

# A New Selection Mechanism Based on Hypervolume and its Locality Property

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**Abstract**—In this paper, we propose a new selection mechanism based on the hypervolume indicator and on its “locality property”, which is incorporated into the SMS-EMOA, giving rise to the so-called improved SMS-EMOA (iSMS-EMOA). Our proposed selection mechanism is validated using standard test functions taken from the specialized literature, having three to six objective functions. iSMS-EMOA is compared with respect to its predecessor SMS-EMOA and with respect to another version of SMS-EMOA that uses the approximation of the hypervolume indicator, instead of its exact calculation. Our preliminary results indicate that our proposed selection mechanism outperforms the selection mechanisms based on the hypervolume indicator that have been proposed in recent years, since it significantly reduces the computational time required by the algorithm without sacrificing quality in the approximation generated.

## I. INTRODUCTION

Many optimization problems arising in the real world involve multiple objective functions which must be satisfied simultaneously. They are generically called *multiobjective optimization problems (MOPs)* and usually their objectives are in conflict with each other. In MOPs, the notion of optimality refers to the best possible trade-offs among the objectives. Consequently, there are several solutions (the so-called *Pareto optimal set* whose image is called the *Pareto front*). The use of evolutionary algorithms for solving MOPs has become very popular and they have two main goals [1]: (i) to find solutions that are, as close as possible, to the true Pareto front and, (ii) to produce solutions that are spread along the Pareto front as uniformly as possible.

There are different indicators to assess the quality of the approximation of the Pareto-optimal set generated by a *multi-objective evolutionary algorithm (MOEA)*. However, the *hypervolume indicator*  $I_H$  is the only unary indicator which is “Pareto compliant” [2].  $I_H$  was originally proposed by Zitzler and Thiele in [3], and it’s defined as the size of the space covered by the Pareto optimal solutions.  $I_H$  rewards convergence towards the Pareto front as well as the maximum spread of the solutions obtained. Fleischer proved in [4] that, given a finite search space and a reference point, *maximizing the hypervolume indicator is equivalent to finding the Pareto optimal set*.

In recent years, MOEAs based on the hypervolume indicator have become relatively popular. This is because the use of Pareto-based selection has several limitations. From them, its poor scalability regarding the number of objectives of a MOP is, perhaps, the most remarkable. The quick increase in the number of nondominated solutions as we increase the number of objective functions, rapidly dilutes the effect of the selection mechanism of a MOEA [5]. However, the hypervolume indicator has one important disadvantage: its high computational cost (the running time for calculating  $I_H$  is exponential in the number of objective functions). Because of this, MOEAs based on hypervolume normally become impractical when we want to solve MOPs with many objectives. Recently, several proposals have been made to address this problem. For example, some authors have proposed to reduce the dimensionality of the MOP [6], others have proposed improvements to the calculation of the contribution of each individual in the population to the hypervolume indicator [7], [8], as well as mechanisms to approximate the contribution of each individual in the population to the hypervolume [9], [10], [11]. In this work, we propose a new selection mechanism based on the hypervolume indicator. Our selection mechanism significantly reduces the computational time required by the MOEA that incorporates it, without sacrificing the quality of the approximation that such an algorithm generates. The proposed mechanism is based on an interesting property of the hypervolume: *locality*.

The remainder of this paper is organized as follows. Section II states the problem of our interest. The previous related work is discussed in Section III. Section IV describes the hypervolume indicator. Our proposal is discussed in Section V. Our experimental validation and the results obtained are shown in Section VI. Finally, we provide our conclusions and some possible paths for future work in Section VII.

## II. PROBLEM STATEMENT

We are interested in the general *multiobjective optimization problem (MOP)*, which is defined as follows: Find  $\vec{x}^* = [x_1^*, x_2^*, \dots, x_n^*]^T$  which optimizes

$$\vec{f}(\vec{x}) = [f_1(\vec{x}), f_2(\vec{x}), \dots, f_k(\vec{x})]^T \quad (1)$$

such that  $\vec{x}^* \in \Omega$ , where  $\Omega \subset R^n$  defines the feasible region of the problem. Assuming minimization problems, we have the following definitions.

*Definition 1:* We say that a vector  $\vec{u} = [u_1, \dots, u_n]^T$  dominates vector  $\vec{v} = [v_1, \dots, v_n]^T$ , denoted by  $\vec{u} \leq_p \vec{v}$ , if and only if  $f_i(\vec{u}) \leq f_i(\vec{v})$  for all  $i \in \{1, \dots, k\}$  and there exists an  $i \in \{1, \dots, k\}$  such that  $f_i(\vec{u}) < f_i(\vec{v})$ .

*Definition 2:* A point  $\vec{x}^* \in \Omega$  is Pareto optimal if and only if for all  $\vec{x} \in \Omega$  we have that  $\vec{x}^* \leq_p \vec{x}$  where  $\vec{x}^* \neq \vec{x}$ .

*Definition 3:* For a given MOP,  $\vec{f}(\vec{x})$ , the Pareto optimal set is defined as:  $\mathcal{P}^* = \{\vec{x} \in \Omega \mid \neg \exists \vec{y} \in \Omega : \vec{f}(\vec{y}) \leq_p \vec{f}(\vec{x})\}$ .

*Definition 4:* Let  $\vec{f}(\vec{x})$  be a given MOP and  $\mathcal{P}^*$  the Pareto optimal set. Then, the Pareto Front is defined as:  $\mathcal{PF}^* = \{\vec{f}(\vec{x}) \mid \vec{x} \in \mathcal{P}^*\}$ .

### III. PREVIOUS RELATED WORK

During the last 10 years, there have been several proposals to incorporate the hypervolume indicator into a MOEA. Knowles and Corne [12] used a bounded archive to save the nondominated solutions found at each generation. When the archive was full and Pareto dominance could no longer discard solutions then, they proposed to use the hypervolume indicator as follows: calculate the contribution of each solution to the hypervolume indicator; if the contribution of the new solution is better than the contribution of the solution with the worst contribution, then the new solution would replace it; otherwise, the archive would remain the same.

Zitzler and Künzli [13] proposed a general selection mechanism based on a performance indicator. The general algorithm was called "IBEA", and it assigns the fitness of each individual using:  $F(\vec{x}) = \sum_{\vec{y} \in P \setminus \{\vec{x}\}} e^{-I(\{\vec{y}\}, \{\vec{x}\})/k}$ , where  $P$  is the population,  $\vec{x}, \vec{y} \in P$  and  $k$  is a scalar fitness factor. When the worst individual is eliminated, IBEA updates the fitness of all other individuals with:  $F(\vec{x}) = F(\vec{x}) + e^{-I(\{\vec{x}'\}, \{\vec{x}\})/k}$ , where  $\vec{x}'$  is the eliminated individual.

Emmerich et al. [14] proposed an algorithm based on NSGA-II and the archived strategies proposed by Knowles, Corne and Fleisher. They called it "SMS-EMOA". SMS-EMOA creates an initial population and in the following, it generates only one solution by iteration using the operators (crossover and mutation) of the NSGA-II. After that, it applies Pareto ranking. When the last front has more than one solution, SMS-EMOA uses the contribution to the hypervolume indicator to decide which solution will be removed. Beume et al. [15] proposed not to use the contribution to the hypervolume indicator when in the Pareto ranking we obtain more than one front. In that case, they proposed to use the number of solutions which dominate to one solution (the solution that is dominated by more solutions is removed). The authors argue that the motivation to use the hypervolume indicator is to improve the distribution in the true Pareto front and then it is not necessary in fronts different of the true Pareto front.

Igel et al. [16] used an evolution strategy with Pareto ranking as a primary criterion selection and crowding or hypervolume as a second selection criterion. This approach operates in a way similar to SMS-EMOA. Mostaghim [17] designed a MOEA based on particle swarm optimization in which the hypervolume indicator was used in the leader selection mechanism.

All the above approaches have an important disadvantage. It is well known that the computation of the hypervolume indicator has a high computational cost. When all the individuals are nondominated, all the above approaches need to calculate  $|P| + 1$  times the hypervolume indicator in order to obtain the contribution of each individual in the population and the contribution of the new individual. This turns these approaches impractical when we want to solve MOPs with many objective functions. In an attempt to reduce the computational cost, Bradstreet [7] proposed a method to calculate the contribution to the hypervolume indicator of each solution in a fast way without calculating the hypervolume for each solution. The main idea is as follows: when we eliminate one solution of the population not all the contributions of the other solutions are affected. Emmerich and Fonseca [8] propose a dimension sweep algorithm for computing all hypervolume contributions in three dimensions with a time complexity equal to  $O(n \log n)$ . Also, they show that for  $d > 3$  (more than three objective functions), the time complexity is bounded below by  $\Omega(n \log n)$ . Bader and Zitzler [10] proposed to assign a fitness to each individual using an approximation of the hypervolume based on the idea that is not necessary to know the exact contribution to the hypervolume of each solution, since we only aim to obtain a good ranking of the solutions in the population. Bringmann and Friedrich [11] also proposed to use the approximation of the hypervolume indicator as a selection criterion in archiving algorithms.

### IV. HYPERVOLUME INDICATOR

The hypervolume indicator ( $I_H$ ) was originally proposed by Zitzler and Thiele in [3], and it's defined as the size of the space covered by the Pareto optimal solutions. If  $\Lambda$  denotes the Lebesgue measure,  $I_H$  is defined as:

$$I_H(A, y_{ref}) = \Lambda \left( \bigcup_{y \in A} \{y' \mid y < y' < y_{ref}\} \right) \quad (2)$$

where  $y_{ref} \in \mathbb{R}^k$  denotes a reference point that should be dominated by all the Pareto optimal points.

Auger et al. [18] did a study about the optimal  $\mu$ -distributions and the choice of the reference point in the hypervolume indicator. In this work, they mentioned one interesting property of this indicator when  $d = 2$  (two objective functions), called **locality** which says: *given three consecutive points on the Pareto front, moving the middle point will only affect the hypervolume contribution that is solely dedicated to this point, but the joint hypervolume contribution remains fixed*. Also, Auger et al. did a similar study for  $d = 3$  in [19] and they mentioned that the optimal placement of a single solution is not determined by only two neighbors, anymore, as it is the case for  $d = 2$ .

### V. OUR PROPOSED APPROACH

As we saw in Section III, the main disadvantage of the selection mechanisms based on the hypervolume indicator that have been proposed so far is that they require calculating the contribution to the hypervolume indicator of each solution in the population and this is a computationally expensive process, particularly when dealing with many-objective

problems. Here, we propose a new selection mechanism that exploits the locality property of the hypervolume indicator as follows. Let's assume that at each iteration of a MOEA, only one solution  $\vec{x}_{new}$  is created. After that, we calculate the Euclidean distance of the new solution to each solution in the current population:

$$dist_i = \|\vec{x}_i - \vec{x}_{new}\| \text{ such that } \vec{x}_i \in P \quad (3)$$

and, we choose the nearest solution:

$$\vec{x}_{near} \text{ such that } dist_{near} = \min dist_i \quad (4)$$

where  $P$  is the current population and  $i = \{1, \dots, |P|\}$ . These two solutions (the new solution,  $\vec{x}_{new}$ , and its nearest neighbor,  $\vec{x}_{near}$ ) compete to survive. The core idea is to move a solution within its neighborhood with the aim of improving its contribution to the hypervolume.

To validate our selection mechanism, we decided to incorporate it into SMS-EMOA. In Figures 1.(a) and 1.(c), we can see the results obtained by our selection mechanism in the WFG1 and WFG4 test problems [20], with two objective functions and twenty-four decision variables. For this experiment, we used 100 individuals and 500 generations<sup>1</sup>. In these figures, we can see that the modified SMS-EMOA is able to generate a good approximation of the Pareto optimal front. However, for the WFG1 test problem, this approach can only produce a portion of the Pareto optimal front and in the WFG4 test problem, some solutions are dominated and the nondominated solutions are not well-distributed. One important thing that we must consider is the case in which the new solution is located in an unexplored region (a region with few solutions). In this case, it is not a good idea to remove the new solution or its nearest neighbor. To address this problem, we propose to choose randomly another solution; this solution will also compete with the other two ( $x_{new}$  and  $x_{near}$ ). In Figures 1.(b) and 1.(d), we can see the results using the random individual. In these figures, we can note that if we use only one random solution, the distribution of the solutions on the Pareto front significantly improves. Therefore, if we use our proposed selection mechanism, we can significantly improve the computational time required by our approach, since we would need to compute the hypervolume only three times for each iteration (regardless of the population size) unlike SMS-EMOA which, in the worst case needs to compute  $|P| + 1$  times the hypervolume. Because of this, this proposed approach is called **improved SMS-EMOA** (iSMS-EMOA). Algorithm 1 shows the complete selection mechanism that we propose. We can note that we leave open the option to use more than one random solution. However, in all the experiments reported here, we only used one random solution. Also, in Algorithm 1, we can observe that our selection mechanism is only based on the hypervolume indicator (Pareto dominance is not used at any time).

#### A. Study of our proposed selection mechanism

We studied the behavior of our selection mechanism with respect to the random solution. Specifically, we wanted to know: how many times is the random solution eliminated?

<sup>1</sup>Each time we generate 100 individuals, this is considered as one generation.

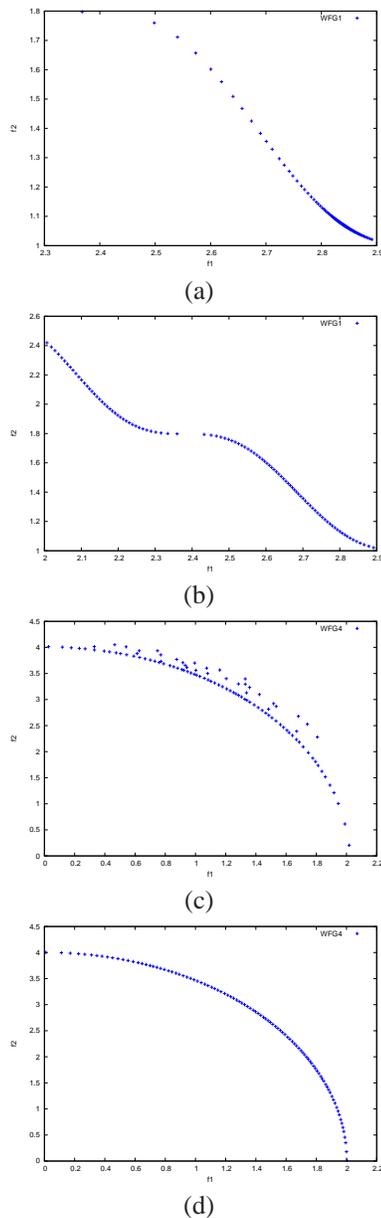


Fig. 1. Pareto fronts obtained with the modified SMS-EMOA algorithm, in the WFG1 and WFG4 test problems with two objective functions and 24 variables. In (a) and (c), we didn't use random solutions. In (b) and (d), we used one random solution.

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**Algorithm 1:** New selection mechanism based on the hypervolume indicator and its locality property.

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**Input** : Current population ( $P_t$ ), the new solution ( $\vec{x}_{new}$ ) and the number of random individuals that we will use ( $n_{random}$ )

- 1  $E = \emptyset$  (candidate solutions to be eliminated);
  - 2 Add to  $E$  the nearest solution to  $\vec{x}_{new}$ ,  
 $E = E \cup \{\vec{x}_{near}\}$ , using Euclidean distance:  
 $\vec{x}_{near} \mid dist_{near} = \min dist_i$ , where  
 $dist_i = \|\vec{x}_i - \vec{x}_{new}\| \mid \vec{x}_i \in P_t$ ;
  - 3 Add to  $E$ ,  $n_{random}$  solutions chosen randomly;
  - 4  $E = E \cup \vec{x}_{new}$ ;
  - 5 Calculate the contribution to the hypervolume indicator of each solution in  $E$  and remove the solution with the minimum contribution.
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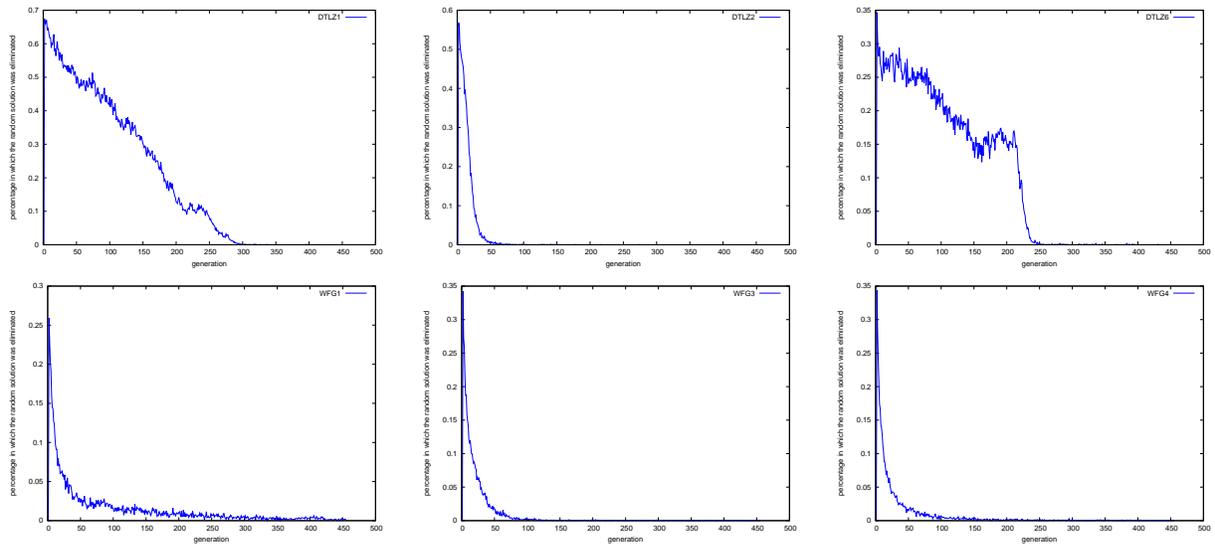


Fig. 2. Mean of 30 independent runs corresponding to the percentage of times in which the random solution was eliminated. We considered problems DTLZ1, DTLZ2, DTLZ6, WFG1, WFG3 and WFG4, all of them with three objective functions.

For this, we performed 30 independent runs for the test problems: DTLZ1, DTLZ2 and DTLZ6 [21] with three-objective functions, seven decision variables for DTLZ1 and DTLZ6 and twelve decision variables for DTLZ2. Also, we used WFG1, WFG3 and WFG4 [20] with three-objective functions and twenty-four decision variables. Per run, we calculate at each generation the number of times in which the random solution is eliminated. Figure 2 shows that the random solutions are eliminated at the beginning of the search process. And, as the search process progresses, the new solution or its nearest neighbor is eliminated more often. This is very important because it experimentally illustrates that the “locality property” is true for  $d = 3$ . Although in [19] it is shown that, for three dimensions, the optimal placement of a single solution is not determined by only two neighbors, we think that this does not mean that the “locality property” is false, but only that the neighborhood of the involved solutions gets larger in this case. Also, it is important to note that although the neighborhood size increases, our selection mechanism still works well. Therefore, we can claim the following: while the Pareto front is being generated, we need of the randomness for exploring the entire Pareto front. Later on, we only need to exploit the “locality property” to maximize the hypervolume indicator and improve the distribution on the Pareto front.

## VI. EXPERIMENTAL RESULTS

We validate the proposed selection mechanism comparing our iSMS-EMOA with respect to the algorithms: SMS-EMOA [15] and a version of SMS-EMOA that uses Monte Carlo simulation to approximate the hypervolume (we called it appSMS-EMOA). For this sake, we used the source code of HyPE available in the public domain [10] adopting  $10^4$  as our number of samples <sup>2</sup>.

<sup>2</sup>The source code of the three algorithms (appSMS-EMOA, SMS-EMOA and iSMS-EMOA) can be provided by the first author upon request.

For our experiments, we used seven problems with 3, 4, 5 and 6 objective functions taken from the Deb-Thiele-Laumanns-Zitzler (DTLZ) test suite [21]. We used  $k = 5$  for DTLZ1, DTLZ3 and DTLZ6 and  $k = 10$  for the remaining test problems. Also, we used six problems with 3, 4, 5 and 6 objective functions, taken from the WFG toolkit [20], with 24 decision variables. For each test problem, we performed 30 independent runs. For all algorithms, we adopted the parameters suggested by the authors of NSGA-II:  $p_c = 0.9$  (crossover probability),  $p_m = 1/n$  (mutation probability), where  $n$  is the number of decision variables. Both for the crossover and mutation operators, we adopted  $\eta_c = 15$  and  $\eta_m = 20$ , respectively. We performed a maximum of 50,000 fitness function evaluations (we used a population size of 100 individuals and we iterated for 500 generations). Only in DTLZ3 we performed 100,000 evaluations (we used a population size of 100 individuals and we iterated for 1000 generations). However, *we adopted four hours as our maximum running time* because we know that the computation of the hypervolume has a very high computational cost. For example, SMS-EMOA with exact hypervolume calculation requires more than five hours per run when dealing with 4 or more objective functions (in fact, a single run takes about one month when using seven objective functions).

### A. Performance Indicators

We adopted only  $I_H$  to validate our results because it rewards both convergence towards the Pareto front as well as the maximum spread of the solutions obtained. Also, SMS-EMOA, appSMS-EMOA and iSMS-EMOA, have as their aim to maximize the hypervolume and, therefore, it makes sense to use this indicator to assess their performance. To calculate the hypervolume indicator, we used the following reference points:  $y_{ref} = [y_1, \dots, y_M]$  such that  $y_i = 0.7$  for DTLZ1,  $y_{ref} = [y_1, \dots, y_M]$  such that  $y_i = 1.1$  for DTLZ(2-6),  $y_{ref} = [y_1, \dots, y_M]$  such that  $y_M = 6.1$  and  $y_{i \neq M} = 1.1$  for DTLZ7. In the case of the WFG test

problems, we generated the reference point using the highest value found for each objective function taking into account all the outputs of the three algorithms (i.e., SMS-EMOA, appSMS-EMOA and iSMS-EMOA).

### B. Discussion of Results

First, we will review the results of the DTLZ test problems. In Table I, we can observe that our iSMS-EMOA is better than SMS-EMOA and appSMS-EMOA because in all cases, it obtained similar results or outperformed the other two algorithms. For example, in DTLZ3 with 3, 4, 5 and 6 objectives, and in some problems with 6 objectives, our iSMS-EMOA significantly outperformed both appSMS-EMOA and SMS-EMOA. Also, we can observe that the quality of the approximated Pareto set decreases if we approximate the hypervolume. Although, in some cases, the appSMS-EMOA performed more objective function evaluations than SMS-EMOA and iSMS-EMOA (this algorithm did not exceed the maximum running time of 4 hours allowed for the other approaches), it was unable to obtain better results than any of the other approaches. Regarding the WFG test problems, in Table II, we can see that our iSMS-EMOA obtained similar results to those of the SMS-EMOA and in WFG1 and WFG4 with 5 and 6 objective functions, our iSMS-EMOA significantly outperformed SMS-EMOA. Again, we can see in this case, that appSMS-EMOA loses quality in their approximations in most cases. In Table III, we can observe the results of a statistical analysis using Wilcoxon’s rank sum. With these results, we can corroborate that our iSMS-EMOA can indeed outperform appSMS-EMOA and SMS-EMOA because only in the problems in which our iSMS-EMOA obtained better results the null hypothesis (“medians are equal”) can be rejected. Now, we will discuss the running time required by each of the three algorithms. Table IV shows that our iSMS-EMOA significantly decreases the running time required by SMS-EMOA. For example, in the DTLZ test problems with five objective functions, SMS-EMOA consumes the maximum allowable time (4 hours) and is unable to converge, whereas our iSMS-EMOA requires a maximum of 55 minutes to converge in any of the DTLZ test problems. Additionally, our proposed approach does not consume the maximum allowable time in any of the DTLZ test problems with 6 objectives. With respect to the WFG test problems, we can also see that our iSMS-EMOA requires less time than SMS-EMOA. However, it is important to note that in this case, SMS-EMOA doesn’t consume the maximum allowable time in the problems with 6 objectives. This indicates that the algorithm uses Pareto ranking most of the time, since otherwise its computational cost would had been much higher. This is interesting, because our iSMS-EMOA always uses hypervolume to discard one individual, but it still requires less running time than SMS-EMOA in the WFG test problems.

With respect to appSMS-EMOA, we can say that, although it requires a much lower running time than the other approaches, its results are of poor quality, and it’s unable to converge in several cases (see the results for DTLZ3 with 3, 4, 5 and 6 objective functions, and for DTLZ6 with 4, 5 and 6 objective functions in Table I). Also, it is important to consider that although in some problems appSMS-EMOA

$f$	appSMS-EMOA	SMS-EMOA	iSMS-EMOA
DTLZ1 (3)	0.314943 (0.000640)	<b>0.317025</b> <b>(0.000026)</b>	0.316981 (0.000074)
DTLZ2 (3)	0.742759 (0.001515)	<b>0.758039</b> <b>(0.000034)</b>	0.757902 (0.000110)
DTLZ3 (3)	0.000000 (0.000000)	0.596142 (0.216365)	<b>0.701242</b> <b>(0.126774)</b>
DTLZ4 (3)	0.744330 (0.001486)	<b>0.758018</b> <b>(0.000046)</b>	0.757923 (0.000094)
DTLZ5 (3)	0.438004 (0.000166)	<b>0.439373</b> <b>(0.000008)</b>	0.439352 (0.000017)
DTLZ6 (3)	0.397337 (0.032957)	0.418621 (0.013549)	<b>0.419454</b> <b>(0.016586)</b>
DTLZ7 (3)	1.707290 (0.333059)	1.859562 (0.271075)	<b>1.929149</b> <b>(0.185694)</b>
DTLZ1 (4)	0.229200 (0.003761)	<b>0.234452</b> <b>(0.000020)</b>	0.234451 (0.000020)
DTLZ2 (4)	1.007769 (0.003697)	<b>1.044589</b> <b>(0.000065)</b>	1.044208 (0.000160)
DTLZ3 (4)	0.000000 (0.000000)	0.671114 (0.314784)	<b>0.999968</b> <b>(0.120025)</b>
DTLZ4 (4)	1.016881 (0.003487)	<b>1.044610</b> <b>(0.000099)</b>	1.044315 (0.000106)
DTLZ5 (4)	0.412361 (0.006446)	<b>0.437300</b> <b>(0.000215)</b>	0.437094 (0.000328)
DTLZ6 (4)	0.069490 (0.024654)	<b>0.408659</b> <b>(0.017136)</b>	0.403198 (0.020206)
DTLZ7 (4)	0.509020 (0.217796)	0.712497 (0.138236)	<b>0.713030</b> <b>(0.187745)</b>
DTLZ1 (5)	0.160703 (0.003344)	0.166703 (0.000020)	<b>0.166730</b> <b>(0.000010)</b>
DTLZ2 (5)	1.239057 (0.005161)	1.294242 (0.000284)	<b>1.295762</b> <b>(0.000140)</b>
DTLZ3 (5)	0.015480 (0.041438)	0.411990 (0.455534)	<b>1.285288</b> <b>(0.031502)</b>
DTLZ4 (5)	1.248092 (0.005374)	1.295125 (0.000222)	<b>1.295819</b> <b>(0.000175)</b>
DTLZ5 (5)	0.404590 (0.007489)	0.445145 (0.000692)	<b>0.446236</b> <b>(0.000570)</b>
DTLZ6 (5)	0.004125 (0.005667)	0.221195 (0.011336)	<b>0.350987</b> <b>(0.022494)</b>
DTLZ7 (5)	0.087509 (0.073320)	0.120929 (0.074349)	<b>0.151319</b> <b>(0.063041)</b>
DTLZ1 (6)	0.099675 (0.027426)	0.000008 (0.000036)	<b>0.117299</b> <b>(0.000006)</b>
DTLZ2 (6)	1.447387 (0.006637)	1.313996 (0.060301)	<b>1.528940</b> <b>(0.000250)</b>
DTLZ3 (6)	0.020706 (0.052084)	0.000000 (0.000000)	<b>1.028456</b> <b>(0.669435)</b>
DTLZ4 (6)	1.468604 (0.006714)	1.228692 (0.070313)	<b>1.529425</b> <b>(0.000253)</b>
DTLZ5 (6)	0.408784 (0.010836)	0.369204 (0.027242)	<b>0.461434</b> <b>(0.000896)</b>
DTLZ6 (6)	0.000312 (0.000839)	0.000000 (0.000000)	<b>0.314153</b> <b>(0.019712)</b>
DTLZ7 (6)	0.012383 (0.013826)	0.005464 (0.005830)	<b>0.025679</b> <b>(0.009017)</b>

TABLE I. RESULTS OBTAINED IN THE DTLZ TEST PROBLEMS. WE SHOW AVERAGE VALUES OVER 30 INDEPENDENT RUNS. THE VALUES IN PARENTHESES CORRESPOND TO THE STANDARD DEVIATIONS.

performs more objective function evaluations than our iSMS-EMOA, it cannot outperform iSMS-EMOA.

Finally, Figure 3 shows the Pareto fronts obtained by the three algorithms in their median with respect to the hypervolume indicator in some of the test problems adopted. Here, we can see again that appSMS-EMOA loses quality in most cases, unlike our iSMS-EMOA which achieves a good distribution even in DTLZ3 where both SMS-EMOA and appSMS-EMOA have difficulties to generate the Pareto front.

## VII. CONCLUSIONS AND FUTURE WORK

We have proposed a selection mechanism based on the hypervolume which was found to outperform other selection

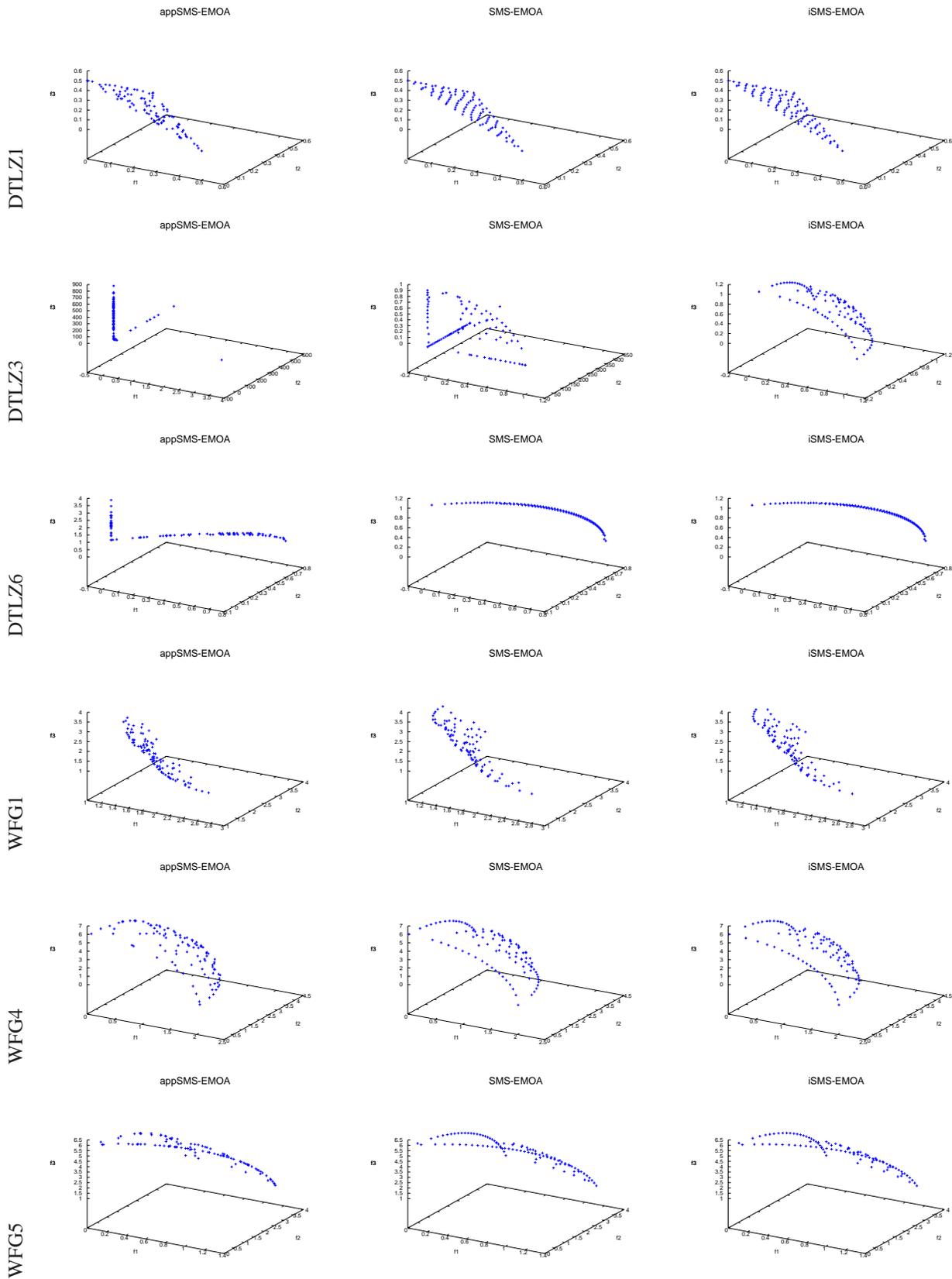


Fig. 3. Pareto fronts obtained by the three algorithms in the median (with respect to the hypervolume indicator) of their independent runs for the test problems: DTLZ1, DTLZ3, DTLZ6, WFG1, WFG4 and WFG5.

$\bar{f}$	appSMS-EMOA	SMS-EMOA	iSMS-EMOA
WFG1 (3)	19.907439 (1.027382)	21.083857 (0.551874)	<b>21.191940</b> <b>(0.150189)</b>
WFG2 (3)	0.115931 (0.011099)	<b>0.121293</b> <b>(0.010681)</b>	0.120384 (0.011041)
WFG3 (3)	0.449310 (0.004032)	0.466021 (0.001275)	<b>0.466647</b> <b>(0.000635)</b>
WFG4 (3)	22.638354 (0.191186)	<b>24.638858</b> <b>(0.070075)</b>	24.594651 (0.084218)
WFG5 (3)	8.072040 (0.026285)	8.449360 (0.002789)	<b>8.449718</b> <b>(0.005162)</b>
WFG6 (3)	0.982610 (0.010233)	<b>1.024750</b> <b>(0.002973)</b>	1.024599 (0.002291)
WFG1 (4)	73.969092 (1.719960)	<b>86.218895</b> <b>(0.469877)</b>	85.914358 (0.532377)
WFG2 (4)	0.016697 (0.001397)	0.017826 (0.001371)	<b>0.018185</b> <b>(0.001116)</b>
WFG3 (4)	0.184779 (0.002217)	<b>0.196333</b> <b>(0.000854)</b>	0.196211 (0.000880)
WFG4 (4)	269.982519 (2.836349)	<b>302.620739</b> <b>(0.707563)</b>	301.375978 (1.112941)
WFG5 (4)	18.754249 (0.209889)	20.359809 (0.019662)	<b>20.363936</b> <b>(0.024180)</b>
WFG6 (4)	0.301178 (0.005629)	<b>0.325112</b> <b>(0.002245)</b>	0.324303 (0.002332)
WFG1 (5)	86.060111 (1.227076)	91.287290 (1.005588)	<b>104.028046</b> <b>(0.528738)</b>
WFG2 (5)	0.001986 (0.000169)	0.002113 (0.000177)	<b>0.002180</b> <b>(0.000183)</b>
WFG3 (5)	0.032774 (0.000767)	<b>0.037038</b> <b>(0.000336)</b>	0.036662 (0.000391)
WFG4 (5)	3014.723575 (42.408502)	3303.742068 (26.484787)	<b>3469.478161</b> <b>(18.564929)</b>
WFG5 (5)	20.172967 (0.309907)	<b>23.177534</b> <b>(0.029596)</b>	23.174481 (0.027510)
WFG6 (5)	0.154873 (0.002405)	<b>0.169363</b> <b>(0.001220)</b>	0.169207 (0.001146)
WFG1 (6)	14.193726 (0.235245)	8.227708 (0.451872)	<b>16.538709</b> <b>(0.144086)</b>
WFG2 (6)	0.000418 (0.000036)	<b>0.000447</b> <b>(0.000037)</b>	0.000444 (0.000033)
WFG3 (6)	0.004292 (0.000157)	<b>0.005211</b> <b>(0.000069)</b>	0.005176 (0.000094)
WFG4 (6)	40122.027459 (718.269874)	33496.820458 (948.442179)	<b>46450.488116</b> <b>(278.139580)</b>
WFG5 (6)	14.877233 (0.370266)	<b>17.972798</b> <b>(0.019577)</b>	17.970919 (0.021764)
WFG6 (6)	0.031139 (0.000644)	0.034996 (0.000433)	<b>0.035027</b> <b>(0.000475)</b>

TABLE II. RESULTS OBTAINED IN THE WFG TEST PROBLEMS. WE SHOW AVERAGE VALUES OVER 30 INDEPENDENT RUNS. THE VALUES IN PARENTHESES CORRESPOND TO THE STANDARD DEVIATIONS.

mechanisms based on this same performance measure in terms of the quality of the approximation that it generates. The proposed approach is also significantly less computationally expensive than the one incorporated in SMS-EMOA as indicated by our preliminary results. It is also worth emphasizing that our proposed approach is very simple, and is based on the sole use of the hypervolume indicator (Pareto ranking is not adopted, as happens in SMS-EMOA). Our proposed approach is designed in such a way that it explores more regions of the Pareto front at the beginning of the search and exploits the locality property of the hypervolume during the entire search process.

As part of our future work, we intend to study other mechanisms to identify the regions in which there are many solutions, and then use this information to improve the success of our approach when we choose a random individual. We also plan to incorporate into our approach other mechanism to identify the stage of the search at which the use of a random solution is no longer required. We believe that this mechanism could improve the convergence

$\bar{f}$	iSMS-EMOA & appSMS-EMOA $P(H)$	iSMS-EMOA & SMS-EMOA $P(H)$
DTLZ1 (3)	0.597312 (0)	0.442896 (0)
DTLZ2 (3)	0.724692 (0)	0.909853 (0)
DTLZ3 (3)	0.002973 (1)	0.279387 (0)
DTLZ4 (3)	0.074090 (0)	0.705912 (0)
DTLZ5 (3)	0.880027 (0)	0.724682 (0)
DTLZ6 (3)	0.279387 (0)	0.724703 (0)
DTLZ7 (3)	0.247205 (0)	0.949853 (0)
DTLZ1 (4)	0.959873 (0)	0.949846 (0)
DTLZ2 (4)	0.705924 (0)	0.979931 (0)
DTLZ3 (4)	0.007068 (1)	0.782000 (0)
DTLZ4 (4)	0.668882 (0)	0.696553 (0)
DTLZ5 (4)	0.450446 (0)	0.969899 (0)
DTLZ6 (4)	0.392388 (0)	0.687322 (0)
DTLZ7 (4)	0.939843 (0)	0.762752 (0)
DTLZ1 (5)	0.705867 (0)	0.420716 (0)
DTLZ2 (5)	0.268358 (0)	0.497008 (0)
DTLZ3 (5)	0.002531 (1)	0.039134 (1)
DTLZ4 (5)	0.668908 (0)	0.801383 (0)
DTLZ5 (5)	0.392388 (0)	0.899907 (0)
DTLZ6 (5)	0.151591 (0)	0.762752 (0)
DTLZ7 (5)	0.513063 (0)	0.919850 (0)
DTLZ1 (6)	0.959871 (0)	0.007630 (1)
DTLZ2 (6)	0.919850 (0)	0.406460 (0)
DTLZ3 (6)	0.016765 (1)	0.002717 (1)
DTLZ4 (6)	0.579956 (0)	0.979930 (0)
DTLZ5 (6)	0.743649 (0)	0.705924 (0)
DTLZ6 (6)	0.020603 (1)	0.002973 (1)
DTLZ7 (6)	0.151591 (0)	0.860223 (0)
WFG1 (3)	0.182448 (0)	0.919850 (0)
WFG2 (3)	0.450446 (0)	0.782000 (0)
WFG3 (3)	0.899907 (0)	0.939843 (0)
WFG4 (3)	0.247205 (0)	0.801383 (0)
WFG5 (3)	0.899907 (0)	0.840506 (0)
WFG6 (3)	0.450446 (0)	0.687322 (0)
WFG1 (4)	0.919850 (0)	0.840506 (0)
WFG2 (4)	0.899907 (0)	0.959871 (0)
WFG3 (4)	0.420829 (0)	0.724703 (0)
WFG4 (4)	0.290721 (0)	0.435492 (0)
WFG5 (4)	0.227249 (0)	0.217713 (0)
WFG6 (4)	0.959873 (0)	0.919847 (0)
WFG1 (5)	0.705924 (0)	0.880027 (0)
WFG2 (5)	0.743649 (0)	0.939838 (0)
WFG3 (5)	0.959874 (0)	0.668908 (0)
WFG4 (5)	0.579972 (0)	0.687322 (0)
WFG5 (5)	0.939843 (0)	1.000000 (0)
WFG6 (5)	0.959874 (0)	0.860223 (0)
WFG1 (6)	0.579972 (0)	0.137759 (0)
WFG2 (6)	0.919843 (0)	0.919801 (0)
WFG3 (6)	0.650690 (0)	0.959874 (0)
WFG4 (6)	0.724703 (0)	0.860223 (0)
WFG5 (6)	0.113009 (0)	0.959874 (0)
WFG6 (6)	0.919850 (0)	0.860223 (0)

TABLE III. RESULTS OF STATISTICAL ANALYSIS APPLIED TO OUR EXPERIMENTS.  $P$  IS THE PROBABILITY OF OBSERVING THE GIVEN RESULT (THE NULL HYPOTHESIS IS TRUE). SMALL VALUES OF  $P$  CAST DOUBT ON THE VALIDITY OF THE NULL HYPOTHESIS.  $H = 0$  INDICATES THAT THE NULL HYPOTHESIS ("MEDIAN ARE EQUAL") CANNOT BE REJECTED AT THE 5% LEVEL.  $H = 1$  INDICATES THAT THE NULL HYPOTHESIS CAN BE REJECTED AT THE 5% LEVEL.

rate of our approach, while also decreasing the number of hypervolume computations that it performs. Finally, it would also be interesting to analyze the behavior of our selection mechanism when adopting an approximate method to compute the hypervolume, with the aim of producing high quality approximations at a very low computational cost.

## REFERENCES

- [1] C. A. Coello Coello, G. B. Lamont, and D. A. Van Veldhuizen, *Evolutionary Algorithms for Solving Multi-Objective Problems*. New York: Springer, second ed., September 2007. ISBN 978-0-387-33254-3.

$f$	appSMS-EMOA	SMS-EMOA	iSMS-EMOA
DTLZ1 (3)	≈ 2 m	≈ 3 m	≈ 3 m
DTLZ2 (3)	≈ 3 m	≈ 5 m	≈ 3 m
DTLZ3 (3)	≈ 5 m	≈ 8 m	≈ 5 m
DTLZ4 (3)	≈ 3 m	≈ 5 m	≈ 3 m
DTLZ5 (3)	≈ 3 m	≈ 3 m	≈ 3 m
DTLZ6 (3)	≈ 3 m	≈ 4 m	≈ 3 m
DTLZ7 (3)	≈ 4 m	≈ 4 m	≈ 3 m
DTLZ1 (4)	≈ 2 m	≈ 23 m	≈ 4 m
DTLZ2 (4)	≈ 3 m	≈ 41 m	≈ 4 m
DTLZ3 (4)	≈ 5 m	≈ 100 m	≈ 6 m
DTLZ4 (4)	≈ 3 m	≈ 42 m	≈ 4 m
DTLZ5 (4)	≈ 4 m	≈ 27 m	≈ 3 m
DTLZ6 (4)	≈ 4 m	≈ 34 m	≈ 4 m
DTLZ7 (4)	≈ 4 m	≈ 22 m	≈ 3 m
DTLZ1 (5)	≈ 2 m	≈ 4 h	≈ 26 m
DTLZ2 (5)	≈ 3 m	≈ 4 h	≈ 28 m
DTLZ3 (5)	≈ 5 m	≈ 4 h	≈ 55 m
DTLZ4 (5)	≈ 3 m	≈ 4 h	≈ 29 m
DTLZ5 (5)	≈ 4 m	≈ 4 h	≈ 26 m
DTLZ6 (5)	≈ 4 m	≈ 4 h	≈ 25 m
DTLZ7 (5)	≈ 5 m	≈ 213 m	≈ 9 m
DTLZ1 (6)	≈ 2 m	≈ 4 h	≈ 4 h
DTLZ2 (6)	≈ 3 m	≈ 4 h	≈ 4 h
DTLZ3 (6)	≈ 5 m	≈ 4 h	≈ 236 m
DTLZ4 (6)	≈ 3 m	≈ 4 h	≈ 4 h
DTLZ5 (6)	≈ 5 m	≈ 4 h	≈ 198 m
DTLZ6 (6)	≈ 5 m	≈ 4 h	≈ 169 m
DTLZ7 (6)	≈ 5 m	≈ 4 h	≈ 92 m
WFG1 (3)	≈ 5 m	≈ 6 m	≈ 3 m
WFG2 (3)	≈ 4 m	≈ 4 m	≈ 4 m
WFG3 (3)	≈ 5 m	≈ 5 m	≈ 4 m
WFG4 (3)	≈ 4 m	≈ 5 m	≈ 3 m
WFG5 (3)	≈ 5 m	≈ 5 m	≈ 4 m
WFG6 (3)	≈ 6 m	≈ 5 m	≈ 4 m
WFG1 (4)	≈ 6 m	≈ 56 m	≈ 4 m
WFG2 (4)	≈ 5 m	≈ 12 m	≈ 2 m
WFG3 (4)	≈ 6 m	≈ 12 m	≈ 2 m
WFG4 (4)	≈ 4 m	≈ 45 m	≈ 3 m
WFG5 (4)	≈ 6 m	≈ 17 m	≈ 1 m
WFG6 (4)	≈ 7 m	≈ 15 m	≈ 1 m
WFG1 (5)	≈ 6 m	≈ 241 m	≈ 34 m
WFG2 (5)	≈ 5 m	≈ 39 m	≈ 3 m
WFG3 (5)	≈ 6 m	≈ 22 m	≈ 2 m
WFG4 (5)	≈ 4 m	≈ 4 h	≈ 27 m
WFG5 (5)	≈ 7 m	≈ 43 m	≈ 2 m
WFG6 (5)	≈ 7 m	≈ 43 m	≈ 2 m
WFG1 (6)	≈ 6 m	≈ 4 h	≈ 4 h
WFG2 (6)	≈ 6 m	≈ 88 m	≈ 5 m
WFG3 (6)	≈ 7 m	≈ 43 m	≈ 2 m
WFG4 (6)	≈ 5 m	≈ 4 h	≈ 4 h
WFG5 (6)	≈ 8 m	≈ 72 m	≈ 3 m
WFG6 (6)	≈ 8 m	≈ 84 m	≈ 4 m

TABLE IV. TIME REQUIRED BY APPSMS-EMOA, SMS-EMOA AND iSMS-EMOA PER RUN, FOR THE TEST PROBLEMS ADOPTED.  $m$  = MINUTES, AND  $h$  = HOURS. THREE ALGORITHMS WERE COMPILED USING GNU C COMPILER AND THEY WERE EXECUTED ON A COMPUTER WITH A 2.66GHZ PROCESSOR AND 4GB IN RAM.

- [2] E. Zitzler, L. Thiele, M. Laumanns, C. M. Fonseca, and V. G. da Fonseca, "Performance Assessment of Multiobjective Optimizers: An Analysis and Review," *IEEE Transactions on Evolutionary Computation*, vol. 7, pp. 117–132, April 2003.
- [3] E. Zitzler and L. Thiele, "Multiobjective Optimization Using Evolutionary Algorithms—A Comparative Study," in *Parallel Problem Solving from Nature V* (A. E. Eiben, ed.), (Amsterdam), pp. 292–301, Springer-Verlag, September 1998.
- [4] M. Fleischer, "The Measure of Pareto Optima. Applications to Multi-objective Metaheuristics," in *EMO 2003* (C. M. Fonseca, P. J. Fleming, E. Zitzler, K. Deb, and L. Thiele, eds.), (Faro, Portugal), pp. 519–533, Springer. Lecture Notes in Computer Science. Volume 2632, April 2003.
- [5] M. Farina and P. Amato, "On the Optimal Solution Definition for Many-criteria Optimization Problems," in *Proceedings of the NAFIPS-FLINT International Conference'2002*, (Piscataway, New Jersey), pp. 233–238, IEEE Service Center, June 2002.
- [6] D. Brockhoff and E. Zitzler, "Improving Hypervolume-based Multiobjective Evolutionary Algorithms by Using Objective Reduction Methods," in *2007 IEEE Congress on Evolutionary Computation (CEC'2007)*, (Singapore), pp. 2086–2093, IEEE Press, September 2007.
- [7] L. Bradstreet, L. Barone, and L. While, "Updating Exclusive Hypervolume Contributions Cheaply," in *CEC'2009*, (Trondheim, Norway), pp. 538–544, IEEE Press, May 2009.
- [8] M. T. Emmerich and C. M. Fonseca, "Computing Hypervolume Contributions in Low Dimensions: Asymptotically Optimal Algorithm and Complexity Results," in *Evolutionary Multi-Criterion Optimization, 6th International Conference, EMO 2011* (R. H. Takahashi, K. Deb, E. F. Wanner, and S. Grecco, eds.), (Ouro Preto, Brazil), pp. 121–135, Springer. Lecture Notes in Computer Science Vol. 6576, April 2011.
- [9] H. Ishibuchi, N. Tsukamoto, Y. Sakane, and Y. Nojima, "Indicator-Based Evolutionary Algorithm with Hypervolume Approximation by Achievement Scalarizing Functions," in *Proceedings of the 12th annual conference on Genetic and Evolutionary Computation (GECCO'2010)*, (Portland, Oregon, USA), pp. 527–534, ACM Press, July 7–11 2010. ISBN 978-1-4503-0072-8.
- [10] J. Bader and E. Zitzler, "HypE: An Algorithm for Fast Hypervolume-Based Many-Objective Optimization," *Evolutionary Computation*, vol. 19, no. 1, pp. 45–76, Spring, 2011.
- [11] K. Bringmann and T. Friedrich, "Convergence of Hypervolume-Based Archiving Algorithms II: Competitiveness," in *GECCO'2012*, (Philadelphia, USA), pp. 457–464, ACM Press, July 2012. ISBN: 978-1-4503-1177-9.
- [12] J. Knowles and D. Corne, "Properties of an Adaptive Archiving Algorithm for Storing Nondominated Vectors," *IEEE Transactions on Evolutionary Computation*, vol. 7, pp. 100–116, April 2003.
- [13] E. Zitzler and S. Künzli, "Indicator-based Selection in Multiobjective Search," in *Parallel Problem Solving from Nature - PPSN VIII* (X. Y. et al., ed.), (Birmingham, UK), pp. 832–842, Springer-Verlag. Lecture Notes in Computer Science Vol. 3242, September 2004.
- [14] M. Emmerich, N. Beume, and B. Naujoks, "An EMO Algorithm Using the Hypervolume Measure as Selection Criterion," in *EMO 2005* (C. A. Coello Coello, A. Hernández Aguirre, and E. Zitzler, eds.), (Guanajuato, México), pp. 62–76, Springer. Lecture Notes in Computer Science Vol. 3410, March 2005.
- [15] N. Beume, B. Naujoks, and M. Emmerich, "SMS-EMOA: Multiobjective selection based on dominated hypervolume," *European Journal of Operational Research*, vol. 181, pp. 1653–1669, 16 September 2007.
- [16] C. Igel, N. Hansen, and S. Roth, "Covariance Matrix Adaptation for Multi-objective Optimization," *Evolutionary Computation*, vol. 15, pp. 1–28, Spring 2007.
- [17] S. Mostaghim, J. Branke, and H. Schmeck, "Multi-Objective Particle Swarm Optimization on Computer Grids," in *2007 Genetic and Evolutionary Computation Conference (GECCO'2007)* (D. Thierens, ed.), vol. 1, (London, UK), pp. 869–875, ACM Press, July 2007.
- [18] A. Auger, J. Bader, D. Brockhoff, and E. Zitzler, "Theory of the Hypervolume Indicator: Optimal  $\{\mu\}$ -Distributions and the Choice Of The Reference Point," in *FOGA '09: Proceedings of the tenth ACM SIGEVO workshop on Foundations of genetic algorithms*, (Orlando, Florida, USA), pp. 87–102, ACM, January 2009.
- [19] A. Auger, J. Bader, and D. Brockhoff, "Theoretically Investigating Optimal  $\mu$ -Distributions for the Hypervolume Indicator: First Results for Three Objectives," in *Parallel Problem Solving from Nature-PPSN XI, 11th International Conference, Proceedings, Part I* (R. Schaefer, C. Cotta, J. Kołodziej, and G. Rudolph, eds.), pp. 586–596, Kraków, Poland: Springer, Lecture Notes in Computer Science Vol. 6238, September 2010.
- [20] S. Huband, P. Hingston, L. Barone, and L. While, "A Review of Multiobjective Test Problems and a Scalable Test Problem Toolkit," *IEEE Transactions on Evolutionary Computation*, vol. 10, pp. 477–506, October 2006.
- [21] K. Deb, L. Thiele, M. Laumanns, and E. Zitzler, "Scalable Test Problems for Evolutionary Multiobjective Optimization," in *Evolutionary Multiobjective Optimization. Theoretical Advances and Applications* (A. Abraham, L. Jain, and R. Goldberg, eds.), pp. 105–145, USA: Springer, 2005.