Infinite Sets

In class we studied some aspects of infinite sets, and the diagonalization technique. This hand-out is to supplement the material that we saw in class[1]. This text have not been properly proof-read, please report if you find mistakes.

1 Countability

The natural numbers originally arose from counting elements in sets. There are two very different possible sizes for sets, namely *finite* and *infinite*, and in this section we discuss these concepts in detail. Finite sets are those for which we can indicate the number of elements. Like the set of chairs in a given room, the set of pencils in a box etc. On the other hand, there exists sets consisting of an infinite number of elements e.g. set of natural numbers, all points on the real line, all circles in the plane, all polynomials with rational coefficients etc. When we say that set is infinite we mean that whatever number of elements are removed from the set still the set has some elements remained in it. We can compare two finite sets by comparing their number of elements. That is we can either count the number of elements in each sets. The other method is that we can also try to establish a correspondence between the elements of these sets by assigning to each element of one of the sets one and only one element of the other. As we have seen earlier, such a correspondence is called a bijection or a 1-1 correspondence. For finite sets if the number of elements of two sets are same then one can define a bijection between them. But is it possible to compare infinite sets in similar fashion. That is, does it make sense to ask which is larger: set of circles in the plane or set of rational numbers on the real line. Unlike finite sets counting the number of elements of two infinite sets are not possible so the other method of establishing a bijection for comparison is possible.

Definition 1. Countable set (denumerable set)*: A set is said to be countable if a bijection can be set between its elements and the set of* natural numbers. *Thus, a countable set is one whose elements can be indexed as follows:* $a_1, \ldots, a_n, \ldots$.

Example 1. The set of all integers $\mathbb{Z}$ is a countable set. To see this we can set up the following 1-1 correspondence between the set of integers $\mathbb{Z}$ and the set of natural numbers $\mathbb{N}$.

0 -1 1 -2 2 . . .
↓ ↓ ↓ ↓ ↓
1 2 3 4 5 . . .
$n \leftrightarrow 2n+1$ if $n \geq 0$ and $n \leftrightarrow -2n$ if $n < 0$.

Example 2. The set of all positive even integers is countable. The correspondence being $n \leftrightarrow 2n$.

Definition 2. *An infinite set which is not countable is uncountable (or non-denumerable)*

Now we illustrate some properties of countable sets.

Property 1. Every subset of a countable set is either finite or countable.

[1] The discussion closely follows the text: *Functional Analysis, Vol:1*, by A.N. Kolmogorov and S.V. Fomin, Graylock Press, New York, 1957

Proof. Let A be a countable set and let B be a subset of A. If we enumerate the elements of the set $A : a_1, a_2, \ldots, a_n, \ldots$ and let $n_1, n_2, \ldots,$ be the natural numbers which correspond to the elements in B in this enumeration, then if there is a largest one among these natural numbers , B is finite, otherwise B is countable. □

Property 2. The sum of arbitrary finite or countable set of countable sets is again a finite or countable set.

Proof. Let $A_1, A_2, \ldots$ be countable sets. All their elements can be written in the form of an infinite table:

$$\begin{array}{lllll} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Where the elements of A_1 are listed in the first row, the elements in A_2 in the second row and so on. Now we enumerate all these elements by the *diagonal method*,i.e. we take a_{11} for the first element, a_{12} for the second, a_{21} for the third, and so on, taking the elements in the order shown in the following table:

$$\begin{array}{cccccccc} a_{11} & \longrightarrow & a_{12} & & a_{13} & \longrightarrow & a_{14} & \cdots \\ & \swarrow & & \nearrow & & \swarrow & & \\ a_{21} & & a_{22} & & a_{23} & & a_{24} & \cdots \\ \downarrow & \nearrow & & \swarrow & & & & \\ a_{31} & & a_{32} & & a_{33} & & a_{34} & \cdots \\ & \swarrow & & & & & & \\ a_{41} & & a_{42} & & a_{43} & & a_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

It is clear that in this enumeration every element of each of the sets A_i receives a definite index, i.e. we have established a bijection between all elements of $A_1, A_2, \ldots$ and the set of natural numbers. □

Property 3. Every infinite set contains a countable subset.

Proof. Let M be an infinite set. Consider an arbitrary element $a_1 \in M$. Since M is infinite, we can find an element $a_2 \in M$ which is distinct from a_1, then an element a_3 distinct from a_1 and a_2 and so on. Continuing this process (which cannot terminate in a finite number of steps since M is infinite), we obtain a countable set $A = \{a_1, a_2, a_3 \ldots, \}$ of the set M. This countable set is the *smallest* subset of the infinite set. □

2 Equivalence of sets

Definition 3. *Two sets M and N are said to be* equivalent *denoted by $M \sim N$, if a one-one correspondence (bijection) can be set up between their elements.*

The concept of equivalence can be applied to infinite as well as finite sets. It is clear that two finite sets are equivalent if and only if they contain the same number of elements. The definition we introduced for countable sets can now be restated as: *a set is said to be countable if it is equivalent to the set of natural numbers.*

Theorem 1. *Every infinite set is equivalent to some proper subset of itself.*

Proof. Let M be a infinite set. From property 3 we can say that M contain a countable subset A, let

$$A = \{a_1, a_2, \dots, a_n, \dots\}.$$

We partition A into two countable subsets as

$$A_1 = \{a_1, a_3, a_5, \dots\} \text{ and } A_2 = \{a_2, a_4, a_6, \dots\}.$$

As A and A_1 are both countable so a one-to-one correspondence can be set up between them. This correspondence can be extended to a one-to-one correspondence between the sets

$$A_1 \cup (M \setminus A) = M \setminus A_2 \text{ and } A \cup (M \setminus A) = M.$$

We already have a 1-1 correspondence between A_1 and A, now setting and identity map in the set $M \setminus A$, we obtain a 1-1 correspondence between M and $M \setminus A_2$. And $M \setminus A_2$ is a proper subset of M. □

3 The set of real numbers is not countable

We already defined uncountable sets, the following theorem asserts the existence of such sets.

Theorem 2. *The set of real numbers in the closed interval* $[0, 1]$ *is uncountable*

Proof. Let us assume that the set of real numbers in the interval $[0, 1]$ is countable. Thus, we can write all the real numbers in $[0, 1]$,which can be expressed in the form of an infinite decimal in the form of a sequence

$$\begin{array}{l} 0.a_{11}\ a_{12}\ a_{13}\ \dots a_{1n}\ \dots \\ 0.a_{21}\ a_{22}\ a_{23}\ \dots a_{2n}\ \dots \\ 0.a_{31}\ a_{32}\ a_{33}\ \dots a_{3n}\ \dots \\ \dots\quad \dots\ \dots\ \dots\dots \\ 0.a_{n1}\ a_{n2}\ a_{n3}\ \dots a_{nn}\ \dots \\ \dots\quad \dots\ \dots\ \dots\dots \end{array} \tag{1}$$

Where a_{ik} is one of the elements of $\{0, 1, 2, \dots, 9\}$. Now let us construct the decimal

$$b = 0.b_1 b_2 \dots b_n \dots$$

such that $b_i = 2$ if $a_{ii} = 1$ and $b_i = 1$ if $a_{ii} \neq 1$. This construction guarantees that b does not belong to any of the decimals listed in Eq. (1). This thus contradicts our assumption that the set of real numbers in the interval $[0, 1]$ is countable. □

4 The Cardinal Number

If two finite sets are equivalent, they consists of the same number of elements. If M and N are two arbitrary equivalent sets we say that M and N have the same cardinal number (the same cardinality, the same potency). Thus, cardinal number is what all equivalent sets have in common. For finite sets the concept of cardinal number coincides with the number of elements in the set. The cardinal number of the set of natural numbers (i.e., any countable set) is denoted by the symbol $\aleph_0$ (read as "aleph zero"). Sets which are equivalent to real numbers are said to have the cardinality of the continuum, which is denoted by $\mathfrak{c}$ (or sometimes as $\aleph_1$).

If a set A is equivalent to some subset of a set B but is not equivalent to B, then we say that the cardinality of A is less than that of B.

We pointed out that the countable sets were the smallest of all infinite sets. We also showed that there exist infinite sets whose infiniteness is of higher order than that of countable sets. But do there exist infinite cardinal numbers exceeding the cardinal number of the continuum. Does there exist some "highest" cardinal number or not? The following theorem provides answer to such questions.

Theorem 3. *Let M be a set with cardinality $\mathfrak{m}$. Further, let $\mathfrak{M}$ be the set whose elements are all possible subsets of the set M. Then $\mathfrak{M}$ has greater cardinality than $\mathfrak{m}$.*

Proof. It is easy to see that the cardinality of $\mathfrak{M}$ cannot be less than the cardinality of M; in fact, those subsets of M each of which contain only one element form a subset of $\mathfrak{M}$ which is equivalent to M. It remains to prove that cardinality of $\mathfrak{M}$ is not same as that of the cardinality of M. Let us assume in the contrary; i.e., we assume that there is a one-to-one correspondence between the elements of M and $\mathfrak{M}$. Let $a \leftrightarrow A$, $b \leftrightarrow B$, $\ldots$, be a bijection between the elements of the set M and $\mathfrak{M}$. Now, let X be the set of elements in M which do not belong to those subsets to which they correspond (for example, if $a \in A$ then $a \notin X$, if $b \notin B$ then $b \in X$, and so forth). X is a subset of M, i.e., it is an element of $\mathfrak{M}$. By assumption X must correspond to some element $x \in M$. Let us investigate whether this element x belongs to the subset X. First let us assume that $x \notin X$. But by definition X consists of all those elements which are not contained in the subset to which they correspond and consequently the element x ought to be in X. Conversely, if we assume $x \in X$, then we conclude that x cannot belong to X since X contains only those elements which do not belong to the subset to which it correspond. Thus the element corresponding to the subset X should simultaneously belong to or not belong to the set X. This implies that such an element does not exist, i.e., a one-to-one correspondence between M and $\mathfrak{M}$ is not possible. This completes the proof. □

Thus for an arbitrary cardinal number we can in reality construct a set of greater cardinality and then a set with a still greater cardinality, and so on, obtaining in this way a hierarchy of cardinal numbers which is not bounded in any way.