On the Optimal Solution Definition for Many-criteria Optimization Problems

M.Farina and P. Amato

Abstract— **When dealing with many-criteria decision making and many-objectives optimization problems the concepts of Pareto optimality and Pareto-dominance are inefficient in modelling and simulating human decision making. Different fuzzy-based definitions of optimality and dominated solution are introduced and tested on analytical test cases in order to show their validity and closeness to human decision making.**

Keywords— **Multi-criteria decision making, Pareto optimality, fuzzy optimality definition.**

I. Introduction

Pareto's definition captures the notion of "optimality" in a narrowly prescribed sense. In fact this definition is relevant and useful for engineering and design problems where typically the objective number is small and the computational cost of each objective is high, but is less suitable for many other kind of problems (especially decision making problems) where the number of objectives may be big (though computationally costless). Let us consider, for example, a minimization problem with 50 objectives $f_1, \ldots, f_{50}$ (a number which is unusual for engineering problems, but common for many real world decision problem), and two point $\mathbf{v_1}$ and $\mathbf{v_2}$ such that in 49 objectives $\mathbf{v_1}$ is better than $\mathbf{v_2}$ (i.e., $f_i(\mathbf{v}_1) < f_i(\mathbf{v_2})$), and in just one objective j it holds $f_j(\mathbf{v}_2) < f_j(\mathbf{v}_1)$ (maybe for a small value ϵ). It is obvious that any person would vote $\mathbf{v_1}$ as a better solution than $\mathbf{v_2}$. However, by Pareto definition they are absolutely equivalent. The combination of fuzzy logic tools and multiobjective optimization or multicriteria decision making has a great relevance in the literature [1],[2] and [3]; fuzzy sets are considered for interactive multiobjective optimization in [4], [5]. Usually the term fuzzy optimization is used when fuzzy objective functions are tackled via crisp optimization strategies; in this work we consider crisp objective function with fuzzy optimality definition. The idea of this work is to generalize the definition of (Pareto) optimality, in order to capture the common-sense undefined concept of optimality for a multi-criteria decision making problem using some fuzzy-reasoning tools.

II. Limits and drawbacks of Pareto optima definition

The following multi-criteria decision making problem or multi-objective optimization problem is considered:

Def. II.1: Let $\mathbf{V}$ and $\mathbf{W}$ be n-dimensional and M-dimensional continuous or discrete vector spaces, $\mathbf{g}$ and $\mathbf{h}$ be two functions defining inequalities and equalities constraints and $\mathbf{f}$ be a function from $\mathbf{V}$ to $\mathbf{W}$. A *Non-linear multi-criteria (minimum) optimization problem* with M objectives is defined as

$$\min_{\mathbf{v}\in\mathbf{V}} \mathbf{f} = \{f_1(\mathbf{v}), ..., f_M(\mathbf{v})\}$$
$$\text{subject to} \quad \mathbf{g}(\mathbf{v}) \leq \mathbf{0} \quad \mathbf{h}(\mathbf{v}) = \mathbf{0}.$$

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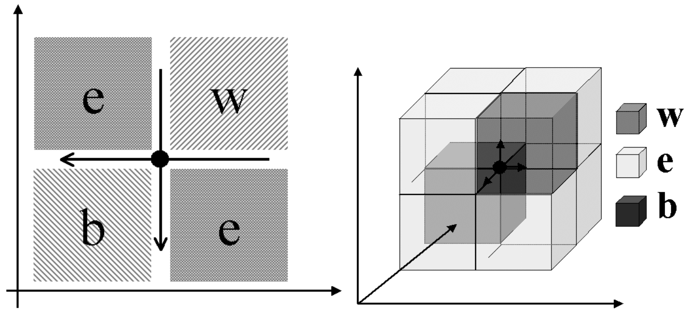


Fig. 1. Schematic view of Pareto-dominance based partial order in 2D and 3D problems when a candidate solution is considered (•): equal (e), better (b) and worse (w) solution regions are shown.

Def. II.2: We call *Design domain search space* $\mathbf{\Omega}$ and *objective domain search space* $\mathbf{\Omega}_O$ the two following sets: $\mathbf{\Omega} = \{\mathbf{v} \in \mathbf{V} \mid \mathbf{g}(\mathbf{v}) \leq \mathbf{0} \wedge \mathbf{h}(\mathbf{v}) = \mathbf{0}\}$ $\quad \mathbf{\Omega_O} = \{\mathbf{f}(\mathbf{v}) \mid \mathbf{v} \in \mathbf{\Omega}\}$.

Both multi-criteria search in a discrete database and multi-objective non-linear constrained optimization in a continuous search space are considered. $\mathbf{\Omega}$ and $\mathbf{\Omega}_O$ can thus be either discrete or continuous spaces. For the convenience of the reader we recall here the well known definition of Pareto-dominance (definition II.3) and Pareto optimum (definition II.4) in a multi-criteria (M criteria) decision making problem on variable $\mathbf{v}$ belonging to the search space Ω.

Def. II.3: For any two points (candidate solutions) $\mathbf{v_1}, \mathbf{v_2} \in \mathbf{\Omega}$, $\mathbf{v_1}$ is said to *dominate* $\mathbf{v_2}$ in the Pareto sense if and only if the following conditions hold:

$$\begin{cases} f_i(\mathbf{v}_1) \leq f_i(\mathbf{v_2}) & \text{for all i} \in \{1, 2, ..., \text{M}\} \\ f_j(\mathbf{v}_1) < f_j(\mathbf{v_2}) & \text{for at least one j} \in \{1, 2, ..., \text{M}\} \end{cases}$$

Def. II.4: $\mathbf{v}^* \in \mathbf{\Omega}$ is *Pareto-optimal (PO)* if there is no $\mathbf{v} \in \mathbf{\Omega}$ such that $\mathbf{v}$ dominates $\mathbf{v}^*$.

Def. II.5: We call *Pareto Optimal Set (POS)* and *Pareto Optimal Front (POF)* the set of Pareto-optimal solutions in design domain and objective domain respectively.

As shown by the simple example in the introduction, the Pareto definition of optimality in a multi-criteria decision making problem can be unsatisfactory due to essentially two reasons: **the number of improved or equal objec-**

tive values is not taken into account, the (normalized) size of improvements is not taken into account. The two aforementioned issues are essential decision elements when looking for the best solution and they are included in the common-sense notion of optimality. The limit of Pareto definition when the first issue is considered can be viewed in the schema shown in figure 1. Since the Pareto dominance gives a partial order of solutions in criteria space, when a vector (a candidate for optimal solution) in the criteria space is considered, all other possible solution can belong to one of the following three different set: better solutions, worse solutions and equivalent solutions. Figure 1 shows such sets for 2 and 3 criteria problems. The portion e of the M-dimensional criteria domain search space that the dominance concept classify as equivalent solutions increases as the number of criteria increases as follows: $e = \frac{2^M - 2}{2^M}$. Thus when M tends to infinity, e tends to 1 (i.e., it is the whole search space). From this it derives that that Pareto definition is uneffective for a large number of objectives, even without considering the second aforementioned issue. In the following sections we will give two more general definition of optimum for a multi-criteria decision making problem, taking into account one issue at a time. As we shall see, Pareto optimum definition is a special case of both definitions.

III. Taking into account the number of improved objectives

In Pareto definition two candidate solutions are equivalent if at least in one objective the first solution is better than the second one, and at least in one objective the second one is better than the first one (or if they are equals in all the objectives). Indeed a more general definition, able to cope with a wider variety of problems, should take into account in how many objectives the first candidate solution is better than the second one and viceversa. To do so, we introduce the following functions which associate to every couple of points in Ω a natural number.

$$\begin{aligned} n_b(\mathbf{v_1}, \mathbf{v_2}) &=_{\text{def}} \quad |\{i \in \mathbb{N} | i \leq M \wedge f_i(\mathbf{v}_1) < f_i(\mathbf{v_2})\}| \\ n_e(\mathbf{v_1}, \mathbf{v_2}) &=_{\text{def}} \quad |\{i \in \mathbb{N} | i \leq M \wedge f_i(\mathbf{v}_1) = f_i(\mathbf{v_2})\}| \\ n_w(\mathbf{v_1}, \mathbf{v_2}) &=_{\text{def}} \quad |\{i \in \mathbb{N} | i \leq M \wedge f_i(\mathbf{v}_1) > f_i(\mathbf{v_2})\}| \end{aligned}$$

For every couple of points $\mathbf{v_1}, \mathbf{v_2} \in \mathbf{\Omega}$, the function n_b computes the number of objectives in which $\mathbf{v_1}$ is better than $\mathbf{v}_2$, n_e computes the number of objectives in which they are equal, and n_w the number of objectives in which $\mathbf{v_1}$ is worse than $\mathbf{v_2}$. To lighten the mathematical notation, from now on we will consider a generic couple of points and we will write simply n_b, n_e and n_w instead of $n_b(\mathbf{v}_1, \mathbf{v}_2), n_e(\mathbf{v_1}, \mathbf{v_2})$ and $n_w(\mathbf{v_1}, \mathbf{v_2})$. A moment's reflection tell us that the following inequalities holds:

$$n_b + n_w + n_e = M \quad 0 < n_b, n_w, n_e < M$$

We now give a first new definition of dominance and optimality namely $(1-k)$-dominance and k-optimality

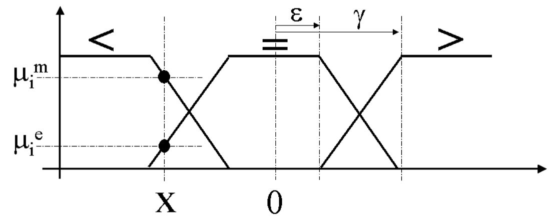


Fig. 2. Linear membership for i-th objective to be used in equation 2 for a fuzzy definition of =, < and >: ε and γ are parameters to be chosen by the decision maker.

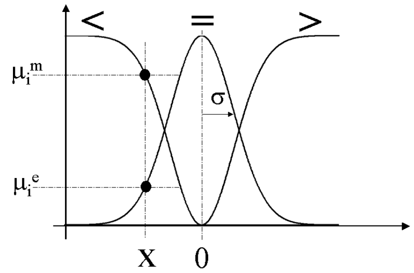


Fig. 3. Gaussian membership for i-th objective to be used in equation 2 for a fuzzy definition of =, < and >: σ is s parameter to be chosen by the decision maker.

Def. III.1 ($(1-k)$-dominance) $\mathbf{v_1}$ is said to $(1-k)$-*dominates* $\mathbf{v_2}$ if and only if:

$$\begin{cases} n_e < M \\ n_b \geq \dfrac{M - n_e}{k+1}, \end{cases} \tag{1}$$

where $0 \leq k \leq 1$.
As can be easily seen, definition III.1 with $k = 0$ corresponds to Pareto-dominance (definition II.3). Ideally k can assume any value in $[0, 1]$, but because n_b has to be a natural number only a limited number of optimality degree need to be considered. In fact in eq (1) the second inequality is equivalent to $n_b \geq \left\lceil \dfrac{M - n_e}{k+1} \right\rceil$. With this new dominance definition the following new optimality can be defined:

Def. III.2 (k-optimality) $\mathbf{v}^*$ is *k-optimum* if and only if there is no $\mathbf{v} \in \mathbf{\Omega}$ such that $\mathbf{v}$ $(1-k)$-dominates $\mathbf{v}^*$
The terms "$(1-k)$-dominance" and "k-optimality" derives respectively from the fact that the former is a loose version of Pareto dominance (1-dominance), while the latter is a strong version of Pareto optimality (0-optimality).

Def. III.3 (k-OS and k-OF) We call *k-optimal set (k-OS)* and *k-optimal front (k-OF)* the set of k-optimal solutions in design domain and objective domain respectively. As evident the Pareto Optimal set (POS) is the 0-OS and the Pareto Optimal Front (POF) is the 0-OF.

IV. Taking into account the size of improvements

A natural way of extending the notion of $(1-k)$-dominance and k-optimality is to introduce fuzzy relations instead of crisp ones. As first step, to take into account in which degree in each objective function a point $\mathbf{v_1}$ is different from (or equal to) a point $\mathbf{v_2}$ we will consider fuzzy number and fuzzy arithmetic. As second step, we will consider the dominance relation itself as a fuzzy relation. A standard way to introduce fuzzy arithmetic on a given universe (here the objective domain search space Ω_O), is to associate to each of its point a triple of fuzzy sets — one for equality (fuzzy number), one for "greater than" and one for "less than". Figures 2 and 3 shows two possible definitions of the fuzzy sets for "equal to 0", "greater than 0" and "less than 0". For coherence with the terminology used so far, we refer to their respective membership function as μ_e,μ_w (where w means "worst", remember that we are talking about minimization problems) and μ_b. Now the fuzzy definition of n_b, n_w and n_e is the following

$$
\begin{aligned}
n_b^F(\mathbf{v_1},\mathbf{v_2}) &=_{\text{def}} \sum_{i=1}^{M} \mu_b^{(i)}(f_i(\mathbf{v_1}) - f_i(\mathbf{v_2}))\\
n_w^F(\mathbf{v_1},\mathbf{v_2}) &=_{\text{def}} \sum_{i=1}^{M} \mu_w^{(i)}(f_i(\mathbf{v_1}) - f_i(\mathbf{v_2}))\\
n_e^F(\mathbf{v_1},\mathbf{v_2}) &=_{\text{def}} \sum_{i=1}^{M} \mu_e^{(i)}(f_i(\mathbf{v_1}) - f_i(\mathbf{v_2}))
\end{aligned}
$$

In order that n_b^F, n_w^F and n_e^F are a sound extension of n_b, n_w and n_e, the membership functions must satisfy Ruspini condition (i.e, in each point they must sum up to 1) [6]. In fact, under this hypothesis the following holds:

$$n_b^F + n_e^F + n_w^F = \sum_{i=1}^{M}(\mu_i^b + \mu_i^w + \mu_i^e) = M$$

In the figures 2 and 3, two possible different membership shapes are considered: linear and gaussian. Both of them are characterized by parameters defining the shape, ε_i and γ_i for the linear one and σ_i for the gaussian one. Although the definition of such parameters is to be carefully considered, their intended meaning is clear. Thus they can be derived from the human decision-maker knowledge on the problem.

• ε_i defines in a fuzzy way the practical meaning of equality and it can thus be considered the tolerance on the i-th objective, that is the interval within which an improvement on objective i is meaningless.

• γ_i can be defined as a relevant but not big size of improvement for objective i.

• σ_i evaluation requires a combination of the two aforementioned concepts of maximum imperceptible improvement on objective i (ε_i) and γ_i; the following formula for the membership μ can be used:

$$\mu_i = \exp\left(-\frac{\ln\chi}{\varepsilon_i^2}(f_1^i - f_2^i)^2\right) \qquad 0.8 < \chi < 0.99 \tag{2}$$

χ being an arbitrary parameter.
With such a fuzzy definition of n_b^F, n_e^F and n_w^F, both dominance and optimality definition can be reconsidered.

A. *k-optimality with fuzzy numbers of improved objectives*

A first extension of the definitions of $(1-k)-$*dominance* and k-optimality can be given if n_b, n_e and n_w are replaced by n_b^F, n_e^F and n_w^F in definitions III.1, III.2 and III.3. The parameter k has the same meaning as in the previous case ($0 \leq k \leq 1$) but now a continuous degree of optimality is introduced: $(1-k)_F -$ *dominance* and k_F-optimality .

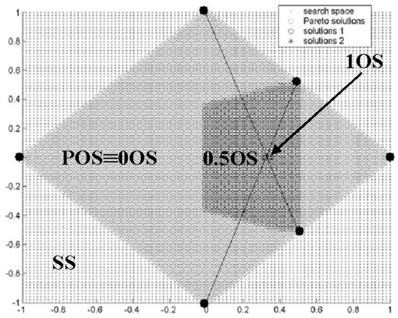


Fig. 4. k-optimality classification on problem 5.Six parabolic functions are centered on • and three different degree of k-optimal solutions are shown together with the whole search space.

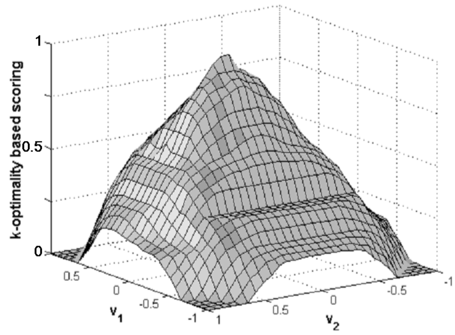


Fig. 5. k_F-optimality classification on problem 5 with linear membership and with $\varepsilon =$ and $\gamma =$

B. *Fuzzy-optimality*

A more general procedure can be introduced by fuzzyfing not only the quantities n_b^F, n_e^F and n_w^F, but also the dominance relation itself.

Def. IV.1 (Fuzzy dominance) Let $\mu_D(\mathbf{v_1},\mathbf{v_2})$ a membership function defined as follow:

$$\mu_D(\mathbf{v_1},\mathbf{v_2}) = f_{\mu_D}(n_b^F(\mathbf{v_1},\mathbf{v_2}), n_e^F(\mathbf{v},\mathbf{v_2}), n_w^F(\mathbf{v_1},\mathbf{v_2})),$$

where f_{μ_D} can be a membership function or a fuzzy system. Then μ_D is a *fuzzy dominance* relation if for any $\alpha \in [0,1]$ $\mu_D(\mathbf{v_1},\mathbf{v_2}) > \alpha$ implies that $\mathbf{v_1}$ $(1-\alpha)_F$-dominates $\mathbf{v_2}$.
For example, two straightforward definition of f_{μ_D} are the following:

$$f_{\mu_D} = \frac{n_w^F(\mathbf{v},\widetilde{\mathbf{v}})}{M} \quad f_{\mu_D} = \frac{n_w^F(\mathbf{v},\widetilde{\mathbf{v}})}{M} + \frac{n_e^F(\mathbf{v},\widetilde{\mathbf{v}})}{2M} \tag{3}$$

However f_{μ_D} could be defined by a whole fuzzy system like:

$$f_{\mu_D} = \overset{*}{\bigvee}((\mu_{n_b^F,i} \overset{*}{\wedge} \mu_{n_e^F,i} \overset{*}{\wedge} \mu_{n_w^F,i}) \overset{*}{\wedge} \mu_{D,k})$$

where $\overset{*}{\vee}$ and $\overset{*}{\wedge}$ are a t-conorm and a t-norm respectively, $\mu_{n_b^F,i}$, $\mu_{n_e^F,i}$ and $\mu_{n_w^F,i}$ are membership functions for n_b^F,

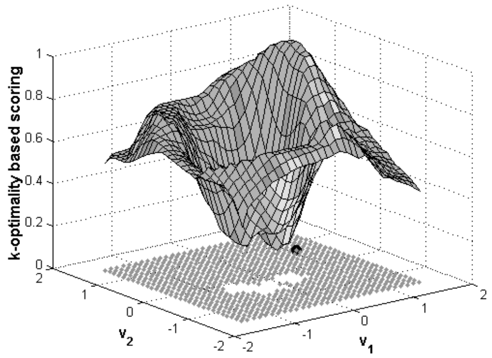


Fig. 6. $(1-k)_F$ dominance based classification with linear membership and corresponding maximum point ($\bullet$); linear membership with $\varepsilon_i = 0.01$ and $\gamma_i = 0.2$, $1 < i < 11$ have been used.

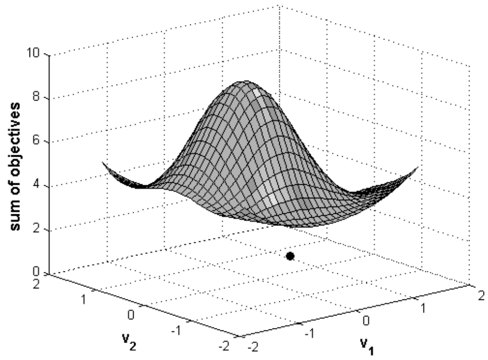


Fig. 7. Classical normalized sum of objectives on problem 6 and corresponding minimum point ($\bullet$)

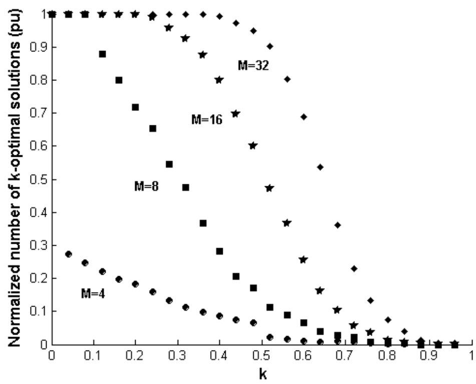


Fig. 8. Normalized number of k_F-optimal solutions versus k for different number of objectives .

n_e^F and n_w^F (antecedents) and where $\mu_{D,k}$ are membership functions for the dominance concept (consequents). As a consequence, a membership function for optimality $\mu_O(\mathbf{v})$ can be implicitly defined through its α-cuts in the following way:

Def. IV.2 (Fuzzy optimality) A membership function represents the *fuzzy optimality* relation if for any $\alpha \in [0,1]$ $\mathbf{v}^*$ belongs to the α-cut if and only if there is no $\mathbf{v} \in \mathbf{\Omega}$ such that:

$$\mu_D(\mathbf{v}, \mathbf{v}^*) > \alpha$$

As will be evident from examples in the following section, when such a membership μ_O for optimality is considered, α-cuts are an extension of *kF-optimal set (k_F-OS)*.

V. Test cases

This last section is devoted to four examples showing the validity of the introduced definitions. The first two are simple MCDM problems that shows in an immediate way the main ideas of the definitions; the third and fourth one are continuous constrained multiobjective optimization problems ($\mathbb{R}^N \to \mathbb{R}^M$).

A. Simple discrete examples

Let us consider the following simple example where five Italian students are scored in five subjects from 10 (best score) to 1 (worst score).

	Student	S1	S3	S2	S4	S5
$\bullet$	A	9	8	5	6	8
	B	6	9	6	8	5
$\bullet\star$	C	5	10	10	9	7
$\bullet$	D	10	4	4	7	9
	E	2	7	2	10	6

(4)

As can be seen most of people would say that the best student is student C. Nevertheless when applying Pareto optima definition as a selection rule all students are equivalent because they all are Pareto optimal. When we apply the crisp k-optimality definition we have the following set: A,C,D. If we need to select further the best student (C) among the three we need k_F-optimality definition with the following parameters: $\varepsilon = 0$ $\gamma = 1.5$ $k = 0.5$ where each of the three parameters listed above have been chosen on the basis of the following common knowledge based assumptions :

- the score from 1 to 10 is a crisp variable with no tolerance $\Rightarrow \varepsilon = 0$
- a 1 point gap is not so big while a 2 point gap significantly big $\Rightarrow \gamma = 1.5$
- with the fuzzy definition a non-integer objective number values can be considered $\Rightarrow \widetilde{n}_b^F = 3.5$

In this very simple example solution C may also be computable as the minimum of the following sum of objectives: $S = [36, 34, 41, 34, 27]$. When more complex problems are considered the normalized weighted sum of objectives gives different and unsatisfactory results as it is already well known from Pareto optima theory and as it is shown in the following subsection . Let us now consider a little bit more complex decision making problem example. Table V-A shows values for five criteria when choosing among 31 houses [7]. When the definitions introduced in previous sections are considered, the DM is asked to give values for parameters ε_i and γ_i or σ_i (where $i = 1:5$); once such values are defined, the classification of houses can be given and a proper choice can be done. Three different classifications are shown in figure 9 corresponding to three different

SIZE	AGE	P	D	T	SIZE	AGE	P	D	T
0.250	48	290	5	4	0.300	26	110	4	2
0.400	22	90	5		22.	60	245	8	5
0.600	25	92	3	2	1.200	7	215	7	4
0.300	45	42	2	1	0.400	11	175	4	3
0.250	16	48	2	1	0.750	15	120	3	2
0.200	34	88	2	1	0.500	3	275	4	3
0.600	12	95	4	2	1.	18	180	5	3
1.330	40	180	7	5	0.350	16	105	4	1
0.300	45	55	3	2	0.450	4	194	3	2
0.400	30	80	3	1	0.200	28	43	3	2
0.600	20	160	5	2	0.850	27	105	5	2
0.350	22	113	4	2	0.500	15	185	5	3
1.250	14	180	3	2	0.250	14	65	4	2
0.600	17	120	6	2	1.750	32	135	4	2
1.	9.	140	6	3	0.400	35	76	3	1
					0.250	7	125	4	2

TABLE I

Buying an house database: values for 5 choice criteria sx:1-15, dx:16-31, P=price, D=bedroom number, T=bathroom number

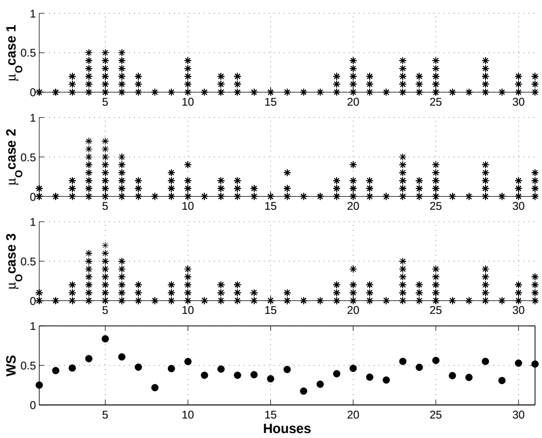


Fig. 9. Different classifications for the house choice problem.

choices for parameter's values. A comparison with a classification based on an equal weights weighted sum equivalent function is also shown. As can be seen the maximum of the WS based classification corresponds to the peak of μ_O based classifications.

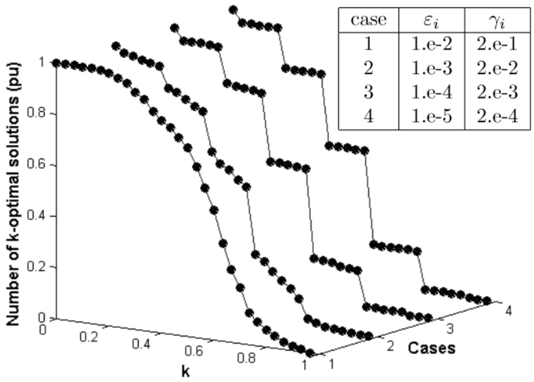
B. Continuous analytical test cases

In the first test case (figure 4 and 5) a 6D problem with only two design variables is considered with parabolic function located in an asymmetric way on a rhombus borders.

$$\begin{cases} \min\limits_{\mathbf{v}\in\mathbb{R}^{\nvDash}} & \mathbf{f} = \{f_1(\mathbf{v}), ..., f_j(\mathbf{v}), ..., f_6(\mathbf{v})\} \\ & f_j(\mathbf{v}) = (v_1 - c_{1,j})^2 + (v_2 - c_{2,j})^2 \\ \mathit{subject\ to} & -1 \le v_1, v_2 \le 1 \end{cases} \tag{5}$$

The coordinates for points $(c_{1,j}, c_{2,j})$ are marked with black bullets and the search space is shown with bright gray dots. As can be seen the region of Pareto optimal solution (k=0) is quite big with respect to the search space. If k is increased (k=.5) a smaller region is selected up to a single optimum for k=1. We thus have four regions, one included in the other: $SS \subset POS \equiv 0OS \subset .5OS \subset 1OS$. If we move from a solution f^1 to a solutions f^2 both belonging to POS we can expect one of the two to be better than the other in at least one objective; if on the other hand we move from a solution f^1 to a solutions f^2 both belonging to $.5OS$ we can expect one of the two to be better than the other in at least two objective. The same may in general hold for $1OS$ but in this example $1OS$ is a single solution. When fuzzy k-optimality is considered on the same test problem a continuous classification of solution can be obtained and is shown in figure 5; the sets $fkOS$ are infinite and corresponds to any real value of k $\in [0:1]$. In the second example (figure 6, 7, and 8) the classification ability of $(1-k)_F$dominance and k_F optimality is shown on a $\mathbb{R}^2 \to \mathbb{R}^{12}$ multiobjective optimization problem on a box-like constrained search space, where almost all points in the search space (a box $\in \mathbb{R}^2$) are Pareto-optimal (see figure 6 where the POS is also shown). No decision could thus be taken with Pareto optimality. In this example, the number of objectives is increased with respect to the previous one and the shape is a little bit more complex. A sampling of 400 points on the search space (candidate solutions) is considered and classified. Classically multiobjective search problems are tackled *via* an equivalent scalar function such as weighted sum of objectives [8],[2]. The limitations of such an approach are the following: only one special Pareto-optimal solution can be computed, preference and tolerance on objectives are difficult to be expressed in a clear way. On the other hand the proposed optimality definition can give a continuous classification of solution, and consequently any number of Pareto optimal solutions with a clearly defined degree of stronger optimality can be easily computed. This possibility is clearly shown in figure 6 where a continuous classification similar to the on in figure 5. Moreover, the point corresponding to the minimum of the sum function is coincident with the maximum of the k_F-optimality-based classification; both points are shown under 3D surfaces. The normalized number of solution $nos(k)$ satisfying a certain k_F-optimality is plotted against k in figure 8 for the two test cases. As can be seen the $nos(k)$ behavior depends on the problem complexity. For problem 2 an high value of k is required in order to select some solutions from the POS. The introduced k_F-optimality can be seen as the limit of crisp k-optimality when $\varepsilon_i, \gamma_i, \sigma_i \to 0$. This can be easily seen form figure 10 where four different $nos(k)$ functions are plotted for the values of membership parameters shown in the table. From a continuous k_F-optimality based classification (case 1), a crisp k-optimality based classification (case 4) can be obtained membership parameters close to zero; some intermediate levels are also shown.

$$\begin{cases} \min\limits_{\mathbf{v}\in\mathbb{R}^N} & \mathbf{f} = \{f_1(\mathbf{v}), ..., f_i(\mathbf{v}), ..., f_M(\mathbf{v})\} \\ & f_i(\mathbf{v}) = \sum\limits_{k=1}^{np} \exp(-c_{i,k} \sum\limits_{j=1}^{N} (v_j + p_{k,j})^2) \\ & i = 1:M \\ \mathit{subject\ to} & L_j \le v_j \le U_j \quad j = 1:N \end{cases} \tag{6}$$



case	ε_i	γ_i
1	1.e-2	2.e-1
2	1.e-3	2.e-2
3	1.e-4	2.e-3
4	1.e-5	2.e-4

Fig. 10. From k_F-optimality to k-optimality for a 4-objectives problem (6) ; membership parameters for the four cases are listed in the table.

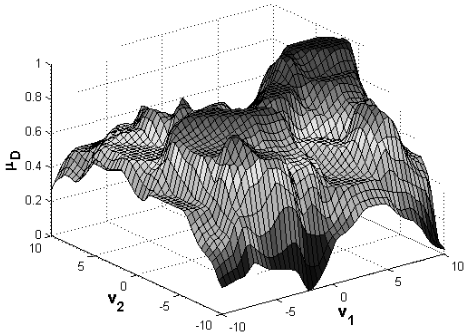


Fig. 11. Membership $\mu_D(v, \widetilde{v})$ for dominance degree with respect to a fixed solution $\widetilde{v}$.

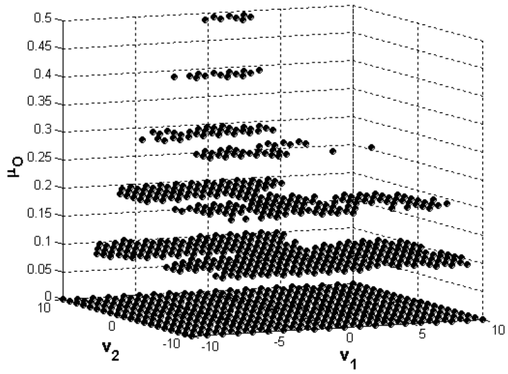


Fig. 12. Optimality membership building through α-cuts, a sampling of 400 solutions in the search space is considered.

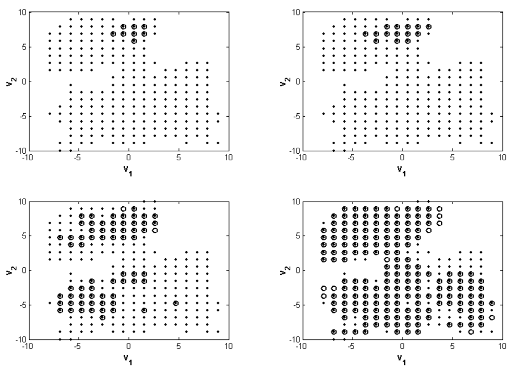


Fig. 13. Comparison of α-cuts ($\circ$) at different α values and crisp Pareto Optimal front ($\cdot$), a sampling of 400 solutions in the search space is considered.

An example of dominance membership with respect to one fixed solution is shown in figure 11; as can be seen... An example of k_F-optimality membership building through α-cuts can be shown in figure 12; as can be seen the obtained maximum optimality degree is 0.5 and not 1. This is a numerical effect due to the poor sampling of the search space (a coarse sampling has been considered for better figure rendering purposes). Moreover, a comparison of two α-cuts (corresponding to two different degrees of optimality) and the crisp Pareto optimal front is also shown in figure 13; as can be seen the fuzzy optimality definition is able to select properly among Pareto optimal solutions some more optimal solutions corresponding to the degree α of optimality. Moreover for high values of α the crisp Pareto Optimal front (which is obtained via a different procedure for comparison and checking purposes) can be properly reconstructed.

VI. Conclusion

The following conclusive remarks can be given. The Pareto optimality definition is unsatisfactory when considering multi-criteria search problems with more than three objectives. Moreover it does not numerically express the common knowledge based criteria that a decision maker would consider when making decision on such a problem. Fuzzy reasoning tool can be profitably used for treating concept like equal, smaller, higher and numerically emulate the decision maker thinking. Moreover the same optimality definition can be applied to both continuous optimization and discrete decision making. Some test cases belonging to both typologies show the validity of definitions and the accordance to common knowledge based reasoning.

References

[1] J. Kacprzyk and S. A. Orlovski editors, *Optimization Models Using Fuzzy sets and Possibility Theory*, Reidel Publishing Company, Dordrecht, 1987.
[2] C. Carlsson and R. Fuller, “Multiobjective optimization with linguistic variables,” *Proceedings of the Sixth European Congress on Intelligent Techniques and Soft Computing (EUFIT98, Aachen, 1998, Veralg Mainz, Aachen,*, vol. 2, pp. 1038–1042, 1998.
[3] R. Fuller and C. Carlsson, “Fuzzy multiple criteria decision making: Recent developments,” *Fuzzy Sets and Systems*, vol. 78, pp. 139–153, 1996.
[4] R. Fuller C. Carlsson and S. Giove, “Optimization under fuzzy linguistic rule constraints,” *Proceedings of Eurofuse-SIC’99, Budapest*, pp. 184–187, 1999.
[5] M. Sakawa, *Fuzzy Sets and Interactive Multiobjective Optimization*, Plenum press, New York and London, 1993.
[6] P. Amato and C. Manara, “Relating the theory of partitions in mv-logic to the design of interpretable fuzzy systems,” in *Trade-off between Accuracy and Interpretability in Fuzzy Rule-Based Modeling*, J. Casillas, O. Cordón, F. Herrera, and L. Magdalena, Eds. Springer-Verlag, Berlin, to be published.
[7] http://www.ccas.ru/mmes/mmeda/, ,” .
[8] K. Miettinen, *Nonlinear Multiobjective Optimization*, Kluwer Academic Publishers, Dordrecht, THE NEDERLANDS, 1999.