DESIGN OF LAYERED STRUCTURES WITH DESIRED DISPERSION PROPERTIES USING A MULTIOBJECTIVE GENETIC ALGORITHM

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ABSTRACT

An important dispersion-related characteristic of waves propagating through periodic materials is the existence of stop bands, that is, ranges of frequencies over which a medium effectively attenuates all incident waves. The width of these bands, and their location in the frequency domain, depend on the layout of contrasting materials and the ratio of their properties. In this work, a multi-objective genetic algorithm (GA) is utilized to design layered material topologies in such a way as to form frequency band structures that result in maximum wave attenuation. Subject to size and manufacturing constraints, Pareto optimal designs are obtained that illustrate the trade-off between the number of layers in a periodic unit cell and the corresponding wave attenuation capacity. Attention is focused on the longitudinal wave propagation in one-dimensional models for cases involving incident time harmonic waves as well as general transient pulses. The presented method has potential use in the development of shock and vibration isolation devices, among other applications.

KEYWORDS

Wave propagation, Dispersion, Stop bands, Periodic media, Genetic Algorithms.

1. INTRODUCTION

Wave propagation in heterogeneous media is dispersive, *i.e.* the shape of a pulse changes as it propagates through a medium. In materials with periodic heterogeneity there are ranges of frequencies, known as *stop bands* or *band gaps*, over which all incident waves are effectively attenuated. This attenuation phenomenon is attributed to a mechanism of destructive interference among the scattered wave field. In a previous work [1] dealing with longitudinal plane wave propagation, one-dimensional infinite periodic structures were optimized to have the width of stop bands maximized across frequency ranges of interest, thereby enhancing their shock and vibration isolation capability. The unit cell of the periodic structures consisted of sub-layers of alternating material types with different thicknesses. The frequency ban structure (the size and location of stop bands) was controlled by varying the configuration of the unit cell (the number and thicknesses of sub-layers in a unit cell).

For a given maximum limit on the number of layers, the optimal design was found by exhaustively searching over all possible cell configurations, under the restriction that layers only could have thicknesses that are multiples of a minimum size. While the exhaustive search guarantees finding a global optimum, it is prohibitively expensive because the number of possible combinations of layer thicknesses increases exponentially with the number of layers. Although a larger number of cell layers and a wider range of admissible layer thicknesses would provide more precise control of the frequency band structure, they will also entail higher manufacturing costs. It is therefore desirable to efficiently quantify the trade-off between the number of layers and the wave attenuation capacity. As such, the present paper poses the problem as a multi-objective optimization where the number of cell layers and their thicknesses are optimized for maximum wave attenuation capacity with minimum number of layers, subject to the constraints on the minimum layer thickness and the total cell length. Two alternative formulations are developed. In the first (mixed-integer programming formulation), the number of layers and the thicknesses of each layer are represented as an integer and continuous variables, respectively. In the second (zero-one integer programming formulation), the unit cell is divided into a fixed number of imaginary divisions and the material type of each division is represented by a binary variable, thereby representing the number and thicknesses of the actual layers. Both formulations are solved using a multi-objective genetic algorithm [2], a heuristic algorithm capable of efficiently generating Pareto optimal (non-inferior) solutions. Examples are shown for the cases involving incident time harmonic waves as well as general transient pulses.

The rest of the paper is organized as follows. Section 2 describes the mathematical formulation of a Transfer Matrix method for computing the frequency spectra (*i.e.*, the frequency versus wave number dispersion curves). In Section 3, the two alternative formulations of the cell design problem are presented, with a brief description of a multi-objective genetic algorithm. Section 4 presents the results of the two case studies, and conclusions are drawn in Section 5.

2. DISPERSIVE WAVE MOTION IN LAYERED MEDIA

Consider a general multi-layered structure (as depicted in Fig. 1) where an arbitrary layer j is shown to be positioned between an adjacent layer j-1 at its left and an adjacent layer j+1 at its right. The j^{th} layer has thickness $d^{(j)}$, density $\rho^{(j)}$, Young's modulus $E^{(j)}$, and longitudinal and transverse velocities, $c_p^{(j)}$ and $c_s^{(j)}$, respectively. For this one-dimensional model of a multi-layered structure, the elastodynamic response is determined using the Transfer Matrix method, which provides an exact elasticity solution [1, 3-4]. The governing equation for longitudinal wave propagation in the x_1 direction is:

$$\ddot{u}_1(x_1,t) = (c_p^{(j)})^2(x_1)\frac{\partial^2 u_1(x_1,t)}{\partial x_1^2} \quad (1)$$

where u_1 is the displacement field, and t denotes time. The boundary conditions that must be satisfied at the layer interfaces are (i) the continuity of the displacement u_1 and (ii) the continuity of the σ_{11} component of the stress tensor $\boldsymbol{\sigma}$. The solution of Eq. (1) in the j^{th} layer can be written as a superposition of forward and backward traveling waves with harmonic time dependence:

$$u_1(x_1,t) = \left[A_+^{(j)}\mathrm{e}^{ik_p^{(j)}x_1} + A_-^{(j)}\mathrm{e}^{-ik_p^{(j)}x_1}\right]\times \mathrm{e}^{-i\omega t} \quad (2)$$

where $i=\sqrt{-1}$, $k_p^{(j)} = \omega / c_p^{(j)}$ and ω is the time frequency. The stress component is given by

$$\sigma_{11}(x_1,t) = E^{(j)}\frac{\partial u_1(x_1,t)}{\partial x_1} = \rho_j (c_p^{(j)})^2 \frac{\partial u_1(x_1,t)}{\partial x_1} \tag{3}$$

Let x_1^{jL} and x_1^{jR} be used to denote the position along the x_1-axis of the left and right boundaries of layer j, respectively. From Eqs. (2) and (3), and using the relation $x_1^{jR} = x_1^{jL} + d^{(j)}$, the values of the displacement u_1 and stress component σ_{11} at x_1^{jL} are related to those at x_1^{jR}. Through a *transfer matrix* $\mathbf{T_j}$, this connection can be repeated in a recursive manner to relate the displacements and the stresses across several layers. For the n layered system shown in **Fig. 1**, a cumulative transfer matrix $\mathbf{T} = \mathbf{T_n T_{n-1}} \cdots \mathbf{T_1}$ is constructed:

$$\begin{bmatrix} u_1 \\ \sigma_{11} \end{bmatrix}_{x_1^{nR}} = \mathbf{T}\begin{bmatrix} u_1 \\ \sigma_{11} \end{bmatrix}_{x_1^{1L}} \tag{4}$$

For an infinite periodic layered medium consisting of repeated unit cells, each of width $d = d^{(1)} + d^{(2)} + \ldots + d^{(n)}$, Floquet's theory is used to relate the time harmonic response at a given cell to that at the adjacent cell:

$$\begin{bmatrix} u_1 \\ \sigma_{11} \end{bmatrix}_{x_1+d} = \exp(ikd)\begin{bmatrix} u_1 \\ \sigma_{11} \end{bmatrix}_{x_1} \tag{5}$$

Coupling Eq. (4) with Eq. (5) results in the eigenvalue problem:

$$[\mathbf{T} - \mathbf{I}exp(ikd)]\begin{bmatrix} u_1 \\ \sigma_{11} \end{bmatrix}_{x_1^{1L}} = 0 \tag{6}$$

which is solved for the dispersion curves. Furthermore, the displacement mode shapes can be obtained using Eqs. (2-6). The reader should refer to [1] for a step-by-step procedure for computing the time-dependent mode shape corresponding to any point in the frequency spectrum of an infinite structure.

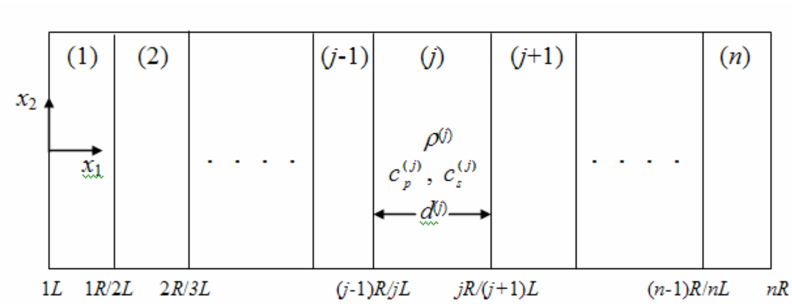

Fig. 1. Unit cell consisting of *n* layers. The layer number is indicated in parenthesis.

For purpose of demonstration, let us consider plane time harmonic wave propagation in a periodic layered medium with an arbitrarily chosen unit cell design, as shown in **Fig. 2.** The unit cell is composed of two parallel layers of stiff (fiber) and compliant (matrix) materials, Using '*f*' and '*m*' to denote fiber and matrix, respectively, the dimensions are $d^{(f)}/d = 0.8$, and the ratio of material properties are $\rho^{(f)}/\rho^{(m)} = 3$ and $E^{(f)}/E^{(m)} = 12$.

The frequency spectrum for longitudinal waves propagating in the direction normal to the layering are computed and plotted in **Fig. 3.** The non-dimensional frequency $\Omega = \omega d / \sqrt{E_m / \rho_m}$ and the non-dimensional wave number $\xi = k \times d$ define the ordinate and abscissa, respectively. The solid lines represent the real part of the dispersion relation, and these appear as multiple branches of *pass band modes* of wave propagation. The dotted lines represent the imaginary part of the dispersion relation, and these too appear as multiple branches, but of *stop band modes.* Within stop band frequencies all incident waves are localized and attenuated in space thus 'forbidding' the effective transmission of energy across the medium.

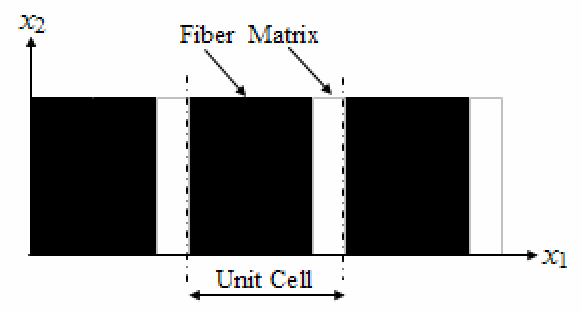

Fig. 2. Infinite layered medium (three unit cells are shown).

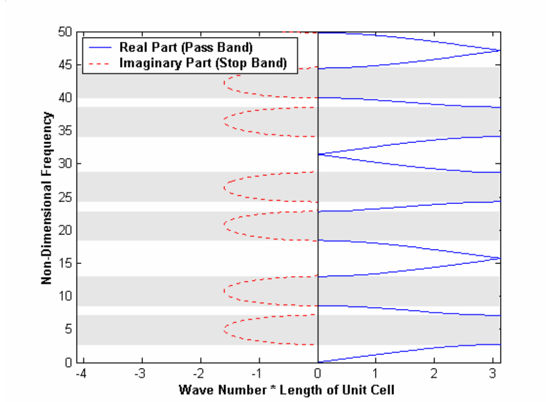

Fig. 3. Dispersion curves for longitudinal wave propagation normal to the layers in the infinite periodic structure in Fig. 2. Stop band regions are shaded.

3. MULTI-OBJECTIVE GENETIC ALGORITHM FOR UNIT CELL DESIGN

3.1. Cell Design Problem Formulations

Varying the number of layers of alternating materials and their thicknesses allows for "shaping" the frequency spectrum of the periodic medium. With this capability, a design can be generated such that stop bands are widened and centered near desired frequencies to enhance the medium's wave attenuation capacity at a specific frequency or a frequency range of interest. The details of such capacity measures are given in Section 4. Let the number and thicknesses of cell layers be n and $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, respectively, where u_i is the thickness of layer i. Since the unit cell is periodic (*i.e.*, repeated infinite times in the composite medium), it is assumed, without loss of generality, that the first layer in the unit cell is always a fiber *'f'* and that n is an even number. Using the integer variable n and continues variable $\boldsymbol{u}$, the problem can be formulated as mixed-integer programming, which shall be refered to as *mixed formulation*:

$$\text{Minimize: } \{ f_1 = Pen(C(n,\boldsymbol{u})),\ f_2 = n \} \tag{7}$$

$$\text{Subject to: } \sum_{i=1}^{n} u_i = 1 \tag{8}$$

$$u_i \geq a \text{ for } i = 1, 2, .., n \tag{9}$$

$$2 \leq n \leq n_{\max}\text{ , } n \text{ is an even number.} \tag{10}$$

$$\boldsymbol{u} \in \mathbf{R}^n,\ n \in \mathbf{Z} \tag{11}$$

where:

f_1: performance objective function, defined as a penalty on the deviation of the wave attenuation capacity of the medium at a specified frequency (or frequency range) from a target value.

f_2: manufacturability objective function, defined as the number of layers in the unit cell.

C: measure of wave attenuation capacity of the medium at a specified frequency (or frequency range) as a function of n and $\boldsymbol{u}$.

a: minimum thickness of a layer given as a fraction of unity.

$n_{\max}$: maximum number of layers.

It should be noted that the total cell length is unity by definition. It is assumed that a and n_{max} are imposed by manufacturing limitations.

In the above formulation, the dimension of vector variable $\boldsymbol{u}$ depends on another variable n, which often causes a difficulty in optimization algorithms. Alternatively, the problem can be formulated in terms of a vector of binary variables $\boldsymbol{b} = (b_1, b_2, \ldots, b_l)$ with a constant dimension l, by assuming a unit cell is divided into l imaginary "slots," each of which can be filled with either fiber 'f' or matrix 'm' [1]:

$$b_j = \begin{cases} 0 & \text{if slot } j \text{ is filled with fiber } f \\ 1 & \text{if slot } j \text{ is filled with matrix } m \end{cases} \tag{12}$$

Contiguous slots filled with the same material form a continuum and are regarded as one cell layer, whose total number is given as

$$n = \sum_{j=1}^{l-1} s_j + 1 \tag{13}$$

where $\boldsymbol{s} = (s_1, \ldots, s_{l-1})$, s_j = XOR(b_j, b_{j+1}). The thickness u_i of layer i can take only discrete values with the multiple of $1/l$, which can be expressed as:

$$u_i = (l_i - l_{i-1}) \times \frac{1}{l} \tag{14}$$

where $\boldsymbol{l} = (l_0, l_1, \ldots, l_n)$ are the indices of s_i with $s_i = 1$, sorted in the ascending order with $l_0 = 0$ and $l_n = l$. For example, if $l = 10$ and $\boldsymbol{b} = (0,0,0,1,1,1,1,0,0,1)$, then $\boldsymbol{s} = (0,0,1,0,0,0,1,0,1)$ and $\boldsymbol{l} = (0, 3, 7, 9, 10)$, and hence n = 4 and $\boldsymbol{u} = (0.3, 0.4, 0.2, 0.1)$, starting with the first layer being 'f' and alternate thereafter.

Using $\boldsymbol{b}$ as an independent design variable, the problem can now be formulated as zero-one integer programming, which shall be refered to as *binary formulation*:

$$\text{Minimize: } \{ f_1 = Pen(C(n, \boldsymbol{u})),\ f_2 = n \} \tag{15}$$

Subject to: $n = \sum_{j=1}^{l-1} s_j$, s_j = XOR(b_j, b_{j+1}) for j = 1, 2, …, l-1 (16)

$u_i = (l_i - l_{i-1}) \times \frac{1}{l}$, l_i for i = 0, .., n are as defined above. (17)

$u_i \geq a$ for i = 1, 2, .., n (18)

$2 \leq n \leq n_{\max}$, n is an even number (19)

$\boldsymbol{b} \in \{0,1\}^l$ (20)

In the previous work [1], the cell design problem was tackled using the binary formulation with l = 10, and this small value allowed for an exhaustive search of 2^{10} alternatives. Since u_i can only be the multiple of 1/l, the resulting optimal cell design could be further improved with a larger l, which makes exhaustive search totally impractical. In the present paper, a multi-objective genetic algorithm is chosen due to its efficiency for global optimization of discrete and/or continuous variables as described in the next section.

3.2. Multi-Objective Genetic Algorithm

A multi-objective genetic algorithm (GA) employed in the following case studies is a variant of Non-dominated Sorting Genetic Algorithm (NDSGA-II) [2, 5, 6], whose steps are outlined below.

1. Create a population P of p random designs and evaluate their objective function values. Also create empty set Q and O.
2. Rank each design c in P according to the number of other designs dominating[1] c (rank 0 is Pareto optimal in P).
3. Store the designs with rank 0 into set O. Update O by removing any designs dominated by others in O. If the size of O reaches a pre-specified number, remove the designs that are similar to others to maintain the size.
4. Select two designs c_i and c_j in P with probability proportional to p-$rank(c_i)$ and p-$rank(c_i)$.
5. Crossover c_i and c_j to generate new design(s) with a certain probability.
6. Mutate the new design(s) with a certain probability.
7. Rapair the new design(s) to maintain their feasibility.
8. Evaluate the objective function values of the new design(s) and store in Q. If the size of Q is less than p, go to 4.
9. Replace P with Q, empty Q, and increment the generation counter. If the generation counter has reached a pre-specified number, terminate the process and return O. Otherwise go to 2.

Due to the differences in the design variables, the two formulations of the cell design problem require different implementations of crossover, mutation, and repair at steps 6, 7, and 8. For the mixed formulation, a design is represented as a pair (n, $\boldsymbol{v}$), where $\boldsymbol{v}$ is a vector of constant size n_{max} with $v_i = u_i$ for i = 1, 2, …, n. The values of v_i for $i > n$ is simply ignored during the evaluation of the objective functions.

The crossover operator produces one new design ($n', \boldsymbol{v}'$) from two "parent" designs (n_1, $\boldsymbol{v}_1$) and (n_2, $\boldsymbol{v}_2$), implemented as a combination of arithmetic and heuristic crossovers [7]:

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[1] For a vector-valued function f = (f1,f2,..,fn) to be minimized, a point x dominates y if fi(x) < fi(y) for all i = 1,2,…,n.

$$n' = n_1 + 2\alpha(n_2 - n_1)\,, \tag{21}$$

$$\boldsymbol{v}' = \boldsymbol{v}_1 + 2\alpha(\boldsymbol{v}_2 - \boldsymbol{v}_1) \tag{22}$$

where α is a random number between 0 and 1. This type of crossover is reported to work well with continuous variables [7], hence it is applied to ***v*** with a high probability. However, it is applied to n with a very low probability to reduce its destructive effects. Then, the mutation operator randomly changes every variable to some value within its allowed range, with a low probability. The repair operator sets all v'_i for $i > n$ to zero, then performs a linear scaling such that $v'_1 + v'_2 + \cdots + v'_n = 1$ and $v'_i \geq a$ for $i = 1, 2, .., n$.

For the binary formulation, a design is simply represented as binary vector ***b***, to which classic multi-point crossover [8] and bit-flip mutation are applied. Similar to the mixed formulation, only even numbers of layers starting with *'f'* are considered without loss of generality. As such, the repair operator rotates the bits in vector ***b*** until $\boldsymbol{b}_1 = 0$ and $\boldsymbol{b}_l = 1$, thereby making the number of layers even and starting with *'f'*.

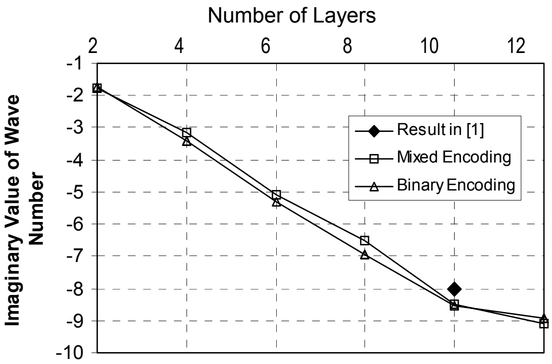
4. CASE STUDIES: CELL DESIGN FOR DESIRED FREQUENCY SPECTRA

4.1. Case 1: Creation of a Wide Stop Band Centered at a Specified Frequency

The objective is to design a cell consisting of *'f'* or *'m'* (same material properties as in Section 2), whose frequency spectrum has a stop band that i) is centered at a predetermined frequency, and ii) has the maximum width (*i.e.*, $\Delta\omega$). From a practical perspective, a finite structure composed of several cells of such a design could be used to attenuate the propagation of a single harmonic wave. Inspection of Fig. 3 and the band diagrams of other designs show that at the center of a stop band, the value of the wave number (which is imaginary) is directly proportional to the width of the band. Furthermore, it is known that the strength of spatial attenuation of an incident wave at a stop band frequency is exponentially related to the value of the corresponding imaginary wave number. On this basis, the value of the imaginary wave number at a predetermined frequency will be taken as the performance objective function f_1 for the design problem.

This design problem is investigated in [1], using the target frequency $\Omega = 20$. The optimal design for the binary formulation with $l = 10$ obtained by exhaustive enumeration is shown as a reference point in **Fig. 4**. The number of layers of the design is 10, and the imaginary value of the wave number at the specified frequency was -8.01. Pareto plots generated by the multi-objective genetic algorithm are shown in **Fig. 4**. The values of $n_{\max} = 16$ and $l = 30$ are used for the mixed and binary formulations, respectively. Both values are chosen to allow a larger design freedom than the case in [1]. It should be noted that $l = 30$ makes an exhaustive search highly impractical since it would require 2^{30} ($\approx 10^9$) function evaluations.

The Pareto plot in **Fig. 4** shows that the multi-objective genetic algorithm found improved designs (imaginary wave number value of about -8.33) using the same number of layers as in [1] as well as even better designs using 12 layers. The unit cell layouts of the best designs are shown in **Fig. 5**. No better designs were found by increasing the number of layers beyond 12. The run-time parameters of the genetic algorithm are given in Appendix.



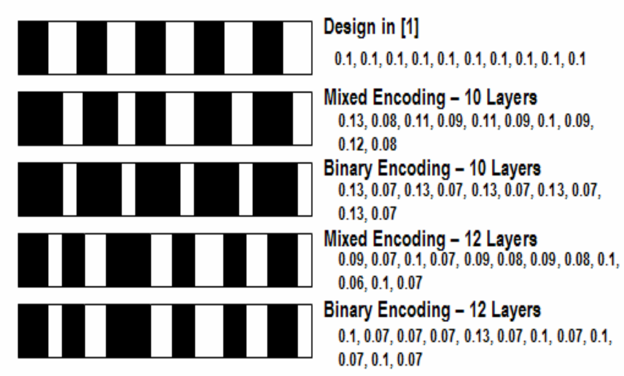


Fig. 4. Case 1 – Pareto Plot **Fig. 5. Case 1 – Best Cell Designs**

4.2. Case 2: Minimization of Transmissibility across a Specified Frequency Range

The objective in this case study is to minimize the transmission of an incident general transient pulse. Transient pulses typically have broad frequency content, and in most practical cases it is unlikely that a stop band can be synthesized with enough width to cover the whole frequency range of the pulse. For this reason, an alternate performance measure, *transmissibility*, is employed; which is defined as the percentage of the sum of frequency ranges where a pass band exists, to the total frequency range of interest. The Pareto plots in **Fig. 6** show the performances of the designs optimized for frequency range of $0 \le \Omega \le 50$, obtained using the mixed formulation, binary formulation with $l = 30$, as well as the previous design found in [1] (which employed exhaustive search with $l = 10$). Again, the multi-objective genetic algorithm found superior designs with the same number of layers as in [1] (6 layers), and even better designs using more layers. The unit cell layouts of the best designs are shown in **Fig. 7**. No better designs were found with more than 10 layers. The run-time parameter values for the genetic algorithm are the same as in Case 1.

In both case studies, the multi-objective genetic algorithms could find better designs, since the solved optimization problems are the relaxation of the ones in [1]. Although the mixed formulation is also a relaxation of the binary formulation, the dependency of the size of $\boldsymbol{u}$ on n decreases the efficiency of the genetic algorithm, a phenomenon known as *epistasis* [8] This is a likely reason why the binary formulation found better designs for a small number of cell layers (4 to 6), than the mixed formulation. Furthermore, from a practical perspective, the binary formulation has an advantage since the optimized layer thickness values require no truncation. On the other hand, for a fixed length binary string, an increase in the number of cell layers implies a loss of resolution because fewer bits are allocated per layer. This in turn limits the performance of the designs with a large number of layers in the binary formulation.

This paper discussed the application of a multi-objective genetic algorithm to the design of material distributions within repeated unit cells of periodic layered structures for maximum wave attenuation and minimum number of cell layers. This was achieved by controlling the size and location of stop bands within the medium's frequency spectrum. Attention was focused on longitudinal wave motion for cases involving incident time harmonic waves as well as general transient pulses. Two alternative formulations of the design problem were presented. In the mixed-integer programming formulation, the number of layers and the thicknesses of each layer are represented as an integer and continuous variables, respectively. In the zero-one integer programming formulation, the unit cell is divided into a fixed number of imaginary divisions and the material type of each division is represented by a binary variable. Two case studies are presented on the maximum attenuation of incident time harmonic waves and of general transient pulses. In both cases, the obtained Pareto plots indicated the increase in the number of cell layers improves the wave attenuation performance. Further, multi-objective genetic algorithm found better unit cell designs than previously reported in the literature. The results also show that the relevant measure of wave attenuation saturated at approximately 12 layers for composite materials designed to stop a time harmonic wave, and 10 layers for isolation of transient pulses spanning a wide frequency range.

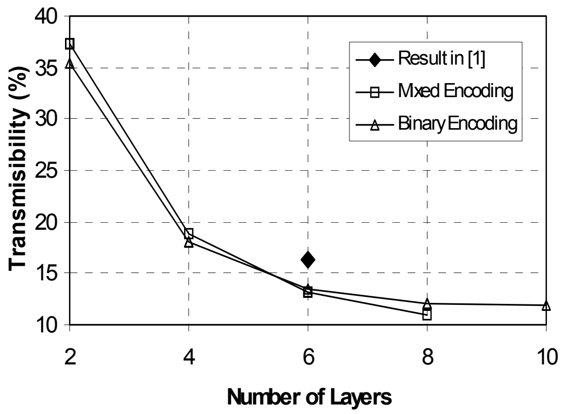


Fig. 6. Case 2 – Pareto Plot

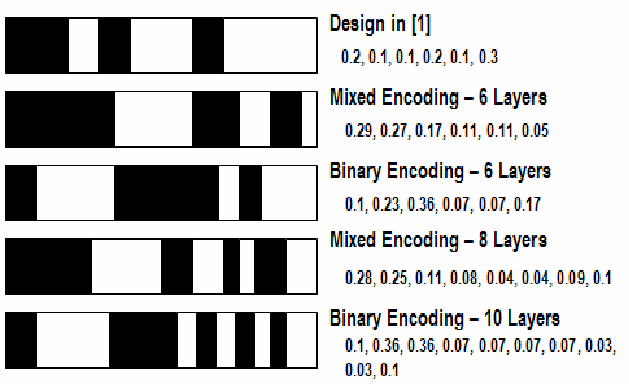


Fig. 7. Case 2 – Best Cell Designs

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APPENDIX

Run-time parameters of Genetic Algorithm use in the case studies

	Mixed formulation	Binary formulation
Population Size	120	100
Number of Generations	80	100
Crossover Probability	0.90	0.90
Mutation Probability	0.02	0.02