix M_φ is an equivalence relation between centered cylinder sets.*

The equivalence classes of M_φ provide important details of the dynamical behavior of reversible automata. We shall prove that the different types of matrices defined by this classes represent several types of transitive behaviors.

Lemma 2. *Every class of M_φ defines an essential and irreducible submatrix of M_φ.*

Proof. Every centered cylinder set of a given class has at least one ancestor and one successor set. Then the submatrix representing this class has non-zero rows or columns and therefore is essential. The images of each centered cylinder set intersect all the other sets of this class as the transitive closure shows. Then each ordered pair of sets is connected by some power of the submatrix, and therefore it is irreducible. □

Lemma 2 characterizes the dynamical behavior of every class of cylinder sets as follows:

Theorem 2. *Every equivalence class of M_φ defines a subset of C such that:*

1. *The subset is topologically ergodic.*
2. *The subset has transitive points.*

Proof. Let N be an equivalence class of M_φ. The submatrix representing N is irreducible by Lemma 2, then for each pair $\mathcal{C}_{[w_1]}, \mathcal{C}_{[w_2]}$ of centered cylinder sets in N, there is an integer $n \in \mathbb{Z}^+$ such that:

$$\Phi^n(\mathcal{C}_{[w_1]}) \cap \mathcal{C}_{[w_2]} \neq \varnothing,$$

therefore N is topologically ergodic. In $\mathcal{C}_{[w_1]}$ there is one configuration whose evolution intersects all the others in the class by the proof of Theorem 1. This configuration is obtained by extensions at both sides of w_1 (Figure 9).

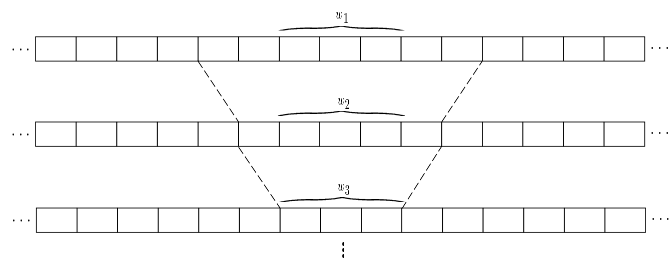

Figure 9: Transition from $\mathcal{C}_{[w_1]}$ to $\mathcal{C}_{[w_3]}$ given extensions of w_1.

In this way, the class is topologically ergodic and each set of this class has transitive configurations.

□

Theorem 2 provides an important result about the global dynamics of a reversible automaton if M_φ has exactly one equivalence class:

Corollary 4. *For any reversible automaton with a Welch index* 1*, if the transitive closure of* M_φ *yields just one equivalence class, then:*

1. *The automaton is topologically ergodic.*
2. *The automaton has transitive configurations.*

The transitive closure of M_φ also provides information of the topologically mixing behavior. First we give a formal definition for a special type of centered cylinder set called recurrent set.

Definition 2. *A centered cylinder set is recurrent if the set is connected with itself in* M_φ*.*

A recurrent set is a diagonal element equal to 1 in M_φ. In a recurrent cylinder set, there are configurations whose evolutions indefinitely intersect the set. In other words, a recurrent cylinder set has images remaining an indefinite number of iterations in the set before leaving it. The irreducible form of each class of M_φ and the existence of recurrent centered cylinder sets yield the following result.

Lemma 3. *For an equivalence class of* M_φ*, if the submatrix* N *corresponding to this class has a recurrent centered cylinder set, then there is some* $n \in \mathbb{Z}^+$ *such that* M^n *is positive.*

Proof. The submatrix N is irreducible. Then every set in the class has images intersecting the recurrent set which is a diagonal element of N. From this diagonal element, we can find a fixed number of steps for connecting each ordered pair of sets in N. In this way, there exists some $n \in \mathbb{Z}$ such that M^n is positive. □

Using Lemma 3, we can show another important property of the submatrix representing an equivalence class with a recurrent set.

Lemma 4. *For an equivalence class of M_φ, if the submatrix N corresponding to this class has a recurrent centered cylinder set, then N is primitive.*

Proof. There exists an integer n_0 such that N^{n_0} is positive by Lemma 3. Besides, N is essential by Lemma 2. Then, for $n > n_0$, any power N^n is the product of non-zero rows of N by positive columns of N^{n-1}. Therefore N^n is positive and N is primitive. □

Lemma 4 provides the following characterization of the global dynamical behavior of some reversible automaton with a Welch index 1.

Theorem 3. *For an equivalence class in M_φ, if the submatrix N corresponding to this class has a recurrent centered cylinder set, then this class is topologically mixing.*

Proof. The submatrix N is primitive by Lemma 4. Then there exists a minimum number of steps connecting every ordered pair of centered cylinder sets in the class. This property follows from Lemma 4 for any greater number of steps, and the class is topologically mixing. □

If there is a unique equivalence class in M_φ, then Theorem 3 gives rise to another important property of the global dynamics.

Corollary 5. *For any reversible automaton with a Welch index 1, if the transitive closure of M_φ yields just one equivalence class and M_φ has at least one recurrent centered cylinder set, then the automaton is topologically mixing.*

5 Examples

This section illustrates the application of the matrix procedure described in this paper. This procedure was implemented in the system RLCAU [15] developed for analyzing reversible one-dimensional cellular automata. The evolutions presented below were obtained from the system NXLCAU developed by Harold V. McIntosh [14].

5.1 Reversible automaton of 2 states, right shift

The simplest reversible automaton is probably a shift of the elements in the initial configuration. Let us take the right shift, the evolution rule is shown in Figure 10.

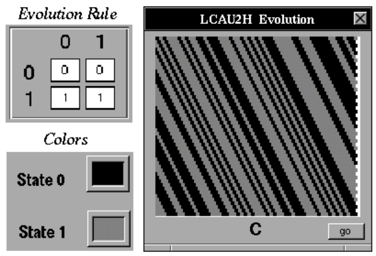

Figure 10: Reversible automaton of 2 states representing the right shift.

The matrix M_φ of this automaton and its transitive closure are presented in Figure 11.

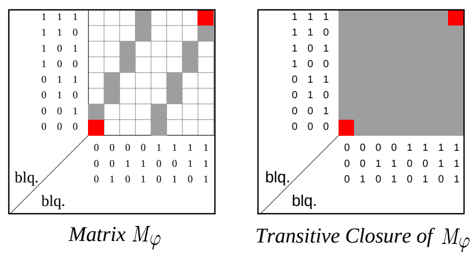

Figure 11: Matrix M_φ of the right shift and its transitive closure.

Figure 11 shows that the transitive closure yields exactly one equivalence class, and the matrix M_φ has 2 recurrent centered cylinder sets ($\mathcal{C}_{[000]}$ and $\mathcal{C}_{[111]}$). In this way, the right shift is topologically ergodic, topologically mixing and has transitive points. A transitive point of the right shift is depicted in Figure 12.

Transitions	*Element of*
$\cdots\, 0\,0\,0\,1\,1\,1\,0\,1\,\boxed{0\,0\,0}\,0\,0\,0\,0\,0\,0\,0\,0\,0 \cdots$	$C_{[000]}$
$\cdots\, 0\,0\,0\,0\,1\,1\,1\,0\,\boxed{1\,0\,0}\,0\,0\,0\,0\,0\,0\,0\,0\,0 \cdots$	$C_{[100]}$
$\cdots\, 0\,0\,0\,0\,0\,1\,1\,1\,\boxed{0\,1\,0}\,0\,0\,0\,0\,0\,0\,0\,0\,0 \cdots$	$C_{[010]}$
$\cdots\, 0\,0\,0\,0\,0\,0\,1\,1\,\boxed{1\,0\,1}\,0\,0\,0\,0\,0\,0\,0\,0\,0 \cdots$	$C_{[101]}$
$\cdots\, 0\,0\,0\,0\,0\,0\,0\,1\,\boxed{1\,1\,0}\,1\,0\,0\,0\,0\,0\,0\,0 \cdots$	$C_{[110]}$
$\cdots\, 0\,0\,0\,0\,0\,0\,0\,0\,\boxed{1\,1\,1}\,0\,1\,0\,0\,0\,0\,0\,0 \cdots$	$C_{[111]}$
$\cdots\, 0\,0\,0\,0\,0\,0\,0\,0\,\boxed{0\,1\,1}\,1\,0\,1\,0\,0\,0\,0\,0 \cdots$	$C_{[011]}$
$\cdots\, 0\,0\,0\,0\,0\,0\,0\,0\,\boxed{0\,0\,1}\,1\,1\,0\,1\,0\,0\,0\,0 \cdots$	$C_{[001]}$

Figure 12: Transitive configuration of the right shift, every transition comprises two evolutions of the automaton.

The recurrent centered cylinder set $\mathcal{C}_{[000]}$ is used in Figure 13 for establishing a transition in n and $n+1$ steps from $\mathcal{C}_{[001]}$ into $\mathcal{C}_{[101]}$, showing the topologically mixing behavior of the right shift.

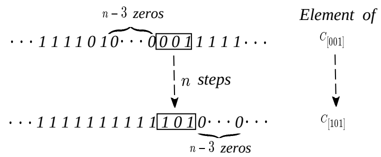

Figure 13: The set $\mathcal{C}_{[001]}$ intersects $\mathcal{C}_{[101]}$ in n steps for $n \geq 6$.

5.2 Reversible automaton of 3 states

Take the reversible automaton of 3 states in Figure 14, with a global behavior a little bit more complicated than a simple shift.

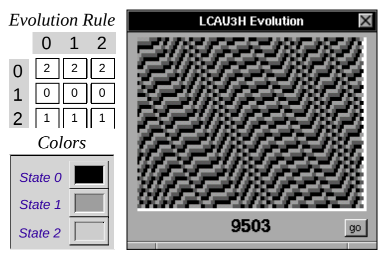

Figure 14: Reversible automaton of 3 states.

The matrix M_φ of this automaton and its transitive closure are shown in Figure 15.

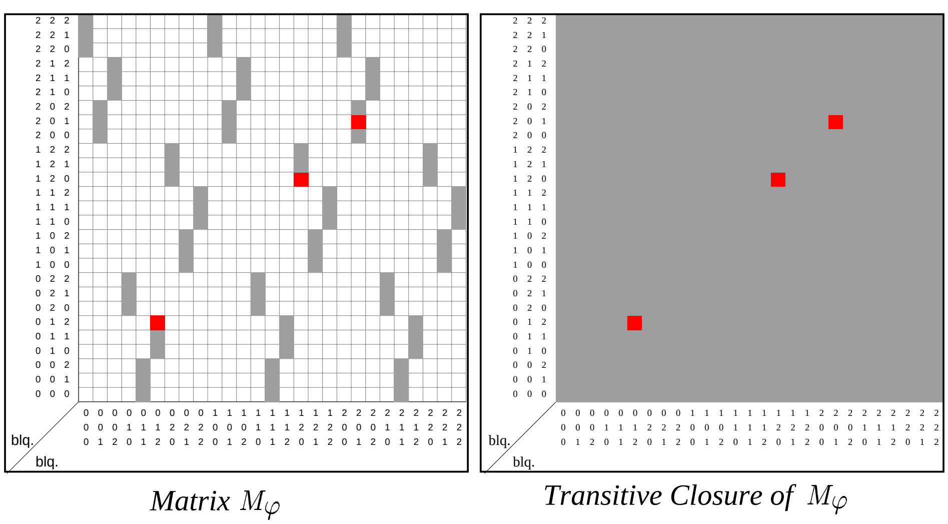

Figure 15: Matrix M_φ and its transitive closure.

Figure 15 presents a unique equivalence class in the transitive closure of M_φ, and this matrix has also recurrent centered cylinder sets. In this way, the automaton is topologically ergodic, topologically mixing and has transitive configurations. A transitive configuration is depicted in Figure 16.

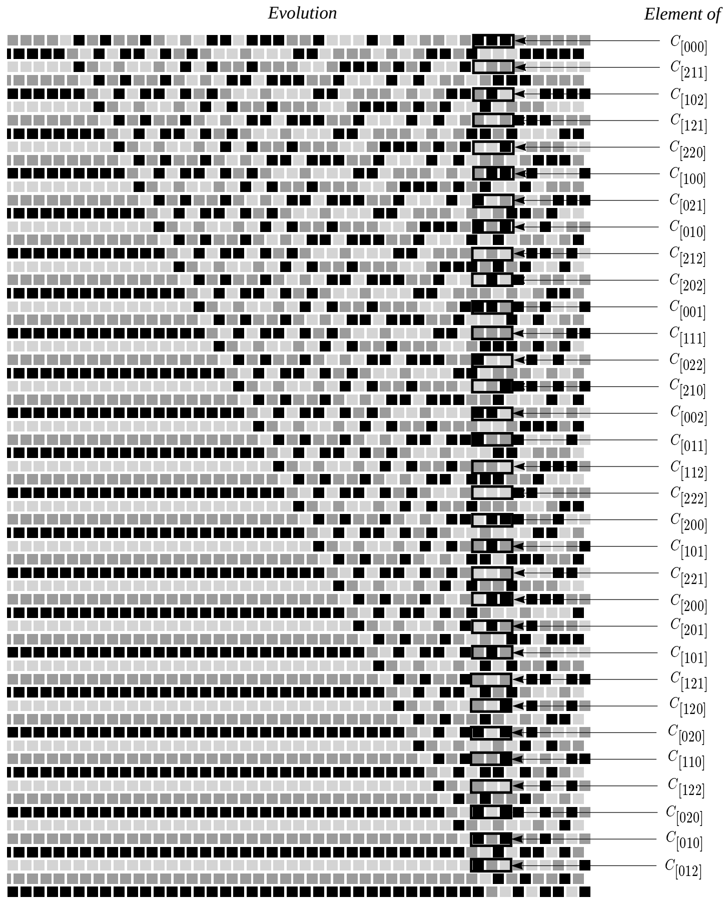

Figure 16: Transitive configuration of the automaton of 3 states.

Transitions from the set $\mathcal{C}_{[121]}$ into $\mathcal{C}_{[220]}$ in six and seven steps are presented in Figure 17.

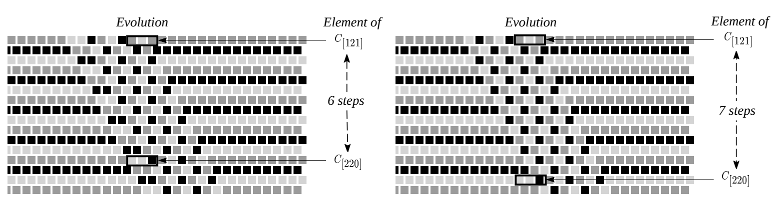

Figure 17: Transition from $\mathcal{C}_{[121]}$ into $\mathcal{C}_{[220]}$ in six and seven steps.

Figure 17, the recurrent set $\mathcal{C}_{[120]}$ has been used for yielding a transition from $\mathcal{C}_{[121]}$ into $\mathcal{C}_{[220]}$ in six and seven steps. In this way, the topologically mixing behavior of this automaton has been sampled.

6 Concluding remarks

The preceding sections have defined a matrix representation M_φ of the dynamical behavior of reversible automata with a Welch index 1 using their characterization by block permutations over centered cylinder sets. The properties of the equivalence classes in M_φ provide important details of the dynamical behavior of these systems. In particular about topologically ergodic, topologically mixing behaviors and transitive configurations.

This matrix procedure was easily implemented on a computational system, detecting and analyzing some examples of these transitive behaviors. This procedure is useful for illustrating some of the theoretical results presented by Cattaneo et al. in [4]. The transitive closure of this matrix truly presents the transitive behavior of the centered cylinder sets by the combinatorial properties of the ancestors in this type of automata. Of course, for automata with a large number of states, this procedure is impractical by the exponential growth of the order of M_φ.

A foregoing work is studying reversible automata where both Welch indices differs from 1. In this case the combinatorial properties of the ancestors are more complicated. Another possible extension of this paper is the use of the theory of groups for providing a better characterization of the dynamical behavior.

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