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Inferring preferences for multi-criteria ordinal classification methods using evolutionary algorithms

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ABSTRACT The problem of assigning actions described by multiple criteria to a set of ordered classes is increasingly mentioned in the literature. Recently, the authors presented two new outranking-based methods able to integrate the treatment of multiple types of uncertainty when addressing such problem. But, as it is well-known, there usually is difficulty when assigning parameter values to methods based on the outranking approach. This difficulty raises not only on the high number of parameters and the decision maker's unfamiliarity to them, but also on imperfectly known (even missing) preference parameter values and criterion values. Therefore, here we address: i) how to elicit the parameter values of the two new methods, and ii) how to incorporate imperfect knowledge during the elicitation. We follow the preference disaggregation paradigm and use evolutionary algorithms to solve it. Our proposal performs very well in a wide range of computer experiments. Some interesting findings are: i) the capacity of the method to restore the assignment examples reaches high values with three profiles in each limiting boundary and the same number of representative actions per class; and ii) the capacity to make appropriate assignments of new actions (not belonging to the training information) can be greatly improved by increasing the number of limiting profiles.

INDEX TERMS Evolutionary algorithms; imperfect information, multiple criteria analysis, multiple criteria ordinal classification, outranking methods.

I. INTRODUCTION

Among the different types of problems addressed by the Multiple-criteria decision analysis (MCDA) approaches, the multiple-criteria ordinal classification, or sorting problem, has received a great interest lately given its interesting theoretical challenges and its applicability in real scenarios. In multiple-criteria ordinal classification, a set of decision alternatives (objects of decision, actions) have to be assigned to a set of classes. These classes, also referred to as categories in the related literature, have been predefined and preferentially ordered by the decision maker (DM). In this paper, we are interested in multi-criteria ordinal classification methods inspired on the outranking approach. In the most popular

methods, the definition of each class can be made through a reference decision action (profile) that can be used as a characteristic action to represent the class as in ELECTRE TRI-C [1] or as a limiting boundary that separates a pair of classes as in ELECTRE TRI-B [2]. Then, in order to perform the assignment of new actions, both the profiles and the actions-to-be-assigned are evaluated by the DM based on a set of conflicting criteria. With the aim to provide a better description of the classes, ELECTRE TRI-C, (respectively ELECTRE TRI-B), was extended to ELECTRE TRI-nC, (resp. ELECTRE TRI-nB), in [3] (resp. [4]); these extensions use a set of profiles in the definition of each class (resp.

boundary). This way, it is possible to consider more pieces of information regarding the relations between actions to be assigned and reference profiles in order to, potentially, provide better decision aid.

In ELECTRE methods, the elicitation of model's parameters is a real concern. When using a direct elicitation method, the DM, commonly aided by a decision analyst, must explicitly set the parameter values representing his/her preferences. Several authors (for example, [5]–[7]) have argued against the direct elicitation since: i) the DM may not be able to completely understand the meaning of the model's parameters; and ii) the DM may not be accessible to involve in a long and complex process of providing appropriate numerical values, which usually are very unfamiliar to her/him; iii) the DM may be a collective entity with conflicting values and ill-defined preferences. Alternatively, when using an indirect elicitation method, (the so-called preference disaggregation analysis), the DM typically uses his/her holistic judgments to provide/accept a set of reference examples inherently containing the DM's preferences; thus, through a regression-inspired procedure, a process of extraction must be performed in order to infer the parameter values underlying the preferences contained in the reference examples. Under some strong simplifications, the extraction of outranking model parameters using an indirect elicitation method can be addressed through classical mathematical programming techniques as in [8]. But such an indirect parameter elicitation becomes a very complex optimization problem when veto thresholds should be inferred. In such cases, evolutionary algorithms should be used as in [7], [9], [10]. Less sophisticated metaheuristic approaches may be used when the preference model does not include veto, as in [11]. Fernández et al. [12] proposed an evolutionary algorithm to infer the whole set of ELECTRE TRI-nB model parameters. To our present knowledge, there is no indirect parameter elicitation method for ELECTRE TRI-nC.

As stated in [13], indirect elicitation methods are generally attractive for the DM, but, to a great extent, their performance is degraded when there is scarce information about the DM's preferences (a relatively small reference set of decision examples): in this case, the indirect elicitation methods often suggest many solutions in the parameter space [14]. All these distributed solutions satisfy the known preferences of the DM. This is imprecise information that should be modeled in an appropriate way.

Thus, imprecise (maybe arbitrary) setting of the outranking model's parameters may be a result of either a direct or indirect elicitation process. For a better model of human hesitancy, many extensions of outranking methods have been proposed that use fuzzy-based approaches (e.g., intuitionistic fuzzy sets, hesitant fuzzy sets, interval-valued fuzzy numbers, etc.) [15]–[25]. It should be underlined that all these fuzzy-based extensions of outranking methods are devoted to solve choosing and ranking decision problems. To the best of our

knowledge, there is no fuzzy extension of the ELECTRE TRI-nC and ELECTRE TRI-nB methods.

In order to deal with imprecise information in model parameters and criterion scores, Fernández *et al.* [26] recently developed two multi-criteria ordinal classification approaches, INTERCLASS-nB and INTERCLASS-nC, which extend ELECTRE TRI-nB and ELECTRE TRI-nC to the interval framework. As Fernández *et al.* [26] argue, there are situations where imperfect knowledge about the parameter values may be characterized in a natural way by interval numbers, which are representations of magnitudes of unknown precise values. Since setting a parameter value as an interval number is easier than as a precise value, INTERCLASS-nB and INTERCLASS-nC reduce the difficulty of direct elicitation of model parameters. However, when the DM is a mythical entity (e.g., the public opinion), or an inaccessible person (e.g., the CEO of a multinational enterprise), INTERCLASS-nB and INTERCLASS-nC should be complemented by an indirect elicitation procedure, which, (learning from decision examples), permits to set the model's parameters as interval numbers.

We present in this paper an evolutionary-based approach to infer the parameters of both the INTERCLASS-nB and INTERCLASS-nC methods. Some parameters may be directly set as interval numbers, whereas others may be inferred. Thus, this paper combines two ways to reduce the difficulty of parameter elicitation. Our approach is extensively assessed, in-sample and out-of-sample, in its ability to restore the assignment examples and the capacity to make consistent assignments of new actions.

The novelty of this proposal rests on the following bases:

- This is the first approach to infer preference model parameters of a multi-criteria ordinal classification method in which classes are described by representative actions, as in ELECTRE TRI-nC;
- Up to now, there were two alternative ways to face the difficulty of eliciting the preference model parameters: modeling imprecision and inferring parameters from decision examples. This paper presents the first approach in which both ways are combined;
- INTERCLASS-nB and INTERCLASS-nC are here fully characterized from a preference disaggregation paradigm, and the effectiveness of both methods are compared in a wide range of problems; interesting conclusions follow from such a comparison.

Our results answer the following questions:

- Concerning INTERCLASS-nB (respectively INTERCLASS-nC), which is the appropriate number of limiting profiles (respectively, characteristic or representative actions) to achieve a good characterization of the limiting boundary between adjacent classes (resp. the related class)?

- How much is the effectiveness of the inference approach depending on the number of criteria and classes?
- How much is the effectiveness depending on the number of assignment examples?

The first question was kept open in [26]. Since finding formal theoretical answers to the above questions may be impossible, we perform a simulation experiment in which a wide range of DM preferences are considered, and the effectiveness of the inference method is characterized for different instances of classes, criteria, and number of assignment examples.

The rest of this document is presented as follows. In Section II, we make a brief description of the INTERCLASS-nB and INTERCLASS-nC methods as well as the interval outranking approach, a fundamental component of these methods. Section III presents our main proposals about how to infer the parameter values of both methods. In Section IV, we describe an extensive computer experiment and its results assessing our elicitation methods. Finally, Section V concludes this paper.

II. BACKGROUND

A. FUNDAMENTAL NOTIONS ON INTERVAL NUMBERS

The main concept of Interval Analysis Theory [27], [28] is the so-called interval number. Let us now present a description of such a concept.

An interval number describes a not-necessarily-defined quantity whose actual value lies within a range of real numbers, $I = [I^-, I^+]$. The limits of this range, $I^-, I^+ \in \mathbb{R}$, are known. Thus, by definition, a real number r can be represented by the interval number $R = [r^-, r^+]$, where $r = r^- = r^+$. Furthermore, any real number $i \in I$ is called a realization of the interval number I . In order to state clearer definitions, in the rest of this document, interval numbers will be denoted by boldface italic letters.

In order to estimate the credibility degree of an interval number $I = [i^-, i^+]$ being greater than or equal to another interval number $J = [j^-, j^+]$, the following possibility function defined in [29] is used by [30]:

$$p(I \geq J) = \begin{cases} 1 & \text{if } p_{\{IJ\}} > 1, \\ 0 & \text{if } p_{\{IJ\}} < 0, \\ p_{\{IJ\}} & \text{otherwise.} \end{cases} \quad (1)$$

Where $p_{\{IJ\}} = \frac{i^+ - j^-}{(i^+ - i^-) + (j^+ - j^-)}$.

Furthermore, if $i^+ = i^-$ and $j^+ = j^-$, then

$$p(I \geq J) = \begin{cases} 1 & \text{if } i^- \geq j^-, \\ 0 & \text{otherwise.} \end{cases}$$

The possibility function defined in (1) indicates that $p(I \geq J)$ is the credibility degree of the assertion “given that both realizations are established, $i \in I$ is not less than $j \in J$ ”. Thus, the possibility function denotes robustness of $I \geq J$, even when these quantities are undetermined.

B. INTERVAL-BASED OUTRANKING APPROACH

Fernández *et al.* [30] proposed an extension of the outranking approach whose main characteristic is its capacity to deal with the imperfect knowledge involved in the decision maker’s preferences and in the actions’ impacts on the criteria. These types of imperfect information can be modeled in such extension using both interval numbers and the traditional pseudo-criteria based on discriminating thresholds (e.g., [31], [32]).

The formal definition of the interval outranking approach depends on the following notation. Let A be a set of potential actions. Each $x \in A$ is evaluated on a family of N coherent criteria (as in the sense of [33]) G , which, without loss of generality, increase with preference. Now, assume that $G_1 \subseteq G$ is the set of criteria whose imperfect knowledge can be modeled by using discriminating thresholds as it is traditionally done by later ELECTRE methods. And that $G_2 \subseteq G$ is the set of criteria whose imperfect knowledge can be modeled using interval numbers; that is, each $g_j \in G_2$ is an interval number of the form $g_j(x) = [g_j^-(x), g_j^+(x)]$. The interval outranking approach requires the assignment of appropriate values to the following parameters to satisfactorily reflect the DM’s preferences:

- $w_j = [w_j^-, w_j^+]$, the weight of criterion g_j ;
- $v_j = [v_j^-, v_j^+]$, the veto threshold of criterion g_j ; and
- $\lambda = [\lambda^-, \lambda^+]$, that reflects a majority threshold.

Where $j = 1, \dots, N$. Furthermore, it is a straightforward work for the DM to assign values to the preference threshold $p_j(\cdot)$, and the indifference threshold, $q_j(\cdot)$; $p_j(\cdot) \geq q_j(\cdot) \geq 0$. As in the classical outranking approach, the interval outranking approach estimates a credibility index, $\eta(x, y) \in [0, 1]$, between pairs of actions of the assertion “ x is at least as good as y ”, xSy . The detailed procedure used by the interval-based outranking approach to estimate this credibility index is described in Appendix C.

The approach assumes that the DM uses a credibility threshold $\delta > 0.5$ such that if $\eta(x, y) \geq \delta$ then the assertion “ x is at least as good as y ” is accepted. Using this threshold, the following relations are defined.

Definition 1. δ - λ -relations

$xS(\delta, \lambda)y \Leftrightarrow \eta(x, y) \geq \delta$ is called δ - λ -interval outranking relation,

$xP(\delta, \lambda)y \Leftrightarrow \eta(x, y) \geq \delta$ and $\eta(y, x) < \delta$ is called δ - λ -interval preference relation,

$xI(\delta, \lambda)y \Leftrightarrow \eta(x, y) \geq \delta$ and $\eta(y, x) \geq \delta$ is called δ - λ -interval indifference relation,

$xR(\delta, \lambda)y \Leftrightarrow \eta(x, y) < \delta$ and $\eta(y, x) < \delta$ is called integrated δ - λ -interval incomparability relation.

The concept of dominance is also extended in [30]. In that work, dominance is not crisp, but there is a “degree of credibility”, α , of the dominance.

Definition 2. Extended dominance

Let $x \neq y$ be two actions and $\alpha \in \mathbb{R}$; then y is α -dominated by x , denoted by $xD(\alpha)y$, if and only if the following conditions are fulfilled:

- i. $g_j(x) \geq g_j(y)$, for all $g_j \in G_I$,
- ii. $\min_{g_j \in G_2} p(g_j(x) \geq g_j(y)) \geq \alpha \geq 0.5$.

C. THE INTERCLASS-NB METHOD

Fernández *et al.* [30] proposed an extension of the outranking approach whose main characteristic is its capacity to deal with the imperfect knowledge involved in the decision maker's preferences and in the actions' impacts on the criteria.

Condition 1.

Let C be a finite set of classes $C = \{C_1, \dots, C_k, \dots, C_M\}$, $M \geq 2$, ordered in increasing preference. In INTERCLASS-nB, the boundaries between categories C_k and C_{k+1} are described by a set of limiting profiles, $B_k = \{b_{k,j}\}$, such that for given $\delta > 0.5$ and $\lambda > [0.5, 0.5]$ the following conditions are fulfilled:

- i. C_k is defined through a set of reference upper limiting profiles, B_k , and through a set of reference lower limiting profiles, B_{k-1} . It is assumed that all $b_{k,j}$ of B_k are in C_{k+1} (that is, all classes are closed from below);
- ii. B_0 (respectively, B_M) is composed of the anti-ideal (respectively, the ideal) action;
- iii. For all k , there is no pair $(b_{k,j}, b_{k,i})$ such that $b_{k,j}P(\delta, \lambda)b_{k,i}$;
- iv. For all k and h such that $k < h$, there is no pair $(b_{k,j}, b_{h,i})$ such that $b_{k,j}P(\delta, \lambda)b_{h,i}$;
- v. For all k and for each limiting action w in B_k , there exists at least one action z in B_{k+1} in such a way that $zD(\alpha)w$, $\alpha \geq \delta$;
- vi. For all k and for each limiting action w in B_{k+1} , there exists at least one z in B_k in such a way that $wD(\alpha)z$, $\alpha \geq \delta$.
- vii. For all k and for each limiting action w in B_k , exists at least one z in B_{k+1} in such a way that $zP(\delta, \lambda)w$, $\alpha \geq \delta$.

The following relations among profiles and decision actions are defined by INTERCLASS-nB:

- $xS(\delta, \lambda)B_k$ if and only if, for all $b_{k,j} \in B_k$, either $xR(\delta, \lambda)b_{k,j}$ or $xS(\delta, \lambda)b_{k,j}$ (the latter should hold for at least one $b_{k,j}$);

- $B_kP(\delta, \lambda)x$ if and only if, for all $b_{k,j} \in B_k$, $b_{k,j}R(\delta, \lambda)x$ or $b_{k,j}I(\delta, \lambda)x$ or $b_{k,j}P(\delta, \lambda)x$ (the latter should hold for at least one $b_{k,j}$).

The assignment procedures constituting the INTERCLASS-nB method are based on the following two logics:

Pseudo-conjunctive procedure

- i. Compare x to B_k for $k = M - 1, \dots, 0$ until the first value, k , such that $xS(\delta, \lambda)B_k$;
- ii. Assign x to class C_{k+1} .

Pseudo-disjunctive procedure

- i. Compare x to B_k for $k = 1, \dots, M$ until the first value, k , such that $B_kP(\delta, \lambda)x$;
- ii. Assign x to class C_k .

D. THE INTERCLASS-NB METHOD

We continue using the previous notation to present now a description of the INTERCLASS-nC method following [30].

In the INTERCLASS-nC method, the set of decision actions characterizing class C_k , $k = 1, \dots, M$, is denoted by $R_k = \{r_{k,j}; j = 1, \dots, \text{card}(R_k)\}$, where $\{R_0, R_1, \dots, R_M, R_{M+1}\}$ is the set of all the characterizing decision alternatives (R_0 , and R_{M+1} are composed of the anti-ideal and ideal actions, respectively). Assume a given $\delta > 0.5$.

Condition 2.

Each element in R_k must fulfill the following conditions:

- i. For all k and for each action w in R_k , there is at least one action z in R_{k+1} such that $zD(\alpha)w$ ($\alpha \geq \delta$).
- ii. For all k and for each action w in R_{k+1} , there is at least one action z in R_k such that $wD(\alpha)z$ ($\alpha \geq \delta$).
- iii. For all k and for each action w in R_{k+1} , there is no action z in R_k such that $zS(0.5, \lambda)w$.

The credibility index of the outranking relation of action x over the subset R_k is defined as follows:

$$\eta(\{x\}, R_k) = \max_{j=1, \dots, \text{card}(R_k)} \{\eta(x, r_{k,j})\}.$$

While the credibility index of the outranking relation of subset R_k over an action x is defined as follows:

$$\eta(R_k, \{x\}) = \max_{j=1, \dots, \text{card}(R_k)} \{\eta(r_{k,j}, x)\}.$$

Such credibility indices allow to build interval crisp outranking relations between decision actions and sets of characteristic actions as follows:

$$xS(\delta, \lambda)R_k \Leftrightarrow \eta(\{x\}, R_k) \geq \delta;$$

$$R_kS(\delta, \lambda)x \Leftrightarrow \eta(R_k, \{x\}) \geq \delta.$$

The selection function is defined as $i(\{x\}, R_k) = \min\{\eta(\{x\}, R_k), \eta(R_k, \{x\})\}$.

The assignments of alternatives to classes are performed in INTERCLASS-nC using two joint rules, called the descending rule and the ascending rule, which should be used conjointly, as in both ELECTRE TRI-C and ELECTRE TRI-nC. We now describe these rules.

Descending assignment rule

- i. Compare x to R_k for $k = M, \dots, 0$, until the first value, k , such that $xS(\delta, \lambda)R_k$;
- ii. For $k = M$, select C_M as a possible class to assign action x .
- iii. For $0 < k < M$, if $i(\{x\}, R_k) \geq i(\{x\}, R_{k+1})$, then select C_k as a possible class to assign x ; otherwise, select C_{k+1} .
- iv. For $k = 0$, select C_1 as a possible class to assign x .

Ascending assignment rule

- i. Compare x to R_k for $k = 1, \dots, M + 1$, until the first value, k , such that $R_kS(\delta, \lambda)x$;
- ii. For $k = 1$, select C_1 as a possible category to assign action x .

For $1 < k < M + 1$, if $i(\{x\}, R_k) \geq i(\{x\}, R_{k-1})$, then select C_k as a possible class to assign x ; otherwise, select C_{k-1} .

III. AN INDIRECT ELICITATION FOR THE PARAMETERS OF THE INTERCLASS-NB AND INTERCLASS-NC METHODS

This section details the main aspects of the approach to infer the parameters of both multi-criteria ordinal classification methods.

A. AN OPTIMIZATION-BASED INFERENCE METHOD OF THE INTERCLASS-NB METHOD

The credibility index of the outranking relation of x over y , $\eta(x, y)$ depends on the values assigned to the parameters of the interval outranking model, $\mathcal{P} = \{\mathbf{w}_1, \dots, \mathbf{w}_n, \mathbf{v}_1, \dots, \mathbf{v}_n, \lambda, \delta\}$; but setting a convenient set of parameters is not trivial. We present here a procedure where, using a set of assignment examples (reference set) provided by the DM, it is possible to assign appropriate values to the parameters of the interval outranking model to satisfactorily represent the DM's preferences. We use the notation of Section II to define such procedure.

Let T be a set of decision actions. We assume that each $x \in T$ is assigned by the DM to a class C_k , $C =$

$\{C_1, \dots, C_k, \dots, C_M\}$, $M \geq 2$. The classes in C are ordered in increasing preference. The assignments of alternatives to classes are holistic decisions made by the DM; thus, his/her multi-criteria preferences are reflected in them. We assume these decisions might be represented by an INTERCLASS-nB model, $\{\mathcal{P}, B_0, \dots, B_M\}$; that is, by the parameters of the interval outranking model, \mathcal{P} , and $M + 1$ sets of limiting profiles. Given that B_0 and B_M are composed, respectively, of the anti-ideal and the ideal actions, we are interested in finding only an approximation to the set of actual preference parameters, $nB_{DM} = \{\mathcal{P}, B_1, \dots, B_{M-1}\}$. Therefore, the inferred preference parameter set that is most appropriate to fit the assignments expressed by the DM, nB_{inf}^* , is the one that minimizes the number of inconsistencies with respect to the expressed preferences. Let

$$x \xrightarrow{nB_{DM}} C_k$$

denote that the DM has assigned x to class C_k ,

$$x \xrightarrow{nB_{inf}} C_k$$

denote that x is assigned to class C_k using the inferred decision model nB_{inf} , and ξ_{nB} be the set of models fulfilling Condition 1 and any constraints established by the DM. The optimization problem of minimizing the number of inconsistencies between nB_{DM} and a given nB_{inf} is equivalent to maximizing the following effectiveness measure:

$$\max_{nB_{inf} \in \xi_{nB}} 1 - \frac{NI(nB_{DM}, nB_{inf})}{\text{card}(T)} \quad (2)$$

where

$$NI(nB_{DM}, nB_{inf}) = \sum_{x \in T} NI(x, nB_{DM}, nB_{inf}), \text{ and } NI(x, nB_{DM}, nB_{inf}) = \begin{cases} 1 & \text{if } x \xrightarrow{nB_{DM}} C_k \text{ and } x \xrightarrow{nB_{inf}} C_h \text{ with } k \neq h, \\ 0 & \text{otherwise.} \end{cases}$$

B. AN OPTIMIZATION-BASED INFERENCE METHOD OF THE INTERCLASS-NC METHOD

In a similar order of the ideas presented in Subsection III.B, we describe here an inference method to elicit the parameter values of the INTERCLASS-nC method. Such method also uses a set of assignment examples where the DM assigns actions to preferentially ordered classes. Let D be this set of actions, where each $x \in D$ is assigned by the DM to one element of the set of classes $C = \{C_1, \dots, C_k, \dots, C_M\}$ or to a range of classes when the ascending and descending assignments are not the same. Thus, our goal is to find a model of the DM's preferences by inferring a configuration of the INTERCLASS-nC method, $nC_{inf}^* = \{\mathcal{P}^*, R_1^*, \dots, R_M^*\}$, that fulfills Condition 2 and is as consistent as possible with the assignments made by the DM.

Nevertheless, defining a fitness function here is not as straightforward as in the previous section. This is because each x is not necessarily assigned to only one class but a set of classes. Thus, if χ_{DM} is the set of classes to which the DM has

v_1	\dots	v_N	w_1	\dots	w_N	$g_1(b_{1,1})$	\dots	$g_N(b_{1,1})$	\dots	$g_N(b_{1,J})$	\dots	$g_N(b_{M-1,J})$	λ
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FIGURE 1. Chromosome of individual representing nB_{inf}

v_1	\dots	v_N	w_1	\dots	w_N	$g_N(b_{1,1})$	\dots	$g_N(b_{1,1})$	\dots	$g_N(b_{M,1})$	\dots	$g_N(b_{M,O})$	λ
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FIGURE 2. Chromosome of individual representing nC_{inf}

assigned x and χ_{inf} is the set of inferred classes, if we define the accuracy as

$$Ac(x, nC_{DM}, nC_{inf}) = \begin{cases} 1 & \text{if } \chi_{DM} = \chi_{inf}, \\ 0 & \text{otherwise;} \end{cases}$$

then, we might be too pessimistic since only one misclassification (perhaps among many classes) would lead to a total error. On the other hand, if we define

$$Ac(x, nC_{DM}, nC_{inf}) = \begin{cases} 1 & \text{if } \chi_{DM} \cap \chi_{inf} \neq \emptyset, \\ 0 & \text{otherwise;} \end{cases}$$

then we might be too optimistic. Therefore, we use here the so-called F₁-score [34], defined through *precision*, P , and *recall*, R , as [35]: $F_1\text{-score} = 2PR/(P+R)$. We adapt it to define the following optimization problem (cf. [11]):

$$\underset{nC_{inf} \in \xi_{nc}}{\text{maximize}} \frac{Ac(nC_{DM}, nC_{inf})}{card(D)}, \quad (1)$$

where

$$Ac(nC_{DM}, nC_{inf}) = \sum_{x \in D} Ac(x, nC_{DM}, nC_{inf}), \text{ and}$$

$$Ac(x, nC_{DM}, nC_{inf}) = \frac{2|\chi_{DM} \cap \chi_{inf}|}{|\chi_{DM}| + |\chi_{inf}|}.$$

C. AN EVOLUTIONARY ALGORITHM FOR ADDRESSING PROBLEMS (2) AND (3)

Given the non-linearity of Problems (2) and (3) and the previous results published in several related research works (e.g., [7], [10], [12], [13]), we implement here a genetic algorithm to address these problems. Even when the main aspects of such algorithm are convenient for addressing both problems, there are some characteristics that are specific of each problem. Thus, we now describe the specific steps to be followed. As in Section II, we assume there are N criteria and M classes.

Specific steps for configuring nB_{inf} :

Individuals are represented by a real-valued vector composed of $K = N(2 + J(M - 1)) + 1$ genes as in Figure 1, where J is the number of profiles used to separate each pair of classes.

Specific steps for configuring nC_{inf} :

For nC_{inf} , individuals are represented by a real-valued vector composed of $N(2 + OM) + 1$ genes as in Figure 2, where O is the number of profiles used to characterize each class.

D. GENERAL STEPS OF THE EVOLUTIONARY ALGORITHM

Each population used in the algorithm contains ps individuals. The individuals in the initial population of the algorithm are randomly generated fulfilling the following constraints (the specific values are established by the DM/analyst pair):

$$v_{imin} \leq v_i \leq v_{imax}, \quad (4)$$

$$\sum w_i = [1, 1],$$

$$g_{jmin} \leq g_i \leq g_{jmax},$$

$$\lambda_{min} \leq \lambda \leq \lambda_{max}.$$

The generation of the weights could be in several ways. In the experiments below, we use the method presented in [36], where $N - 1$ numbers, u_1, \dots, u_{N-1} , are uniformly randomly generated in $(0,1)$; later, these numbers are ordered in ascending order to calculate N values as $\omega_1 = u_1 - 0$, $\omega_i = u_i - u_{i-1}$ ($i = 2, \dots, N - 1$) and $\omega_N = 1 - u_{N-1}$. Such method ensures $\sum_{i=1}^N \omega_i = 1$. Finally, the weights used in the integrated outranking approach could be defined as $w_i = [(1 - \bar{w}_i)\omega_i, (1 + \bar{w}_i)\omega_i]$; where \bar{w}_i is a value that copes with the imperfect knowledge in the DM's mind about the actual weight of the i th criterion.

In our algorithm, the selection of parents is by binary tournament, and we adopt one-point crossover. In order to keep consistency in the weights, we assume that all the genes corresponding to weights form an indivisible unit. Thus, there are $K - N$ possible crossover points as shown in Figure 3 for configuring nB_{inf} and $NOM + 1$ for configuring nC_{inf} as shown in Figure 4. Once the two selected parents are crossed in a randomly selected crossover point, one offspring individual is generated; such individual is then mutated with a given probability. The mutation of an individual consists in the random generation of each unity of genes fulfilling the constraints set (4). At each generation of the algorithm, ps (population size) offspring individuals are generated and (possibly) mutated. All the offspring and parent individuals are introduced within a pool from where $ps - 1$ individuals are randomly selected to form the population in the next generation of the algorithm. We perform elitism in one

		
v_1	...	v_N	w_1	...	w_N	$g_1(b_{1,1})$...	$g_N(b_{1,1})$...	$g_N(b_{M-1,1})$...	$g_N(b_{M-1,J})$	λ

FIGURE 3. Possible crossover points for configuring nB_{model}

		
v_1	...	v_N	w_1	...	w_N	$g_1(b_{1,1})$...	$\textcolor{red}{g}_N(b_{1,1})$...	$\textcolor{red}{g}_N(b_{M,1})$...	$\textcolor{red}{g}_N(b_{M,O})$	λ

FIGURE 4. Possible crossover points for configuring nC_{model}

individual per generation. The fitness of each individual is assessed from the objectives in Problem (2) or Problem (3), as corresponds. After a given number of generations, the algorithm returns the chromosome that represents the feasible solutions with the best fitness values in the population; let these solutions form a set called $best_{known}$. The chromosome representing $best_{known}$ is obtained as the centroid (average of the parameters) of individuals within $best_{known}$. If the centroid reaches the best fitness value already known, it is considered as the best solution. If not, then the solution in $best_{known}$ closest to the centroid is considered as the best solution of the preliminary run. Such distance is calculated as the normalized Euclidean distance of the parameters' central values. In order to reduce the effects of randomness, we perform twenty consecutive preliminary runs. From the second run, we include the best solution of the previous run in the initial population.

As one of the classical algorithm configurators of the related literature, we use ParamILS [37] to set our own algorithm's main parameters; that is, the population size, the number of generations, the crossover probability and the mutation probability. The values found by such configurator are, respectively, 200, 200, 60% and 2%. Thus, such values are used in the present work.

This procedure is formalized in Algorithm 1.

Algorithm 1. Genetic Algorithm proposed to address Problems (2) and (3).

Require: A set of reference examples, T

Ensure: ρ_{final} , individual representing the population with the best fitness values

```

1:  $i \leftarrow 1$ 
2:  $\rho \leftarrow null$ 
3:  $g \leftarrow 0$ 
4:  $P_g \leftarrow \text{create-Initial-Population}()$ 
{Evolving the solutions for 1000 generations}
5: for  $g < 1000$  do
6:    $H_g \leftarrow \text{create-Offspring}(P_g, \text{selection, crossover, mutation})$ 
7:    $P_{g+1} \leftarrow \text{generate-Population}(P_g \cup H_g)$ 

```

```

8:    $g \leftarrow g + 1$ 
9: end for
10:  $best_{known} \leftarrow \text{find-Best}(P_g)$ 
11:  $\rho \leftarrow \text{find-Centroid}(best_{known})$ 
12: if  $\rho$  is-best( $best_{known}$ )
13:    $\rho_{final} \leftarrow \rho$ 
14: else
15:    $\rho_{final} \leftarrow \text{find-closest}(\rho)$ 

```

IV. NUMERICAL EXPERIMENTS

In this section, we present the procedure used to assess our inference approach. Such assessment intends to demonstrate our approach's ability to infer the parameter values of the INTERCLASS-nB and INTERCLASS-nC models by establishing its effectiveness to i) reproduce the DM's reference examples, and ii) appropriately make new assignments.

The procedure to assess our approach is, first, to simulate a decision maker who is compatible with the pseudo-conjunctive INTERCLASS-nB (respectively INTERCLASS-nC), and whose preference model parameters are known; second, by using the known model of preferences, to assign a set of reference actions to ordered classes; third, to exploit the evolutionary algorithm of Subsection III.C addressing Problems (2) or (3) in order to infer the parameter values of the pseudo-conjunctive INTERCLAS-nB or the INTERCLAS-nC methods; fourth, to obtain an in-sample effectiveness firstly using the inferred parameters to assign the reference actions to classes and, later, measuring the proportion of coincidences; fifth, to obtain an out-of-sample effectiveness by generating new actions and assigning them to classes using the known model parameters and the inferred parameters, finally measuring the proportion of coincidences.

A. EXPERIMENTAL INSTANCES

We create a set of experimental instances that would allow us to obtain sound conclusions. Each instance i used in the experiments below is constituted of a) an INTERCLASS-nB or INTERCLASS-nC model, b) a reference set T_i containing m_i assignment examples. So, each instance represents

different decision maker preferences. We use 20 instances ($i = 1, \dots, 20$) to determine the results shown below. Furthermore, we define a wide variety of values in the experiment's configuration as shown in Table 1.

TABLE 1. Configurations of the experiments

Aspect of the experiment's configuration	Notation	Values used
Number of criteria	N	3, 5, 7
Number of classes	M	2, 3, 5
Number of assignment examples per class	n_{class}	3, 5, 9, 12
Cardinality of the set of assignment examples	$card(T)$	$M \cdot n_{class}$
Number of limiting profiles per boundary	$card(B_k), k = 1, \dots, M-1$	1, 3, 5
Number of characterizing profiles per class	$card(R_k), k = 1, \dots, M$	1, 3, 5
Number of out-of-sample validation actions	n_{os}	400
Number of simulated decision-makers	n_{DMs}	20

In our experimental settings, the values of the model parameters, shown in the set of Equations (4), were randomly generated fulfilling:

$$[2,2] \leq v_i \leq [4,4], \quad (5)$$

$$\omega_i \leq w_i \leq \bar{\omega}_i,$$

$$[0.5,0.5] \leq g_j \leq [7.5,7.5],$$

$$[0.51,0.51] \leq \lambda \leq [0.66,0.66].$$

Where $\underline{\omega}_i = \left[\frac{1}{N} - \left(\frac{1}{N} \cdot 0.2 \right), \frac{1}{N} - \left(\frac{1}{N} \cdot 0.2 \right) \right]$ and $\bar{\omega}_i = \left[\frac{1}{N} + \left(\frac{1}{N} \cdot 0.2 \right), \frac{1}{N} + \left(\frac{1}{N} \cdot 0.2 \right) \right]$.

The definition of limiting profiles (respectively characteristic actions) must fulfill Condition 1 (resp. Condition 2).

B. ASSESSMENT PROCEDURE

We use the following assessment procedure:

1. Use different number of criteria (N), classes (M), assignment examples per class (n_{class}) and limiting profiles per boundary (for the INTERCLASS-nB, $card(B_k)$) or characterizing actions per class (for the INTERCLASS-nC, $card(R_k)$); namely, $N = 3, 5, 7$; $M = 2, 3, 5$; $n_{class} = 3, 5, 9, 12$; $card(B_k) = 1, 3, 5$; and $card(R_k) = 1, 3, 5$. For each of these, follow the next steps.
2. Use 20 instances and, for each instance, randomly generate an INTERCLASS-nB simulated DM model, nB_{DM} , with five limiting profiles in each boundary (excepting B_0 and B_M) or an INTERCLASS-nC simulated DM model, nC_{DM} , with ten characterizing profiles per class (excepting R_0 and R_{M+1}). There will be a total of 2,160 instances for each method.

3. Create reference decision alternatives x by randomly generating the values $g_i, i = 2, \dots, N$. Each g_j is randomly generated in $[0.5, 7.5]$.
4. Use the DM's set of model parameters (nB_{DM} or nC_{DM}) to create the set of assignment examples (T or D) by assigning the reference actions to classes. The assignment policy used by the nB_{DM} model is the pseudo-conjunctive procedure.
5. Obtain, using the approach of Section III, a set of parameters nB_{inf}^* or nC_{inf}^* as consistent as possible with the assignments made by the corresponding simulated DM model. The maxima consistency is identified with the optimal solution to Problem (2) or Problem (3) and the optimization is performed using Algorithm 1.
6. Assign the actions in T to classes according to nB_{inf}^* (using the pseudo-conjunctive procedure) or the actions in D to classes according to nC_{inf}^* . Determine the in-sample effectiveness of nB_{inf}^* using the accuracy measure presented in Problem (2). Similarly, the in-sample effectiveness of nC_{inf}^* is determined through Problem (3).
7. Create a new set of potential actions and assign them to classes using nB_{DM} and nC_{DM} , and later determine the approach's out-of-sample effectiveness similarly to step 6.

C. RESULTS

Results are presented per method (INTERCLASS-nB and INTERCLASS-nC) and per type of experiment (in-sample and out-of-sample). Since the out-of-sample results can be considered as more illustrative of the approach effectiveness, we present some graphs of these results in the main text; the graphs for in-sample results are shown in the appendices.

1) INTERCLASS-nB

It is important to keep in mind that the main objective of eliciting preference parameters is not to find the exact values of such parameters (if they even exist), but to determine those (not necessarily unique) values that reproduce the expressed preferences of the DM as well as possible.

We provide the results in terms of in-sample and out-of-sample effectiveness in reproducing the assignments from the set of simulated decision-maker preferences.

i. In-sample effectiveness

Results are shown in Figures A.1 and A.2 of Appendix A, where the error bars are equivalent to twice the standard deviation of the corresponding averages. More concentrated results are given in Tables 2-5.

TABLE 2. Average effectiveness in dependence of the number of criteria

Number of criteria (N)	Effectiveness	Standard deviation
3	0.997	4.0E-4

5	0.997	4.0E-4
7	0.9961	4.0E-4

Given the large number of experiments, the mean values of effectiveness should be distributed normally. In the following, we use the 2-sample t-test with a level of significance of 0.05 and with the null hypothesis “(H0:) The means of the results of two rows of the table are the same”. The null hypothesis was not rejected in comparing any pair of rows in Table 2, which provides evidence on the robustness of the approach regarding the number of criteria.

The effectiveness in dependence of the number of classes is provided by Table 3. The difference between each pair of these effectiveness was significant. These results provide slight evidence indicating that increasing the number of classes has a negative effect on the effectiveness of the approach.

TABLE 3. Average effectiveness in dependence of the number of classes

Number of classes (M)	Effectiveness	Standard deviation
2	0.9995	1.00E-04
3	0.9987	2.00E-04
5	0.992	6.00E-04

The effectiveness in dependence of the number of assignment examples per class is shown in Table 4. There is a statistical difference among all the effectiveness values in this table, showing that the effectiveness is a decreasing function of the number of objects per class.

TABLE 4. Average effectiveness vs the number of assignment examples

Number of Objects per class (n_{class})	Effectiveness	Standard deviation
3	0.9992	3.00E-04
5	0.998	4.00E-04
9	0.9959	5.00E-04
12	0.9938	6.00E-04

The effectiveness of the approach in dependence of the number of limiting profiles is shown in Table 5. There is something interesting with the effectiveness values shown in this table. Statistical analysis shows that there is significant improvement in effectiveness when going from one to three profiles or when going from one to five. However, there is no evidence that increasing from three to five profiles improves effectiveness; therefore, the decision analyst should consider that it may be not worth increasing the cognitive effort of the DM.

TABLE 5. Average effectiveness vs $card(B_k)$

Number of profiles ($card(B_k)$)	Effectiveness	Standard deviation
1	0.9933	6.0E-4

3	0.9985	3.0E-4
5	0.9984	3.0E-4

ii. Out-of-sample effectiveness

Table 6 exhibits the effectiveness in dependence of the number of criteria. Statistical analyses to the values in this table show that the effectiveness of the approach is a decreasing function of the number of criteria, which is intuitive since higher numbers of criteria imply higher complexities of the problem.

TABLE 6. Out-of-sample effectiveness vs number of criteria

Number of criteria	Effectiveness	Standard deviation
3	0.8617	0.0035
5	0.8216	0.0042
7	0.7862	0.0045

Table 7 analyzes the effectiveness in dependence of the number of classes. There is a statistical difference between each pair of values in this table. Here, it is quite interesting how the effectiveness goes up when increasing from two to three classes but goes down when increasing from three to five classes. This behavior is maintained regardless of the number of profiles or the number of assignment examples (see Figures 5 and 6). Thus, one could ask if going from two to three classes increases the reference information providing more learning capacity to the approach without considerably increasing the complexity of the problem, but the new reference information does not compensate the increment in the complexity when going from three to five classes. Such a hypothesis will be evaluated in future works.

TABLE 7. Out-of-sample effectiveness vs number of classes

Number of classes	Effectiveness	Standard deviation
2	0.8217	0.0047
3	0.8758	0.0034
5	0.7721	0.0037

The effectiveness of the approach in dependence of the number of assignment examples per class is provided by Table 8. According to the statistical analysis, the hypothesis “the means of the results of two rows of the table are the same” is rejected for all pairs of rows, except when the first two rows

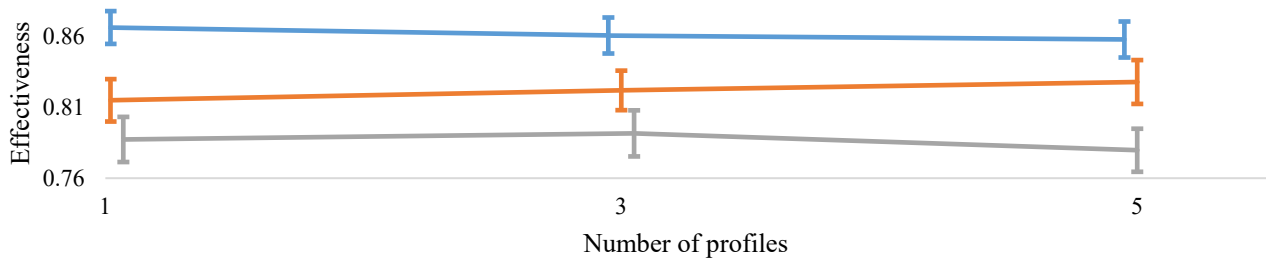


FIGURE 5. Effectiveness vs. number of profiles in the context of number of criteria

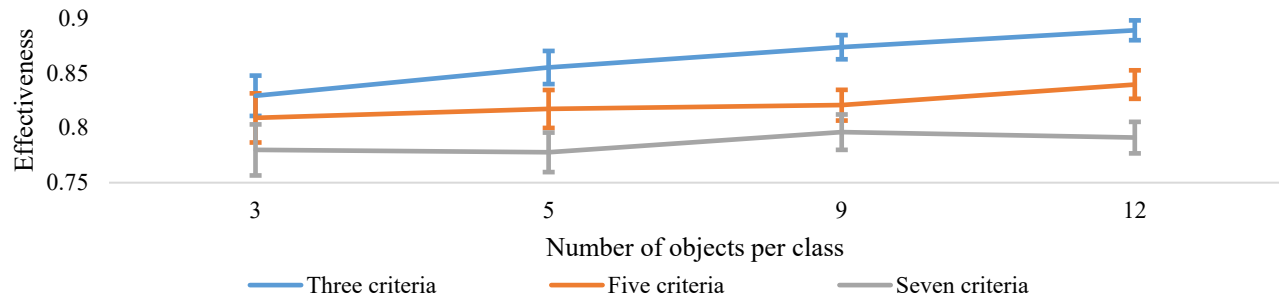


FIGURE 6. Effectiveness vs. number of objects per class in the context of number of criteria

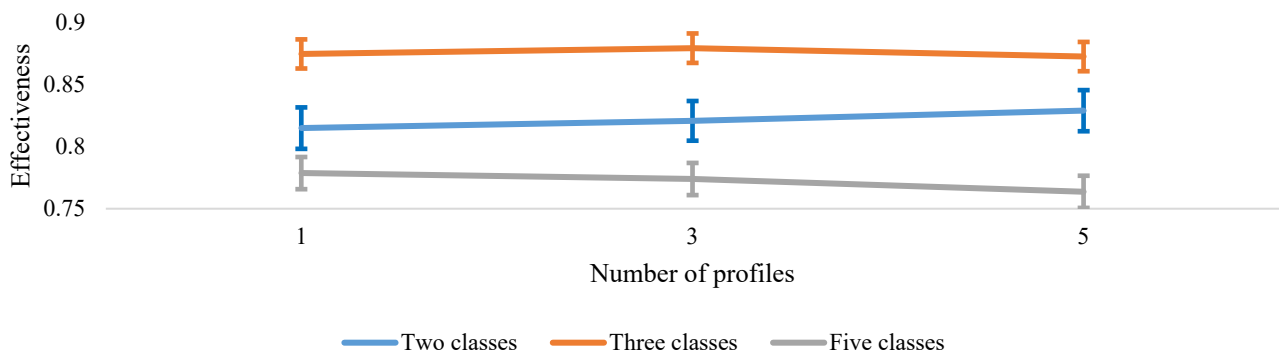


FIGURE 7. Effectiveness vs. number of profiles in the context of the number of classes

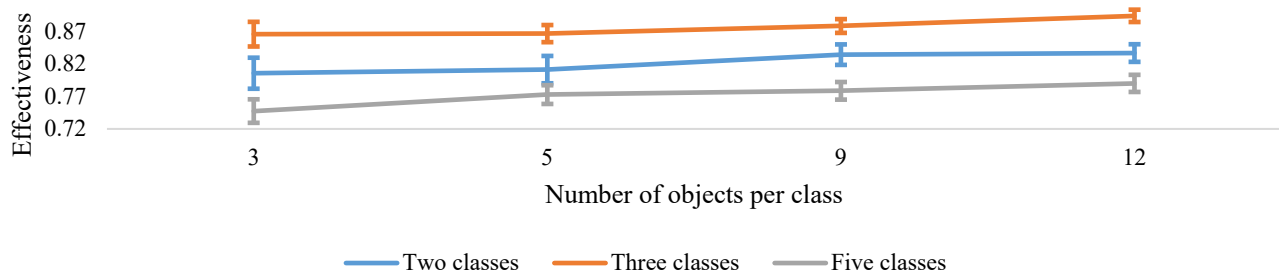


FIGURE 8. Effectiveness vs. number of objects per class in the context of number of classes

are compared. Therefore, there is clear evidence indicating that the effectiveness increases as the number of objects per

class also increases.

TABLE 8. Out-of-sample effectiveness vs number of assignment examples per class

Number of objects per class	Effectiveness	Standard deviation
3	0.8061	0.0062
5	0.8166	0.005
9	0.8301	0.0042
12	0.8399	0.0039

Table 9 shows the effectiveness in dependence of the number of limiting profiles per boundary. There is no statistically significant difference between the average values in Table 9. To a great extent, this is a surprising result since an INTERCLASS-nB model with 5 profiles, although more complex, should “learn” better the decision policy that is implicit in the training set.

TABLE 9. Out-of-sample effectiveness vs $\text{card}(B_k)$

Number of profiles	Effectiveness	Standard deviation
1	0.8229	0.0042
3	0.8248	0.0042
5	0.8218	0.0043

Figures 5-8 present the previous results with more detail addressing questions like “given a number of criteria, what is the effectiveness of the approach in dependence on the number of profiles?”.

ii. Discussion of the results on INTERCLASS-nB

The average in-sample effectiveness reaches values very close to 1. This proves that the evolutionary algorithm used by the inference procedure behaves quite satisfactorily. The effectiveness is statistically independent of the number of criteria and is a decreasing function with the number of classes. It is also a decreasing function with the number of assignment examples per class, perhaps because the increment of this number increases the difficulty of the optimization problem related to the parameter inference. Three limiting profiles in each boundary gives better results than a single profile, but a subsequent increment is not necessary.

Since the inferred model will be used to assign new actions, the analysis of the out-of-sample effectiveness is perhaps more relevant. We can establish the following concluding remarks:

- The values of the effectiveness are slightly higher than the results reported by [12] for ELECTRE TRI-nB.

- The effectiveness is an increasing function of the number of assignment examples per class; this is consistent with the hypothesis of “having more training examples allows a better learning process”.
- The effectiveness is degraded by the increment of the number of criteria; this could be a consequence of the increased number of model parameters and the difficulty to “learn” such a more complex model.
- The dependence of the number of classes has no clear explanation, since it is non-monotonic; more classes imply more complex assignment problems, so it is reasonable that the effectiveness with $M = 5$ is lower than with $M = 3$. However, the effectiveness with $M = 3$ is higher than with $M = 2$. This effect may be related to the total number of assignment examples, because with $M = 3$, there are more training examples.
- The out-of-sample effectiveness does not improve with the number of limiting profiles; this result differs from the one obtained in [9] for ELECTRE TRI-nB. It seems that a single “well-designed” limiting profile suffices to reach acceptable effectiveness, and more profiles do not increase the learning ability of the model, at least within the analyzed range of training examples. More research is needed to reach a comprehensive understanding of this result.

2) INTERCLASS-nC

This section focuses on the effectiveness of INTERCLASS-nC.

i. In-sample effectiveness

Table 10 shows the effectiveness of the approach given different numbers of criteria. The null hypothesis was rejected when comparing all the pairs of rows in Table 10. So, the effectiveness does not vary monotonically with the number of criteria. It is degraded from $N = 3$ to $N = 5$ and improved from $N = 5$ to $N = 7$.

TABLE 10. In-sample average effectiveness vs number of criteria

Number of criteria	Effectiveness	Standard deviation
3	0.9903	6.00E-04
5	0.9874	7.00E-04
7	0.9923	5.00E-04

Table 11 shows the in-sample effectiveness depending on the number of classes. Again, a statistically significant difference was found in the comparison of all the pair of rows in Table

11. The in-sample effectiveness is degraded when the number of classes increases.

TABLE 11. In-sample average effectiveness vs number of classes

Number of classes	Effectiveness	Standard deviation
2	0.9999	0
3	0.9971	2.00E-04
5	0.973	8.00E-04

The effectiveness of the approach regarding the number of assignment examples per class is presented in Table 12. Again, the null hypothesis was rejected in the comparison of all the pair of rows in Table 12.

TABLE 12. Effectiveness vs number of assignment examples per class

Number of Objects per class	Effectiveness	Standard deviation
3	0.9974	3.00E-04
5	0.9931	6.00E-04
9	0.987	8.00E-04
12	0.9826	9.00E-04

Table 13 shows the effectiveness in dependence of the number of characterizing profiles per class. There is statistically significant difference between the average values except when $card(R_k) = 3$ and $card(R_k) = 5$. Therefore, the decision analyst could ask the DM to provide three characteristic actions per class, but there is no evidence that increasing this number will provide higher effectiveness.

TABLE 13. Effectiveness vs number of characteristic actions per class

Number of profiles	Effectiveness	Standard deviation
1	0.9827	0.001
3	0.993	6.00E-04
5	0.9924	6.00E-04

ii. Out-of-Sample effectiveness

The out-of-sample effectiveness of the approach in the context of the number of criteria is shown in Table 14. The statistical analyses to the values in this table show that there is only difference regarding seven criteria. The increment in effectiveness when going to seven criteria is counterintuitive, although consistent with the in-sample effectiveness provided by Table 10. This effect is also seen when the effectiveness is broken down in dependence on different numbers of profiles and characterizing objects (see Figures 9 and 10). A more in-depth analysis is deferred for future work.

TABLE 14. Out-of-sample average effectiveness vs number of criteria

Number of criteria	Effectiveness	Standard deviation
3	0.9192	0.0023
5	0.9167	0.0023
7	0.9384	0.0021

Table 15 presents the effectiveness of the approach regarding the number of classes. The statistical tests found a statistically significant difference between all the pair of rows in Table 16. Thus, here, as in the case of INTERCLASS- nB , the effectiveness of the approach is a decreasing function of the number of classes.

TABLE 15. Out-of-sample average effectiveness vs number of classes

Number of classes	Effectiveness	Standard deviation
2	0.9758	0.0012
3	0.9259	0.002
5	0.8726	0.0021

Table 16 shows the effectiveness in dependence on the number of assignment examples per class. The only pairs for which the difference is not significant are when $n_{class} = 3$ and $n_{class} = 5$, and when $n_{class} = 9$ and $n_{class} = 12$. Therefore, the decision analyst can consider requiring for the DM to assign up to nine examples per class.

TABLE 16. Out-of-sample effectiveness vs number of assignment examples per class

Number of Objects per class	Effectiveness	Standard deviation
3	0.9153	0.0033
5	0.9194	0.0027
9	0.9308	0.0021
12	0.9336	0.002

Table 17 shows how going from one to three profiles significantly improves the effectiveness but going from three to five profiles does not.

TABLE 17. Effectiveness vs the number of characteristic actions per class

Number of profiles	Effectiveness	Standard deviation
1	0.9117	0.0025
3	0.9283	0.0026
5	0.9307	0.0027

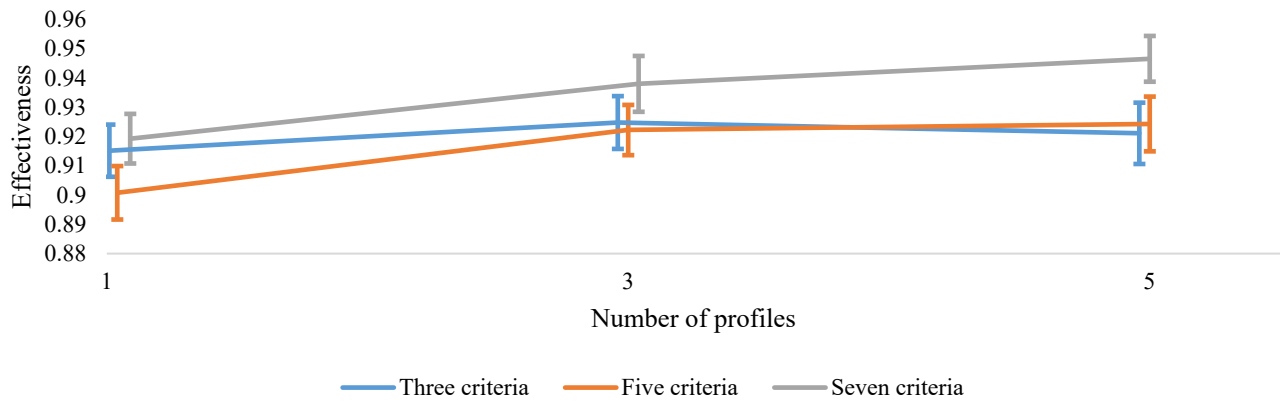


FIGURE 9. Effectiveness vs. number of profiles in the context of number of criteria

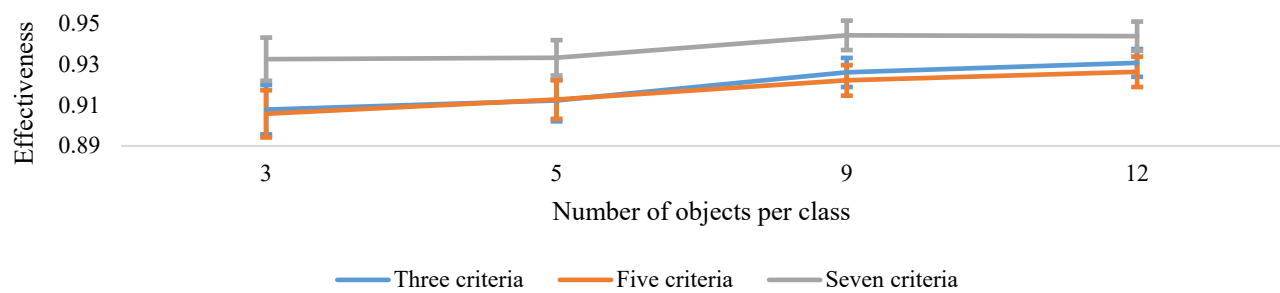


FIGURE 10. Effectiveness vs. number of objects per class in the context of number of criteria

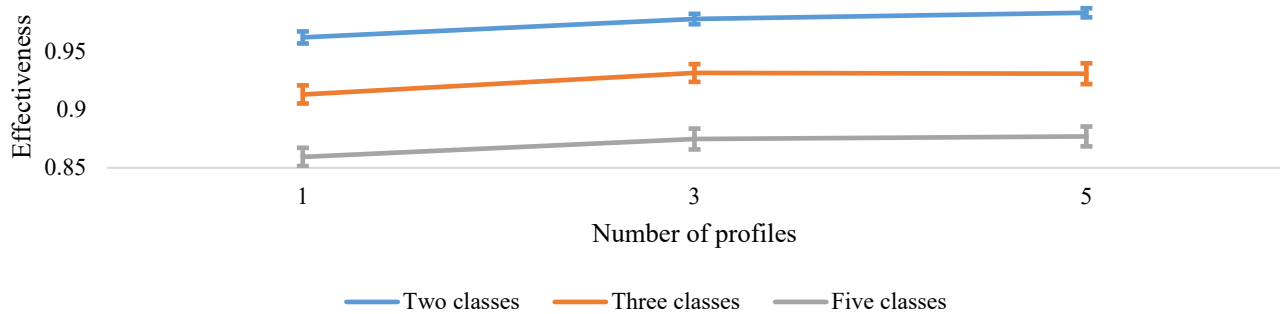


FIGURE 11. Effectiveness vs. number of profiles in the context of number of classes

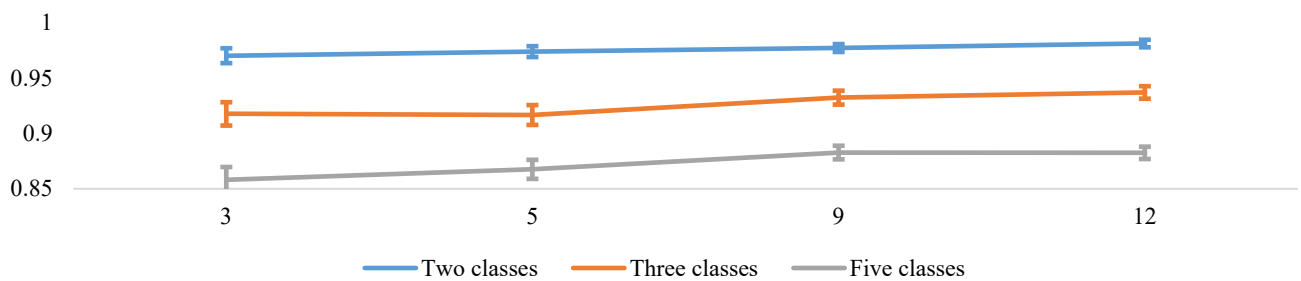


FIGURE 12. Effectiveness vs. number of objects per class in the context of number of classes

Figures 9-12 present the previous results with further details and comparisons.

iii. Discussion of the results on INTERCLASS-nC

Concerning the in-sample effectiveness, we detect several similar features like the INTERCLASS-nB. The average effectiveness reaches values very close to 1 and is a decreasing function with the number of classes. It is also a decreasing function with the number of assignment examples per class, maybe since having more examples makes more complex the optimization problem through which the model parameters are inferred. To use three characteristic actions per class is better than a single action but having more than three actions does not seem to be necessary.

The out-of-sample effectiveness reaches values higher than 0.9, clearly greater than INTERCLASS-nB. Its most interesting features are:

1. It shows an increasing dependence of the number of assignment examples per class. This means that the learning capacity of the methods does not reach a plateau within the range of assignment examples that was analyzed in our experiments.
2. The effectiveness is significantly improved when the number of characteristic actions increases from one to three. From three to five, there is no significant improvement.
3. It is a decreasing function with the number of classes, which is a consequence of a higher difficulty of the assignment problem.
4. The effectiveness seems to slightly increase with the number of criteria. This behavior is the opposite to the one observed on INTERCLASS-nB. Its explanation may be related to the way of defining the measure of effectiveness (see Equation 3). As the number of criteria increases, there could be more incomparability among actions and characteristic subsets R_k ; therefore, $|\chi_{DM} \cap \chi_{inf}|$ could be increased, thus producing an improvement of the effectiveness measure in Equation 3.

V. CONCLUSIONS

A novel approach has been proposed to infer the whole set of model's parameters on the two recently published interval-based multi-criteria classification INTERCLASS-nB and INTERCLASS-nC methods. Given a set of assignment examples, a regression-inspired optimization problem is solved by an evolutionary algorithm, which permits to handle the non-linear complexity of the interval outranking model; additionally, evolutionary optimization tools are more robust than conventional non-linear programming techniques when the number of parameters, criteria and classes increase. In this way, the cognitive effort required to the DM in the parameter elicitation process is strongly diminished.

Two basic issues should be considered for an appropriate setting of the parameters of a multi-criteria classification model via preference disaggregation analysis:

- a) The capacity to restore the known assignment examples (in-sample effectiveness)

- b) The capacity to suggest new assignments that are considered as appropriate by the DM (out-of sample effectiveness).

Most related papers concentrate on point a). Using evolutionary algorithms and assignment examples coming from simulated decision models, high values of the in-sample effectiveness prove that the algorithm finds solutions which are close to the optimal one. Perhaps the analysis of the out-of-sample effectiveness is even more important; it measures the capacity of the method "to learn" what assignments are considered appropriate by the DM, thus being able to suggest appropriate decisions on new actions, that is its real application.

In this paper, the quality of solutions is characterized by measures of both effectiveness measures. Their dependence of the number of limiting profiles (in INTERCLASS-nB), the number of characteristic actions (in INTERCLASS-nC), the number of assignment examples per class, the number of classes, and the number of criteria, have been described.

Some common features of both methods and their inferred model's parameters are:

- A) The in-sample effectiveness reaches very high values using three limiting profiles and the same number of characteristic actions;
- B) The in-sample effectiveness decreases with the number of classes and the number of assignment examples per class;
- C) The out-of-sample effectiveness is improved by the increment of the number of assignment examples per class.

Point A) is coincident with the results reported for ELECTRE TRI-nB by [12]. Point B) is a consequence of an increasing complexity of the optimization problem from which the model's parameters are inferred and increasing difficulty of the assignment problem. Point C) confirms the premise of more information is usually better than less.

Several different behaviors follow:

1. The INTERCLASS-nC out-of-sample effectiveness seems to be higher than the one on INTERCLASS-nB; however, this comparison is not fair because the definition of effectiveness differs from Equation 2 to Equation 3. The definition in Equation 2 requires an exact coincidence between the assignment from the DM and the assignment from the inferred model, whereas the measure in Equation 3 is laxer.
2. As one could assume, the INTERCLASS-nB out-of-sample effectiveness is degraded when the number of criteria increases, contrarily to INTERCLASS-nC;
3. As one would wait, the INTERCLASS-nC out-of-sample effectiveness tends to improve with the number of characteristic actions; contrarily, this measure seems to be independent on the number of limiting actions in INTERCLASS-nB.

Concerning Point 2 above, the surprising performance of INTERCLASS-nC could be explained by the increment of incomparabilities between actions and representative subsets of classes, as was discussed in Section IV, last paragraph.

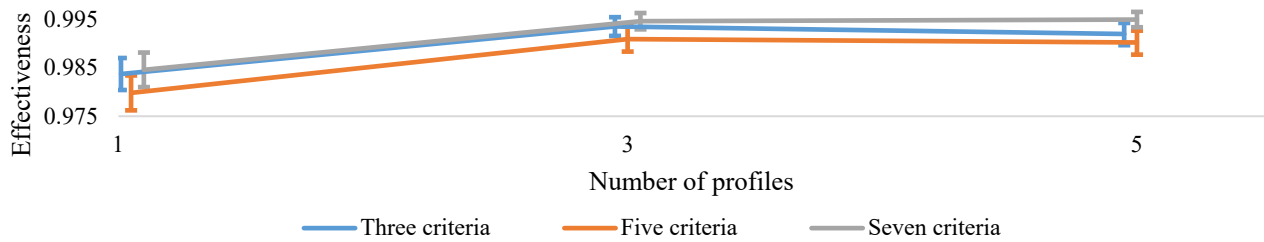


FIGURE A.1. Effectiveness vs. number of profiles in the context of number of criteria

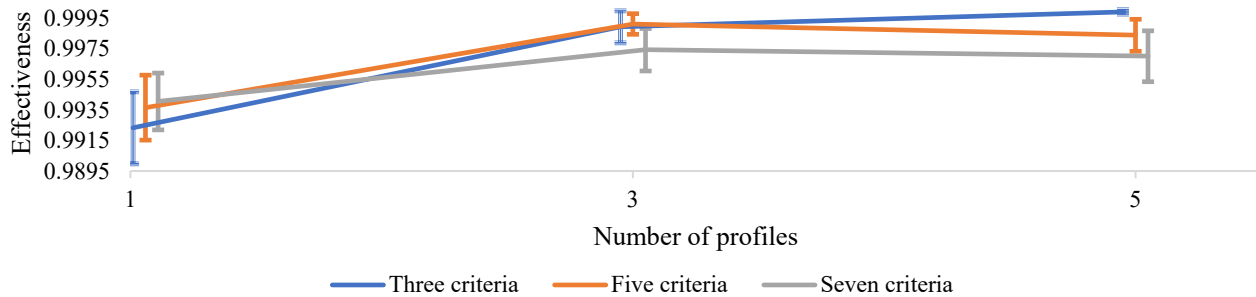


FIGURE A.2. Effectiveness vs. number of profiles in the context of number of classes

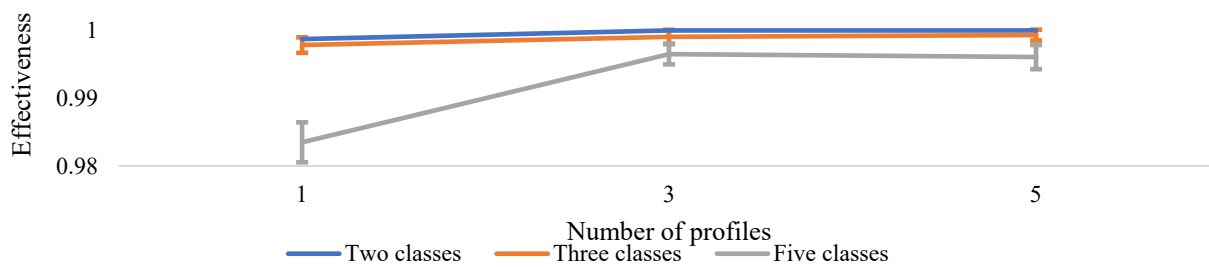


FIGURE B.1. Effectiveness vs. number of profiles in the context of number of criteria

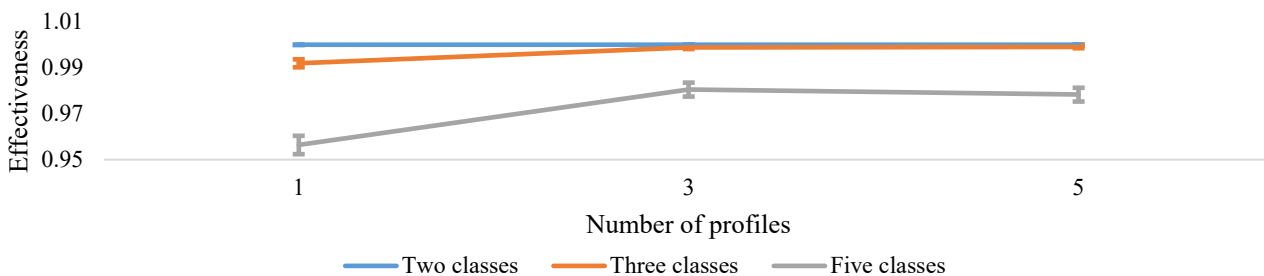


FIGURE B.2. Effectiveness vs. number of profiles in the context of number of classes

Convincing quantitative explanations of the above different characteristics should be found by future research works. The question about which method performs better in a preference disaggregation context is kept open as another avenue of future research. We should remark that INTERCLASS-nC uses more information than the pseudo-conjunctive INTERCLASS-nB; this comes from two different outranking relations (we mean “ x outranks the representative

set R_k ” and “ R_k outranks x ”), whereas the pseudo-conjunctive INTERCLASS-nB uses only “ x outranks the limiting boundary”. Handling more information could bring a higher learning capacity. A fair comparison should deal with a limiting boundary-based method similar to INTERCLASS-nB, but handling “ x outranks the limiting boundary B_k ”, and “ B_k outranks x ”, perhaps with the use of a co-joint assignment rule coming from descending and ascending procedures.

APPENDIX

A. IN-SAMPLE EFFECTIVENESS OF INTERCLASS-nB

Results about the effectiveness of the approach regarding the in-sample effectiveness of INTERCLASS-nB are shown in Figures A.1 and A.2.

B. IN-SAMPLE EFFECTIVENESS OF INTERCLASS-nC

Results about the effectiveness of the approach regarding the in-sample effectiveness of INTERCLASS-nC are shown in Figures B.1 and B.2.

C. PROCEDURE FOLLOWED BY THE INTERVAL-BASED OUTRANKING APPROACH

Let us assume the notation described in Subsection II.2.

The marginal credibility index of x being at least as good as action y on the j th criterion, $\alpha_j(x, y)$, depends on the strength of the arguments provided by such criterion to state that “ x outranks y on this criterion”. On the one hand, if $g_j \in G_1$, then $\alpha_j(x, y)$ is defined as:

$$\alpha_j(x, y) = \begin{cases} 0 & \text{if } g_j \in C(yPx), \\ \frac{g_j(x) - g_j(y) + p_j(\cdot)}{p_j(\cdot) - q_j(\cdot)} & \text{if } g_j \in C(yQx), \\ 1 & \text{if } g_j \in C(xSy). \end{cases}$$

The discordance coalition is defined as $C(yPx) = \{g_j \in G_1: g_j(y) - g_j(x) \geq p_j(\cdot)\}$; $C(yQx) = \{g_j \in G_1: g_j(y) - p_j(\cdot) \leq g_j(x) \leq g_j(y) - q_j(\cdot)\}$; and $C(xSy) = \{g_j \in G_1: g_j(x) - g_j(y) \geq -q_j(\cdot)\}$.

On the other hand, if $g_j \in G_2$, then $\alpha_j(x, y)$ is defined as:

$$\alpha_j(x, y) = p(g_j(x) \geq g_j(y)).$$

If we now consider a given credibility threshold γ , then the set of all the criteria for which $\alpha_j(x, y) \geq \gamma$ is true is called γ -possible concordance coalition and is denoted as $C(xS_\gamma y)$. Conversely, all criteria in $G/C(xS_\gamma y)$ compose the γ -possible discordance coalition, which is denoted as $D(xS_\gamma y)$. In order to ensure that there are some realizations of the criteria weights for which $\sum_{j=1}^N w_j = [1, 1]$ is true, the following constraints are imposed:

$$\sum_{j=1}^N w_j^- \leq 1, \\ \sum_{j=1}^N w_j^+ \geq 1.$$

The concordance index of $xS_\gamma y$, $c(x, y) = [c^-(x, y), c^+(x, y)]$, is then defined as follows:

$$c^-(x, y) = \sum_{g_j \in C(xS_\gamma y)} w_j^-,$$

if

$$\sum_{g_j \in C(xS_\gamma y)} w_j^- + \sum_{g_j \in D(xS_\gamma y)} w_j^- \leq 1, \text{ and}$$

$$\sum_{g_j \in C(xS_\gamma y)} w_j^- + \sum_{g_j \in D(xS_\gamma y)} w_j^+ \geq 1.$$

Otherwise, $c^-(x, y)$ is defined as

$$1 - \sum_{g_j \in D(xS_\gamma y)} w_j^+.$$

Similarly,

$$c^+(x, y) = \sum_{g_j \in C(xS_\gamma y)} w_j^+$$

only if

$$\sum_{g_j \in C(xS_\gamma y)} w_j^+ + \sum_{g_j \in D(xS_\gamma y)} w_j^- \leq 1, \text{ and} \\ \sum_{g_j \in C(xS_\gamma y)} w_j^+ + \sum_{g_j \in D(xS_\gamma y)} w_j^- \geq 1.$$

Otherwise, $c^+(x, y)$ is

$$1 - \sum_{g_j \in D(xS_\gamma y)} w_j^-.$$

As in the classical outranking approach, its interval-based extension also considers the arguments against the outranking relation through a credibility index of the statement “the j th criterion vetoes the assertion x outranks y ”, which is denoted as $d_j(x, y)$ and is defined as follows. For each $g_j \in G_2$, $d_j(x, y) = p(g_j(y) \geq g_j(x) + v_j)$, where v_j is the interval number representing the veto power of criterion g_j . For each $g_j \in G_1$, $d_j(x, y)$ can be calculated by one of two ways, depending on the information available about thresholds. First, if the veto power of the j th criterion is precise, that is v_j is a real number, and there is a discordance (pre-veto) threshold, $u_j \leq v_j$, then $d_j(x, y)$ is (cf. Mousseau and Dias, 2004; Roy and Słowiński, 2008):

$$d_j(x, y) = \begin{cases} 1 & \text{if } g_j(y) - g_j(x) \geq v_j, \\ \frac{g_j(y) - g_j(x) - u_j}{v_j - u_j} & \text{if } u_j < g_j(y) - g_j(x) < v_j, \\ 0 & \text{if } g_j(y) - g_j(x) \leq u_j. \end{cases}$$

Second, if the veto power of the j th criterion is imperfectly known, that is, v_j is an interval number, then $d_j(x, y) = p(g_j(y) \geq g_j(x) + v_j)$.

Let Γ be the set $\{\alpha_j(x, y) \in \mathbb{R}: p(g_j(x) \geq g_j(y)) = \alpha_j(x, y), j = 1, \dots, N\}$. For each $\gamma \in \Gamma$, x outranks y with marginal credibility index η_γ , and majority strength $\lambda > [0.5, 0.5]$, (with $\lambda \geq 0.5$) if and only if

$$\eta_\gamma = \min \left\{ \gamma, p(c(x, y) \geq \lambda), \left(1 - \max_{g_j \in D(xS_\gamma y)} d_j(x, y) \right) \right\}.$$

Each η_γ is the credibility degree of the conjunction between i) “the γ -possible concordance coalition is strong enough” and ii) “the γ -possible discordance coalition does not exert veto”. η_γ is interpreted in [30] as a marginal outranking credibility index. Therefore, x outranks y with credibility index $\eta(x, y) \in [0, 1] = \max\{\eta_\gamma\} (\gamma \in \Gamma)$. $\eta(x, y)$ is the interval

outranking credibility index. If Γ is an empty set, then $\eta(x, y)$ is zero.

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