RIAS

# FLEXAGONS

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# Contents

## Acknowledgements

In 1956, when the authors first learned of flexagons, nothing like the present work was conceived of. Even when in the following summer, on the basis of encouraging theoretical and practical developments, a paper was begun on the basis of encouraging and practical developments, nothing like the final size was envisioned. However, as work progressed, things rapidly mushroomed; what seemed like minor difficulties often developed into major extensions of the subject. This continued during several summers. Finally, the last few years have seen the reigning in of all the loose ends and coordination of the finished text. Drawings were prepared by Anthony Conrad, Bari Thompson, and artists of the Martin Company.

RIAS in Baltimore has contributed in several practical and organizational ways to the completion of the paper.

Mike Schlesinger has constructed many models of flexagons which proved useful in the research.

The authors wish especially to thank the many flexagon enthusiasts with whom they have conferred or communicated. These include Harold V. McIntosh, C. O. Oakley and R. J. Wisner, L. B. Tuckermann, Arthur Stone, and John Tukey.

Finally, the authors would appreciate news of new developments which may reach the ears of the reader; correspondence may be addressed simply Baldwin, Maryland.

Anthony S. Conrad
Daniel K. Hartline
Cambridge, Massachusetts
May 5, 1962.

Flexagons are ostensibly a mathematical recreation, and as is frequently the case turn out not only to have a substantial underlying theory, but a close relation to other seemingly unrelated topics as well. At first, they have been studied simply because they were interesting, and that along is sufficient justification. It is accordingly gratifying to find that the theory is closely related to certain types of programming languages, because in this way those who feel that mathematical research must have an application can take the same delight in a beautiful theory brought to an elegant conclusion, that I have found in supporting this study on its own merits.

Harold V. McIntosh
Baltimore, 14 May 1962.

# Introduction: The story of the Flexagon

We may safely conjecture that such a simple object as the first flexagon must have been discovered long ago, and perhaps many times since. For example, there are reports of such a device existing in elementary schools in pre-war Vienna.

The work on the flexagon came as a result of its discovery at Princeton in 1939. The story has it that Arthur Stone, an English graduate student, was in the practice of doodling with the strips of paper that he cut from around the edges of his notebook paper. American paper was too large for his English binder. One of the constructions arising from this happy misfortune attracted his attention in particular. A committee of graduate students formed to solve the mystery of the "flexible hexagon", or, as it soon became know, the "flexagon".

The members of this group - Richard P. Feynman, Bryant Tuckerman, John W. Tukey, and, of course, Arthur H. Stone - had laid the groundwork for all consequent study, through developing their yet unpublished theory, by the early 1940's. When the group disbanded, the flexagon was left, nearly forgotten, for ten years. Then, toward the beginning of the fifties, it received slight publicity with several very brief articles in mathematics magazines.

It is, however, chiefly due to three influences that flexagons are as well known as they are today. The first of these is Professor Louis B. Tuckerman, the father of one of the original investigators, who has demonstrated items of flexagon theory to winners of the Westinghouse Science Talent Search in Washington, D. C., each year for several years. The present authors trace their introduction to the flexagon back to this source. The second important influence came through Martin Gardner, who told the story of the flexagon, with instructions for building, to a large audience in a pair of articles in Scientific American during 1956 and 1957. More lately, C. O. Oakley and R. J. Wisner have published an article in the American Mathematical Monthly, and R. F. Wheeler has also contributed an article, in The Mathematical Gazette. These articles, although reaching a smaller audience than the Gardner article, contain more technical material. They are the only articles of a technical nature on the topic of flexagons know to the authors. The problems which concern Oakley and Wisner and something of their method of dealing with them are described herein in the section on the pat structure.

Although flexagons have on several occasions been used as cards or announcements, the commercial possibilities they seem to present apparently have never been exploited.

In this article, the authors hope to bring together much of the information that has been distributed concerning flexagons, along with the results of their own investigations. In this way, it may be possible to present a reasonably comprehensive treatment of the flexagon.

# Chapter 1

# Building and Operating the Flexagon

The first flexagon was a hexagonal object folded from a straight strip of paper. This simplest model is quite easily constructed.

Fold off and cut out a sequence of equilateral triangles (see figure 1.1a). Nine of these are to be used in the flexagon. Folding up the flexagon is accomplished by folding consistently clockwise or counterclockwise at every third hinge between triangles (see figure 1.1). If the direction of folding is not consistent, the flexagon will loose both its symmetry and its stability.

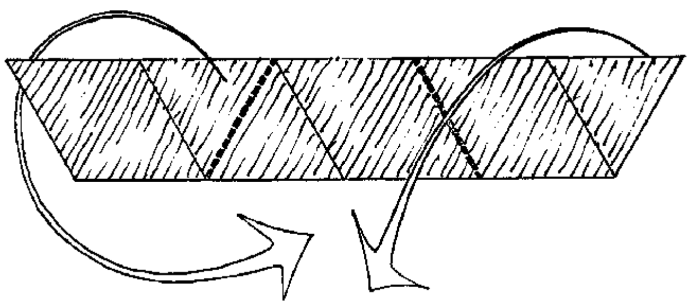


Fig. 1.1a

It will not hold together. When this process has been completed successfully and the ends have been taped in place, the resulting object should resemble that shown in figure 1.1b.

It can now be seen that this flexagon is a kind of three twist Moebius band. It has one surface and one edge. In order to lay it flat, it has been symmetrically squashed out, giving six distinct hinged sections, each of which is called a "pat".

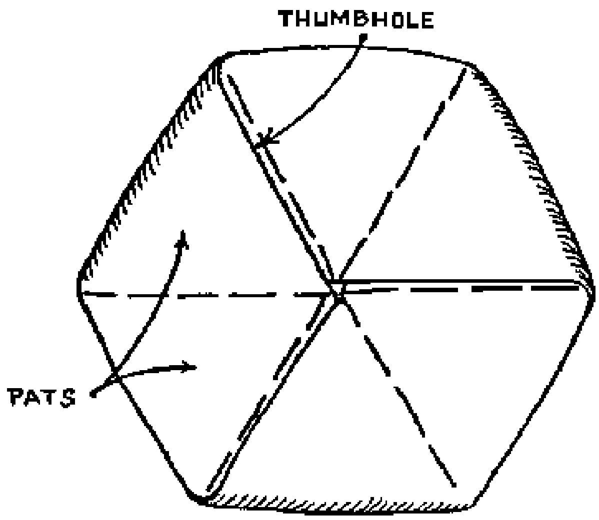



Fig. 1.1b

Each triangle of paper making up the pats is called a "leaf". It will be found that there are two kinds of pats. Some of the pats are one leaf thick, others two. Since the flexagon has tripole symmetry, any pair of adjacent pats that one should happen to choose would have one of each kind of pat. Such a pair will be referred to as a "unit".

This is all very interesting, you may say, but what of the supposed "flexibility"? Before investigating this, we must study the Moebius band a moment longer. Anyone familiar with the Moebius band must know that the twist is in a sense independent of the strip of paper; that is, the twist may be made to pass along the band, or, relative to the twist, the strip may be made to rotate about its open center while the twist remains stationary. The same is the case with our three twist flexagon. We way perform this operation, with some difficulty, after a slight forcing of the ideally rigid leaves. This same operation, carried out in an easier fashion and without bending the leaves, is know as "flexing".

In order to flex the flexagon, one must first find a hinge between two pats at which the thumb of finger can be passed between the two leaves of the larger pat. Such a hole has been named a "thumbhole" (see figure 1.1b). When a thumbhole hinge has been found, the two pats on either side should be pushed together. In doing this, fold the pats down away from the thumbhole hinge. It may now be seen that, since alternate hinges are thumbhole hinges, each of the three units may be folded together in this manner. When each units has been folded, the result is a sort of three bladed affair. (see figure 1.2a).

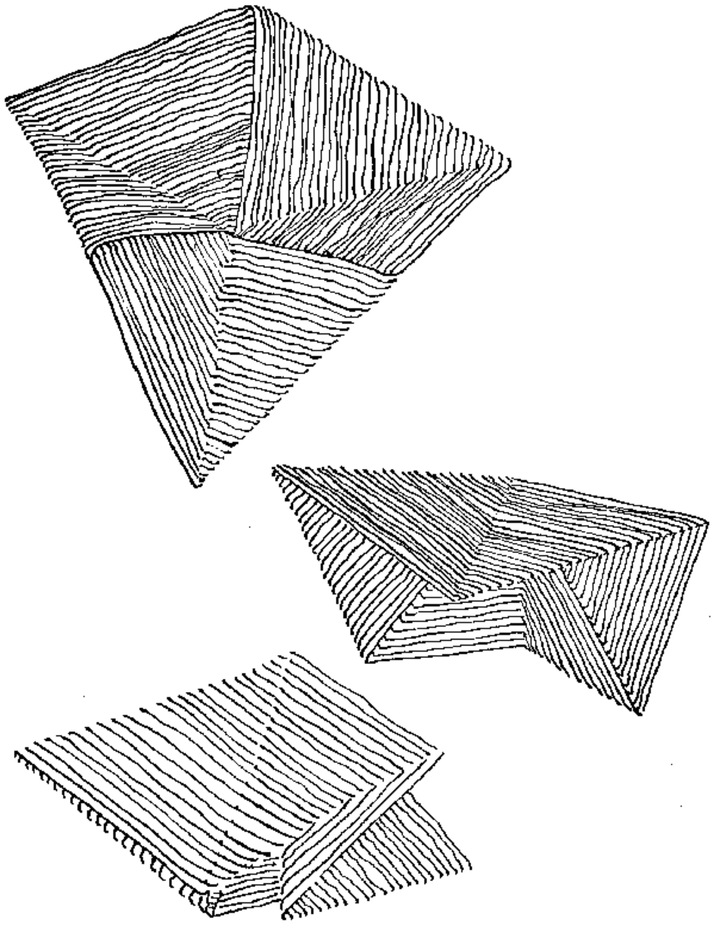


Fig. 1.2a

Now enters the most interesting aspect of the flexagon. Due to double hinging, our three bladed object may be opened out again from either the top or the bottom end. One can, of course, lay the flexagon flat by performing the reverse of the process just completed. But this reverse process may also be carried out from the downward end of the flexagon, exposing the faces that had been hidden in the flexagon's folds (see figure 1.2b). This latter operation is the essential part of the operation know as "flexing", and is found to correspond exactly to the Moebius band type of operation.

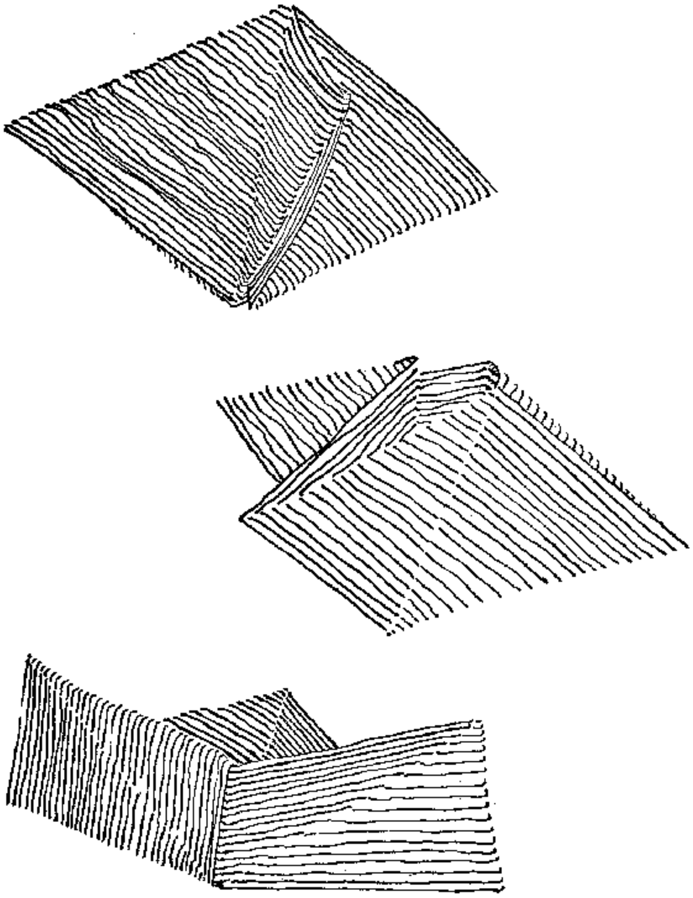


Fig. 1.2b

Now that the flexagon is again flat, is may be rotated 60° about its center to the right position for flexing again. This completes the full "flexing" operation. Thus the process can be carried out again, and then again. We can flex in this manner indefinitely, just as we can perform upon the Moebius band indefinitely. Facility in flexing may be developed with very little practice.

It is much easier to appreciate the flexagon if we mark it in some way so that we can see just what is happening. A number of interesting marking devices are used. Probably the most common and also the most sensational is color. Several different coloring schemes may be used. We shall use the simplest arrangement possible. Other systems will be shown as the need for them arises. Color the top surfaces of each of the six pats red. Turn the flexagon over and paint the exposed surfaces yellow. Color the remaining surfaces blue after one flexing.

It has not doubt been noticed how carefully it was stressed that the thumbhole hinges be creased upward. The possibility of folding the thumbhole hinges down, and thereby folding, the other hinges up, has not been considered. Since everything we learn about the flexagon involves breaking former rules, we will of course try doing this. We meet immediate discouragement, however, for we can no longer flex. But since we are not easily dissuaded, we notice that if we were to create a new thumbhole under the thin pats, we might then flex anyway. Indeed the thumbhole seems simply to be a slash sideways through each thick pat. If one could effect such a slitting through the single pats with a sharp knife, it might be possible to create the desired thumbhole. A much more feasible plan involves adding in three extra leaves in the proper places, modeling the new pats on the already-double pats (figure 1.3a). When the color pattern has been repaired and the long sought after flexing has been performed, a new surface or, as it is called, a new "side", will be exposed.

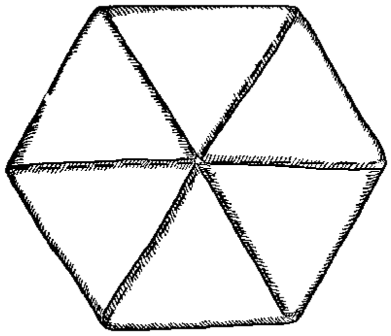


Fig. 1.3a

This seems natural enough in itself, but it leads to an interesting conclusion. Whenever we come to a position from which we can flex along only one set of hinges, we can always rotate an extra 60° between flexings, then add in three new leaves and flex anyway. Therefore, we can now build a flexagon of any arbitrary size, with the sides appearing in any desired order. As a matter of fact, we could say that the three "sided" flexagon was derived in this manner, by slitting, from an ordinary hexagon.

Now that we can build inductively any flexagon, let us reexamine the flexagon of "order" four –that is, the four "sided" flexagon we obtained by slitting the first flexagon. The experimenter may wisely have decided to cut a hinge and lay the flexagon out flat, exhibiting what is know as its "plan". The term "plan" refers to the pattern of the paper strip from which the flexagon is folded up. In this way can bee seen what has happened to the shape of the strip during the slitting. The strip is no longer straight. It is made up roughly "C" - shaped groups of triangles (see figure 1.3b). One triangle of each "C" may be folded over an adjacent triangle, similarly in each "C", to yield the straight strip of nine triangles that we know can be would up into a three "sided" flexagon. This operation is the inverse of the slitting operation.

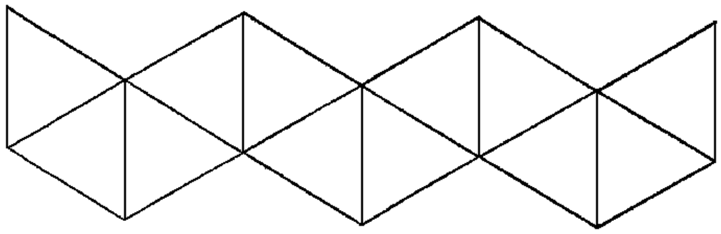


Fig. 1.3b

Now that we have learned the "plan" for this particular type of flexagon, we can make as many more like it as we please. If we build new flexagons by "slitting", or creating thumbholes, we will be able to increase the number of flexagons whose plans we know how to build to any desired extent.

Some flexagons will be easier to build than others, for the simple reason that their plans will be more easily cut out. There is a certain family of flexagons, for example, all of whose members are formed from straight strips of paper. When we folded the "C" of the order four flexagon into a straight strip, every third triangle of the straight strip came out double. If we double the first two in every three triangles (see figure 1.4), this corresponds to another slitting and results in a flexagon of order 5. If all nine triangles are doubled, the resulting flexagon is of order 6 (figure 1.5a). Opening out this last flexagon, we see that the uniform doubling of all the triangles produces another straight strip, twice as long as the first (figure 1.5b).

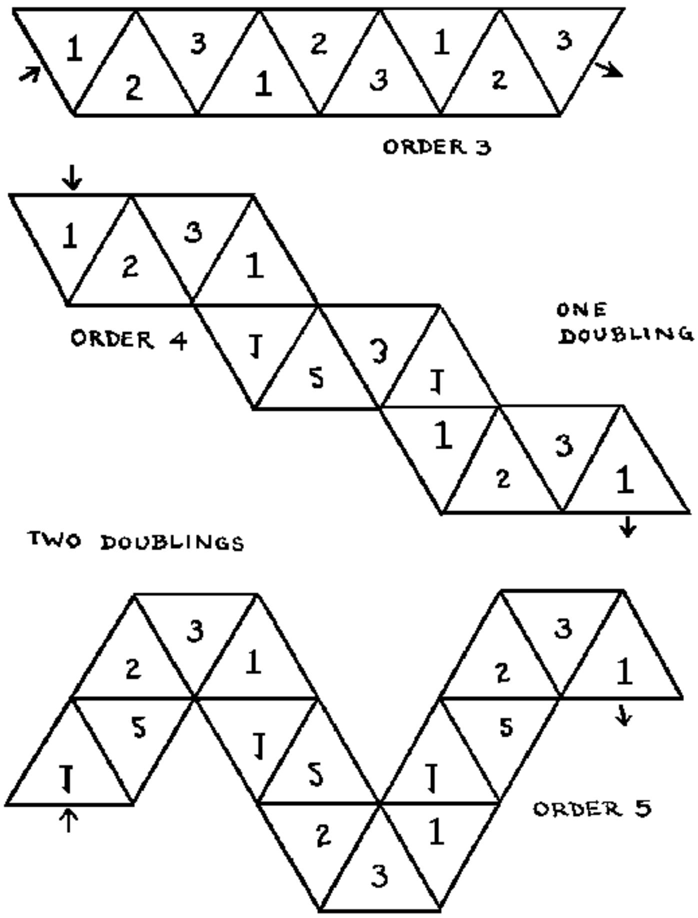



Fig. 1.4

Are there more "straight strip" flexagons? Clearly we could again double the number of triangles, obtaining a twelve sided flexagon made from a straight strip 36 triangles long. In this case, each of the nine triangles in the original strip would have become a stack of four. This doubling process may be continued until we run out of paper. Each flexagon will be of order $3 \cdot 2^n$, the number of triangles used will be $9 \cdot 2^n$, and the number of leaves heaped up under each of the nine final triangles in the straight strip will be $2^n$. The nth flexagon will always be formed from the $(n-1)st$.

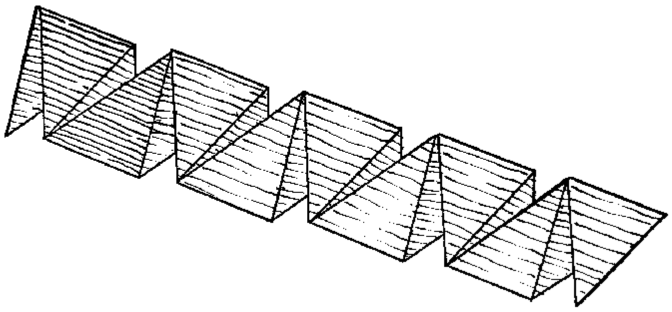


Fig. 1.5a

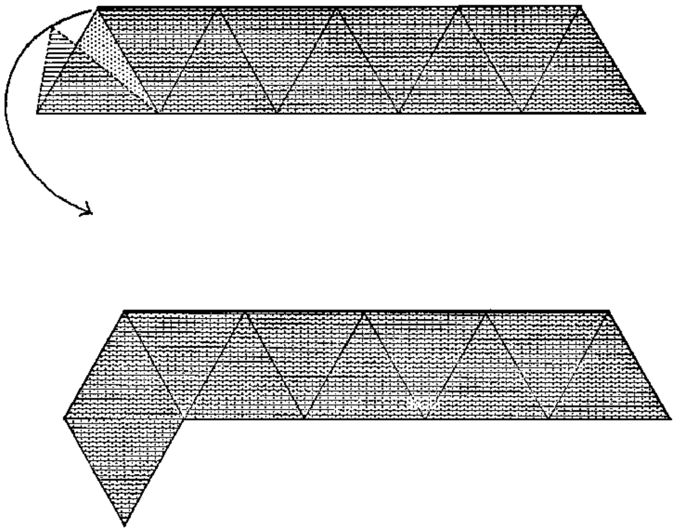


Fig. 1.5b

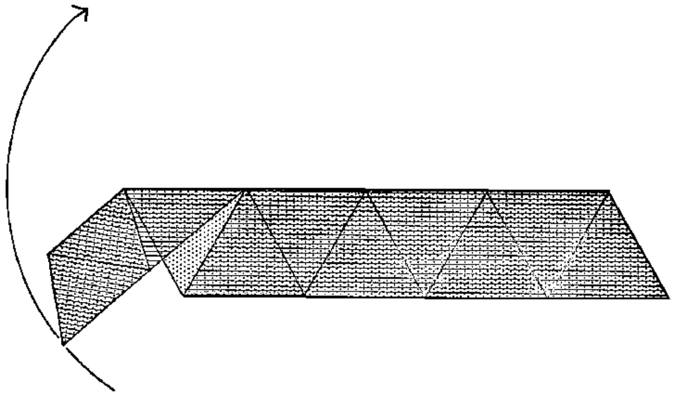


Fig. 1.5b2

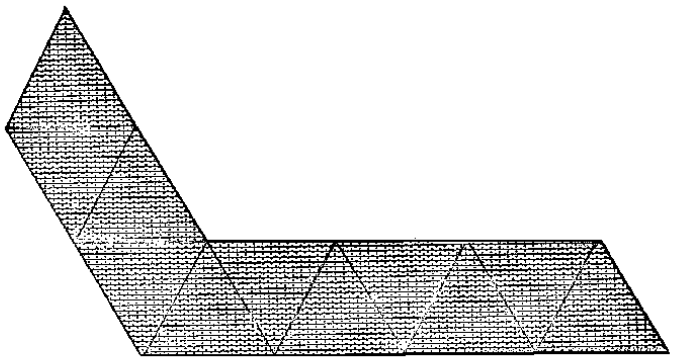


Fig. 1.5b3

Let us return to the straight strip of 36 triangles which has been twisted double twice to form the straight strip of nine heaps (order = 12). These heaps of four triangles each have an interesting property that allows us to create many more straight strip flexagons than we can using the doubling method alone. We observe that if we replace a heap of four triangles by a single leaf, or viceversa, the straightness of the strip remains unimparied (see figure 1.6). The significance to be found in this is that, at any three positions that will be separated by the same number of leaves when the ends of the plan have been joined, we can substitute a heap of four triangles for a single leaf, and still keep a straight strip plan. This adds three new sides to the flexagon. Therefore we can now construct flexagons having order of the form $3n$ from a straight strip.

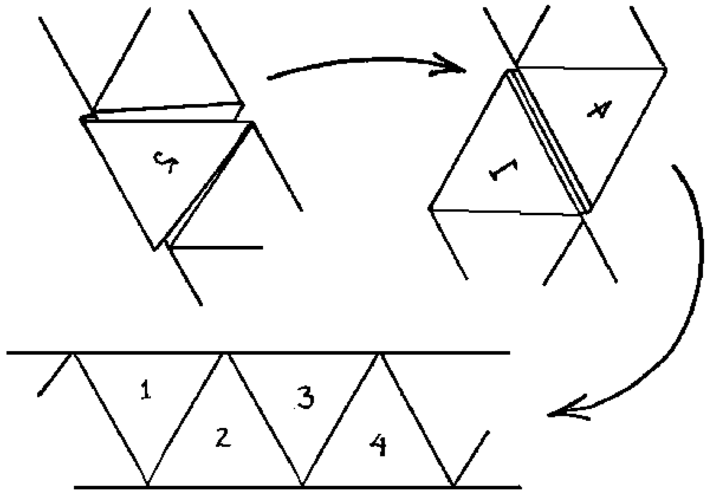



Fig. 1.6

An interesting approach to the non-straight strip flexagon is its consideration not as a slitting product of lower order flexagons, but as a higher order straight strip flexagon, some of whose sides have been unslit, i.e., glued shut.

"Straight strip" flexagons have in general been given the appellation "regular", although the only things distinguishing them from other, "irregular", flexagons are the ease with which they are built and a certain amount of symmetry which certain of them possess in respect to operation.

Now that any required flexagon can be built, certain purely practical problems begin to arise. Since these may be of interest to the reader, several such details have been brought forth and discussed in Appendix A.

We can at this point perceive a number of elementary relationships that are basic to further study of the flexagon. First, it is easily shown that the total number of leaves must always be three times the order of the flexagon, since every side of every leaf is used. This gives the six leaf faces required for each "side". Second, every flexagon plan must be reducible by folding to a straight chain of nine piles of triangles, from which the flexagon can be made. To make our third observation, we first place some form of marking device at each of the three corners of one or more of the top leaves of each side, as shown in figure 1.7. As the flexagon is now flexed and turned over, and the various sides are approached via various other sides, the marking devices, or "dots", indicate that different angles of the leaves are brought to the center of the flexagon at different times. Interesting results have been obtained with this phenomenon through the use of pictures or designs as "dots"; an example of this due to Alan Phillips appears in figure 1.8.

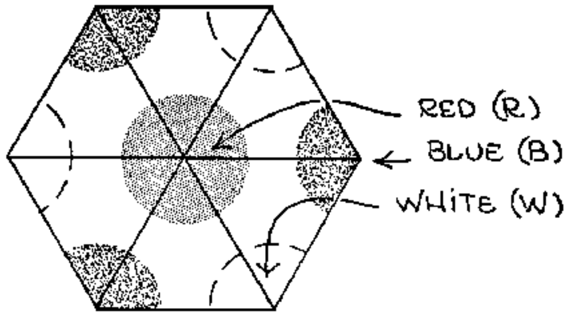



Fig. 1.7

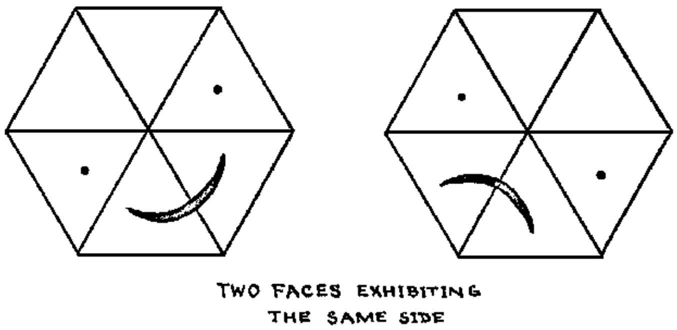



Fig. 1.8

Since there are two different kinds of pats, arranged alternately about the center of the flexagon, there are in a sense two different kinds of leaf faces making up each side. A flexing will then, fold together pairs of similarly colored leaf faces, but only one each of the two types, because it always folds together pairs of adjacent leaf faces. We are assuming, of course, that each "side" is marked with a single color. Also each color used in the flexagon is related to at most two other colors, in the sense that the two will occupy opposite faces of the same leaf. In fact, the number will always be two, if the flexagon has more than two sides. These two facts lead to the empirically substantiated deduction that, if one were to fold together one "side" of the flexagon, so that only one color showed, and then read off the colors in the order in which they would appear as one passed through each stack of triangles, this order would be found invariant, for any given flexagon. The side showing at each position where this constant order of colors was recorded would both begin and end the sequence in each case, but the colors would remain in the same positions relative to one another. This important observation should be carefully noted by the reader, since it provides a foundation for several later considerations. The immediate application is the observation that, if we are given a properly colored plan, we can, by folding together adjacent leaf faces which are colored the same, eventually assemble the flexagon. Faces of one selected color are not to be folded together; this will be the color showing when the process has been completed.

There remains a relatively well known but much dreaded aspect of the flexagon. This is the distortion of the heretofore constant order of the sides in the flexagon structure, which is usually accomplished accidentally, and which often results in a disassembly and consequent reassembly of the flexagon involved. The process is aptly called "distortion", for it involves forcing of the supposedly rigid leaves. The easiest method of distortion is that in which the flexagon is folded double in a downward direction along a diagonal (see figure 1.9a). The center is the opened out to form a six faced cup (figure l.9b). The two sets of leaves at one end of the cup are then folded in upon the middle pair (figure 1.9c). The flexagon now opens out disclosing a set of "sides" in which two pats are of one color and the remaining four of another (see figure 1.9d). This is a specialized operation, and it requires a specialized pat structure. In order to distort a flexagon in this manner, alternate pats must be made up of at least three leaves, with the thumbhole at least two leaves down away from the top of the pat. The remaining pat must be of at least double thickness. The flexagon of order 5 is the simplest flexagon that can be made to meet this requirement. However, a flexagon-like object which will not flex until distorted may be made from a straight strip of ten triangles. After distortion, it becomes of order 3. This is the object shown operating in figure 1.9a, if all the subpats A, B, C, etc. are of a single thickness only. Note that the requirements for doing this operation backwards (which is much harder) are not so demanding. A few more details on the subject of distortions may be found in Appendix D.

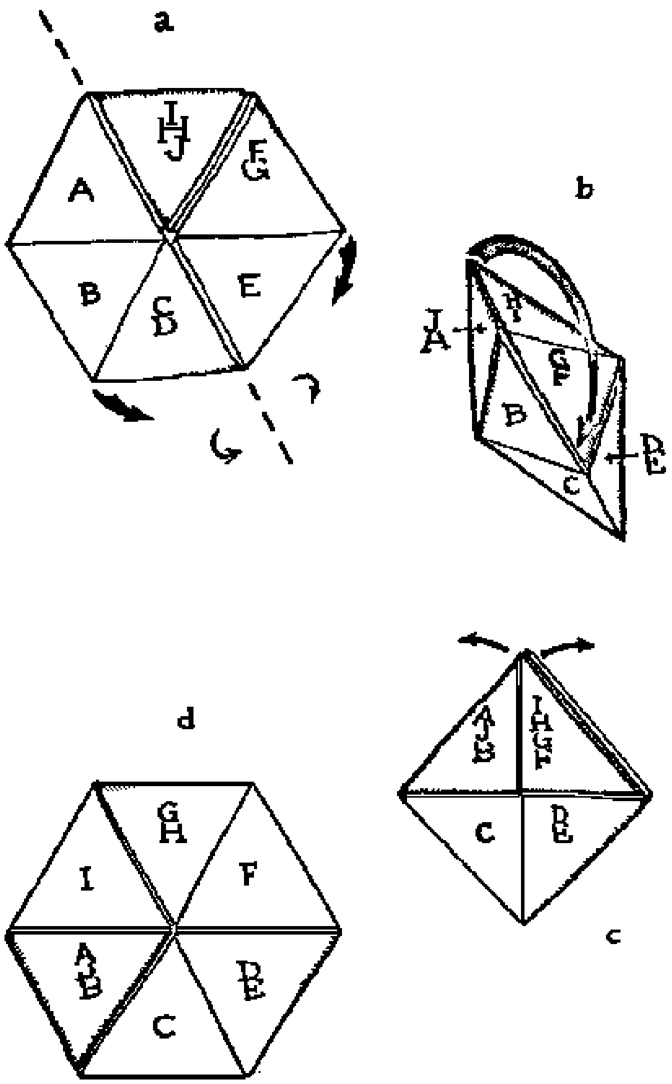



Fig. 1.9

# Chapter 2

# The Covering Space Representation of Flexagon Operation

Although the mode of building and operating the flexagon has been demonstrated, it may be noted that some further knowledge is necessary, for, through random flexing, it is found difficult to "visit" a desired side or even to find out if there are other sides. Our present interest is to devise a methodical system for finding any given side, useable in operating any flexagon.

As with flexagon construction, we start with the simplest true flexagon, which is of order 3. Due to the nature of the flexagon's operation, we visit the sides consecutively, and, unless the flexagon is turned over, always in the same order:

$$1\ 2\ 3\ 1\ 2\ 3\ 1\ 2\ 3 \ldots .$$

This is represented graphically by a triangle, as shown in figure 2.1. We visit side 1, then side 2, then side 3; then we go back to side 1 again. If the flexagon is turned over the sequence is reversed:

$$3\ 2\ 1\ 3\ 2\ 1\ 3\ 2\ 1\ \ldots .$$

It is clear that this also may be represented by figure 2.1. The problem now is this: If we are not sure whether or not the flexagon has been turned over, and we desire to know which side will next turn up, how can we tell? We may see that the side showing is marked 1. We can now read another number, say 2, off the back of the flexagon. Then the next side turned up from side 1 will be marked 3, since 3, 2, and 1 make up the triangular system of figure 2.1.

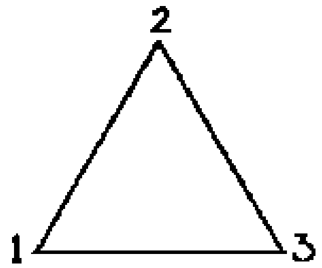



Figure 2.1

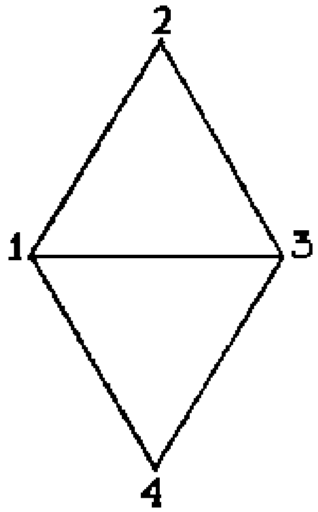



Figure 2.2

If we choose, we may instead try to flex the flexagon along the wrong pair of hinges, as mentioned in Chapter I. That is, we may decide to slit the three single pats and make a flexagon of order 4, in which we could also move from 2 on the back and 1 on the front to a new side, 4. It should now be noted that, in the process of flexing, the side that was last showing goes to the bottom surface of the flexagon, so that 4 will be on top and 1 on the bottom in the new position. The Moebius band type of operation between sides 1 and 4 was identical to that between sides 1 and 3, so that we should now be able, with another flexing operation, to obtain a new position, with side 2 on top and 4 on the bottom. Using this conclusion, we can construct a "map" of the new flexagon (shown in figure 2.2). The new map operates just as the simpler order 3 map did, and as all subsequent maps will; i.e., the side on the back of the flexagon, the side on the front, and the side next to come up are represented by the vertex of a single triangle. This essential property of maps will clearly be uninfluenced by simple transformations such as rotations, reflections, and such that may act upon it.

Any desired flexagon may be produced by the process of slitting along hinges. This corresponds to the addition of a new face on the map, between two others, in the manner just shown. Thus any flexagon built by slitting will correspond to a unique map, of a more or less complex type. It is thus possible to obtain the map of a given flexagon through experimentation, by seeing which positions of the flexagon admit of two possibilities for the next side to turn up, and noting down this information in map form. Then, by following routes which the map says are possible, we can expose any given side at will.

Doubling the straight strip from which a previous straight strip flexagon was made corresponds to adding in a new side between each pair of sides on the map of the old flexagon wherever possible, or adding on a new map triangle at each exposed edge of the map, as shown in figure 2.3. The overlapping the map of the flexagon of order 24 has no inmediate significance. It is worth noticing that the flexagon map leaves room upon its periphery for the point representing each side. Forms such as that shown in figure 2.4, in which triangle vertices are entirely enclosed, are impossible.

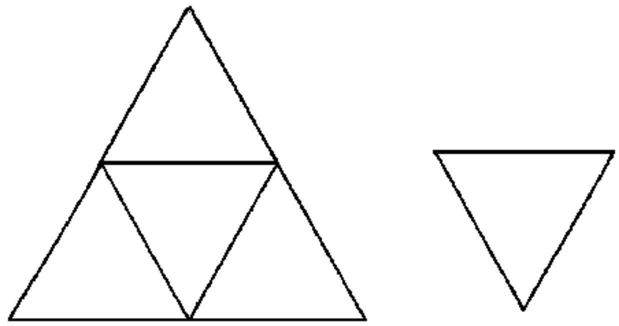


Fig. 2.3 a & b

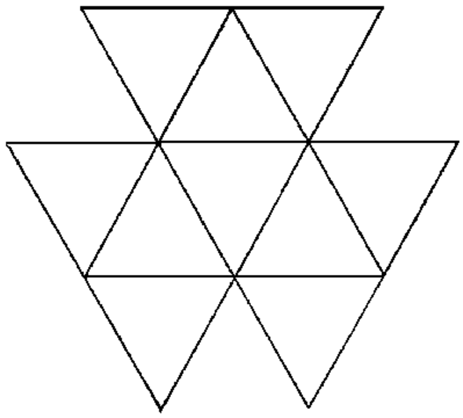


Fig. 2.3c

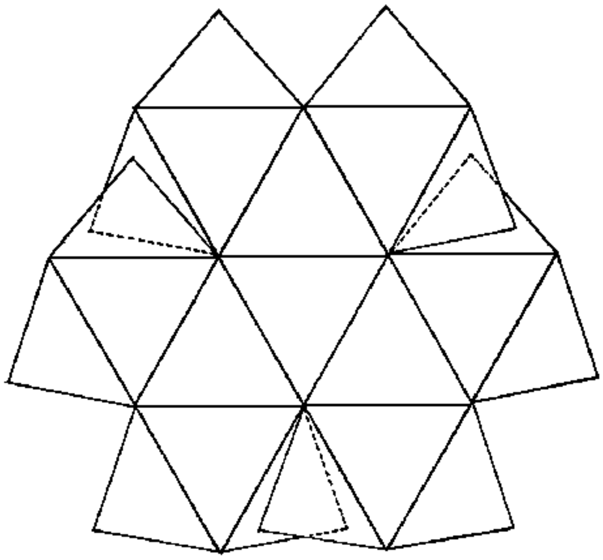


Fig. 2.3d

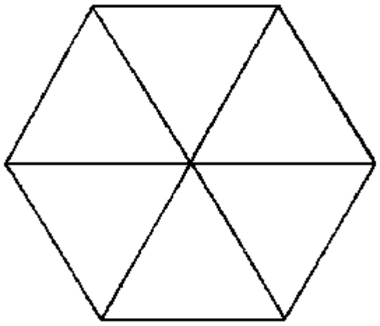


Fig. 2.4

The three-fold symmetry of the maps of the above examples of straight strip flexagons indicates a fine coloring system which makes memorization of the map an easy matter, hence speeding up the study or observation of any given flexagon. The three flexagon sides represented by the central triangles in figure 2.3 are painted red, yellow, and blue, respectively. The other flexagon side in the map triangle containing both red and yellow is painted with a mixture of the two; orange. Similarly, other sides are colored by mixing colors of sides already painted (fig.2.5a). In this manner as many colors as the eye can distinguish from one another may be painted on large flexagons. The same scheme may also be used on flexagons other that straight strip flexagons, these then being treated as deletions of higher order straight strip flexagons. An example of this coloring scheme is given in figure 2.5b.

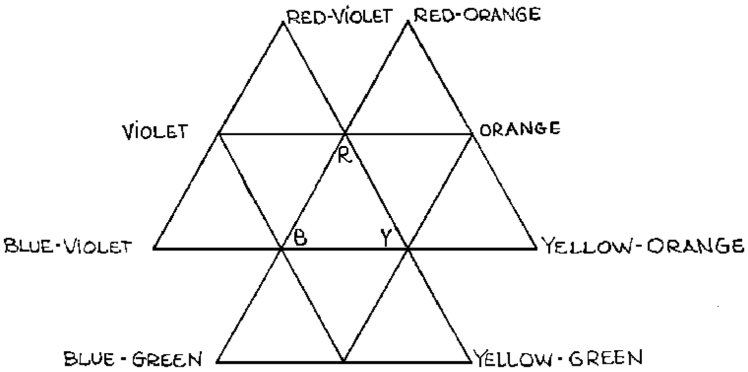



Fig. 2.5a

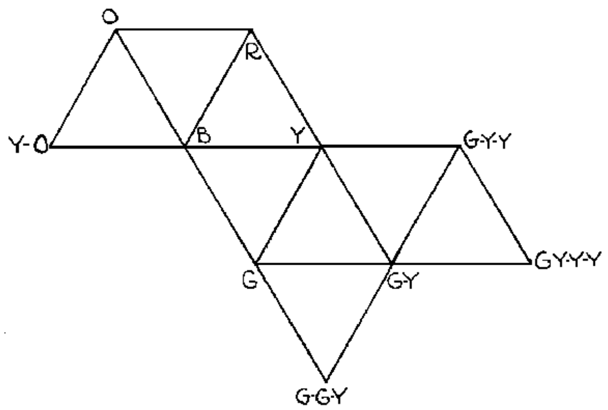



Fig. 2.5b

Suppose that a 12 sided straight strip flexagon has been made and colored, and that we desire to see the yellow orange side. Suppose, moreover, that the sides now showing are blue green, on top, and green, underneath. The path we would then follow to reach yellow-orange would be that shown in figure 2.6.

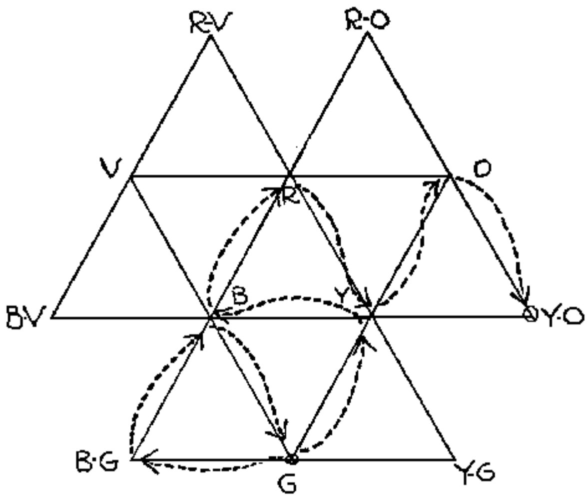



Fig. 2.6

The map system of flexagon representation gives us a method which determines any flexagon uniquely by means of an array of triangles. If we further let each map triangle be replaced by its midpoint, and then connect the midpoints as the triangles were connected, a branched affair results which is also characteristic of the respective flexagon. This is know as the Tuckerman tree, named after Professor L. B. Tuckerman, Sr., its inventor (see figure 2.7).

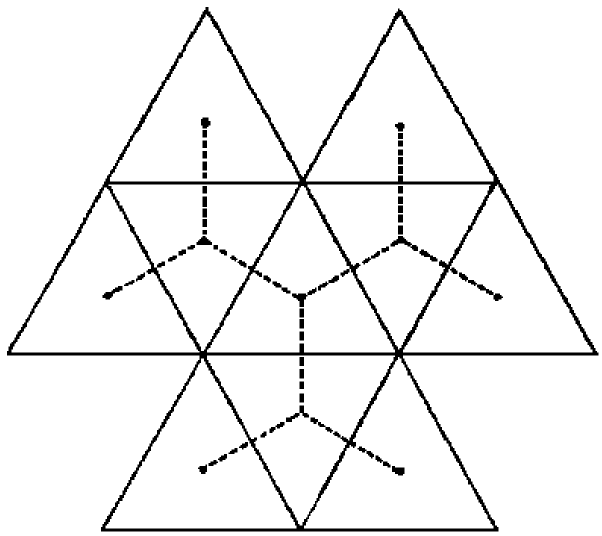


Fig. 2.7

Each operation or group of operations performed on the flexagon can be represented in terms of each of its various "maps" The chart shown in figure 2.8 gives each of the basic operations and its various representations. The Tukey triangle network, or hinge network, is included, although it is to be discussed in Chapter III. The rules used in applying these other representations of the flexagon operations to working the flexagon may be derived directly from the rule used for the map, which is that the points representing the three sides shown over the course of one flexing, including sides shown on the bottom of the flexagon, must lie at the vertices of a single map triangle. The term "path" in figure 2.8 refers to a line segment joining two adjacent points in the tree or map.

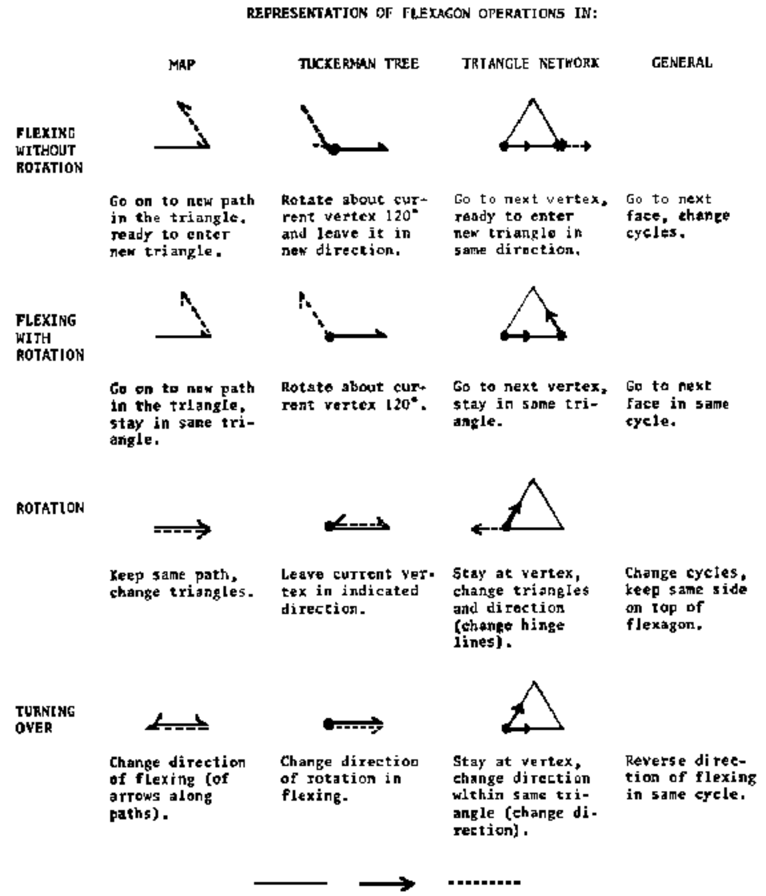



Fig. 2.8

Since the various forms of the map merely provide different graphical representations of the two operations of flexing and rotating, it can be generalized that every flexagon may be characterized as a space in which we can perform certain unique combinations of these operations. Every flexagon, then, can be described as some portion of a covering space that represents all combinations of these operations. Such a covering space may be constructed as follows: Within a circle, construct three arcs tangent to each other at the circumference, as shown in figure 2.9a. To fill the covering space we reflect all that lies without the circle of any given inner arc through the inner arc into the circle out side the arc (see figure 2.9b). If this reflecting operation is repeated indefinitely upon all the arcs produced, the reflections will approach the circumference of the original circle. A later step in the production of the covering space is shown in figure 2.9c. When the curves making up some portion of this covering space are bent straight and made equal in length, a flexagon map results in each case. Thus the covering space as a whole represents a hypothetical "ultimate" flexagon, from which any other flexagon could be produced by deleting various sides.

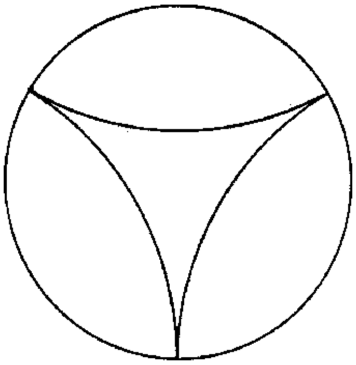


Fig. 2.9a

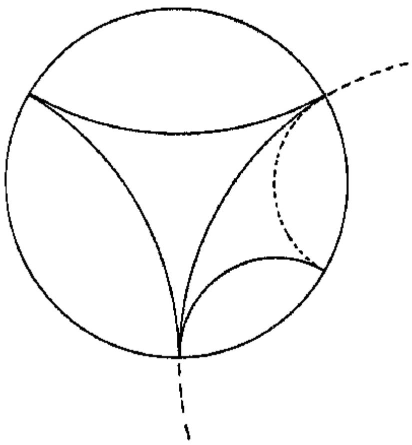


Fig. 2.9b

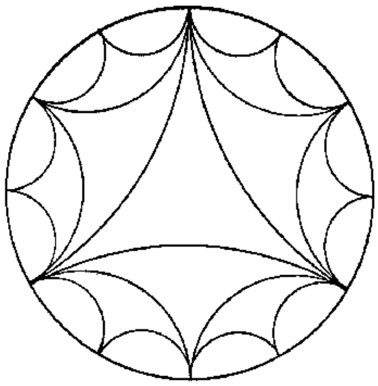


Fig. 2.9c

To interpret the covering space, we notice that the reflecting operation, which produces an identity when performed twice in succession through the same mirror or arc, represents flexagon rotation, which also produces an identity when performed twice successively, and the choice of a curve through which to reflect, which is equivalent to rotating from arc to arc within any one small area of the covering space, and thereby produces an identity when performed three times successively, corresponds to flexing, which similarly produces an identity when performed three times in succession. This device indicates that, as we perform the operation of rotating the flexagon or splitting it to find a new side, which operation we identify with a new reflection in the covering space, we enter a new area or subspace within the covering space. To enter a space within another arc of the covering space, we must first regain entry into the old space. To demonstrate what is meant by this, we may refer to the traverse of the order 12 flexagon which was discussed earlier. From the outer space (BG - B - G; see figure 2.6) it was first necessary to reenter the intermediate space G - B - Y and then the central space R - Y - B before reaching yellow orange, which lies between red and yellow; i.e., within a different primary arc on the covering space. We see from symmetry that the primary or original area in the covering space can be identified with any area in the covering space, since, when the covering space is completed, all are equivalent.

It will be noticed that one can never travel in both directions along the same path in a map without turning the flexagon over. That is, if some side A is on top and another side B is on the bottom, the only way in which B can be placed on top and A on the bottom is by turning over the flexagon. Therefore, for any given position, and eliminating the operation of turning over, a system of arrows can be drawn along the various lines in the map, showing the direction of travel on these paths. In the map shown in figure 2.10a, side 1 can be approached from at most three different sides. The arrows indicating this are drawn as shown. If the flexagon is turned over, all arrows are reversed. Just three approaches are still possible in the present case (figure 2.10b). Note that marked paths connecting a given map vertex with the adjacent vertices are directed alternately toward and away front the given point.

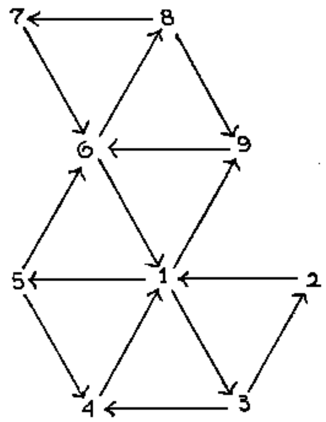



Fig. 2.10a

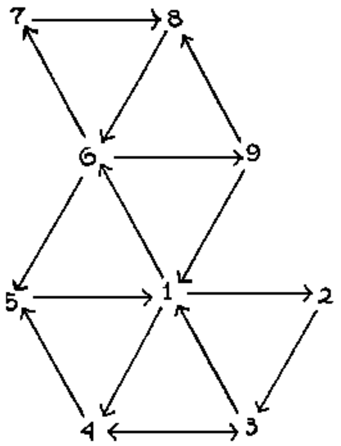



Fig. 2.10b

Now let us follow an individual angle of a leaf as the flexagon is operated. For this purpose, a dot is placed at the vertex that is located at the center of the flexagon, of one of the leaves showing at some face. when the flexagon is then flexed, the dot is then found to lie on the outside edge of the flexagon. With every rotating and flexing operation, the dot is moved about the midpoint of the leaf 120° more in respect to the center of the leaf. If flexing is performed without rotation, the dot is moved back 120° to its previous position. This is explained by the fact that, with a continuous flexing operation, in which the same hinge is maintained between pairs of changing pats, the only possible change in the angle of a leaf toward the center always switches the two at the ends of the remaining hinges. These angles pass between alternate stages of being at the center and at the edge of the flexagon. When the constant flexing is interrupted, the third angle goes to the center. To make all of this easily seen in a flexagon, fold all the pats together, forming one large pat-like structure, and dip each angle in ink or dye, so that each is a different color from the others. When completed and opened out, such a flexagon should resemble figure 1.7. Suppose, for example, that the colors used for the dyeing have been red (R), white (W), and blue (B). Then a map may be drawn in which the color at the center of the flexagon for each position is indicated at the corresponding position on the map. Such a map is shown in figure 2.11. It can be shown, using the facts mentioned above, that all parallel lines in the map will be of the same color. It can now be found at what time any given dot will lie in any given position. This flexagon further demonstrates that, just as the order of the leaves is constant, so is the relative position of the angles and hinges.

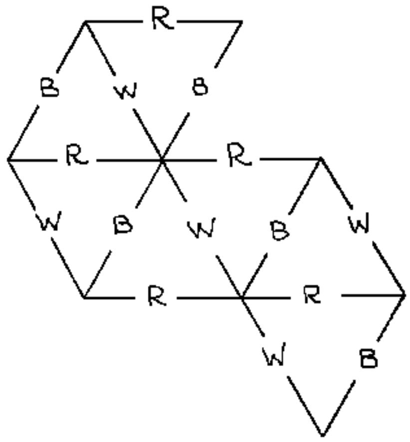



Fig. 2.11

The simplest method, independent of the map, or when the map is unknown, or finding every face of a given flexagon, and, in so doing, of traveling over every path in the map of the flexagon in the shortest possible time, is a method know as the Tuckerman traverse, named after Bryant Tuckerman. The rule is that flexes are to be performed consecutively as long as possible, followed by a rotation only when necessary. It can be seen that, in a flexagon of order three, the only possible sequence of operations is the original 123123... , and hence this will be the path followed by the Tuckerman traverse. Following the rule in higher order flexagons, we discover that the paths traversed during one "roll" (one uninterrupted sequence of flexes, along a given hinge) form the pattern shown in figure 2.12 in the map, first turning one way and then the other.

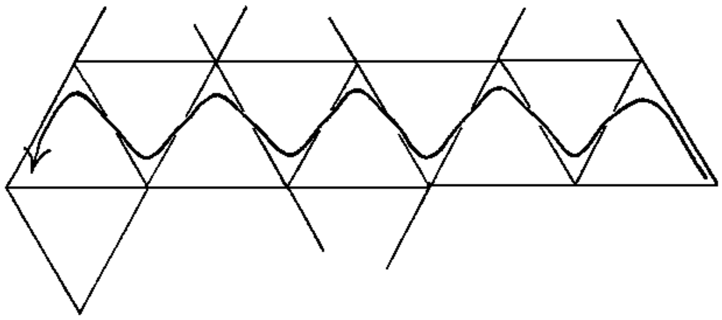


Fig. 2.12

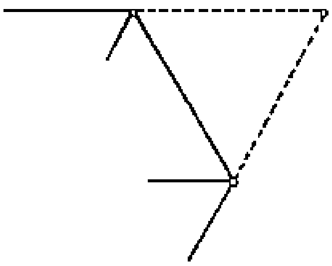


Fig. 2.13a

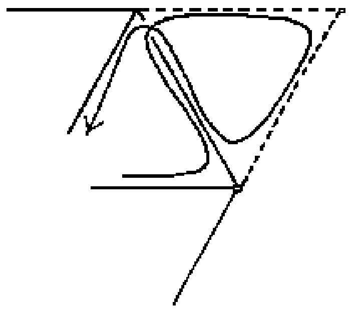


Fig. 2.13b

Each higher order flexagon may be formed from a flexagon of lesser order by the addition of pairs of paths such as that shown in figure 2.13a. The traverse follows each of these pairs in the manner shown in figure 2.13b. Then three extra flexes and one extra rotation are required for each pair added (Fig. 2.14). Since three flexes and three rotations are required in the flexagon of order three, it may be stated that, for any given flexagon, every side may be visited and every path used in $3N-6$ flexes and $N$ rotations, where $N$ is (as throughout) the order of the flexagon. Also, since each side added requires that one half twist be added into each unit of the flexagon, the number of half twists in the band making up the flexagon will be $3N-6$ (no twists are needed for the first two sides), which happens to be the same as the number of flexes used in following the Tuckerman traverse. In the course of the Tuckerman traverse, some sides will be noticed to turn up oftener than others. This is because they are touched by more paths in the map.

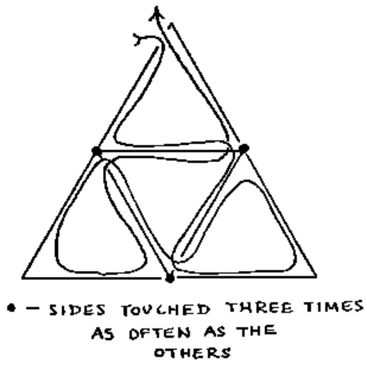



Fig. 2.14

# Chapter 3

# Maps and Plans

There is a relationship between the map and the plan of the flexagon. If the plan changes, the map also changes, and usually the converse is also true. The number of leaves in the plan of one unit equals the order of the flexagon, which is equal to the number of vertices in the map.

Since all the thumbholes pass through the flexagon in a given direction, clockwise or counterclockwise (the direction of winding up the flexagon is constant for a given flexagon), if it is impossible to flex without rotating at any given position, this means that the pat that would have to be slit in order to allow us to proceed is of a single thickness. On the plan, then, the new color would be represented by a pair of leaves, on the backs of which would lie the two colors representing the two sides which the flexagon had exhibited before flexing. This is shown in relationship to the map in figure 3.1. It may be seen that this relationship provides an inductive method for building a flexagon having any desired map. First, however, it is convenient to have a system of orientation for working with the plan. We orient the triangles in the following manner: Draw arrows about the sides of one plan triangle, head to tail, and, for the surrounding triangles, draw arrows about each of these triangles in the direction indicated by the arrow on the side of this triangle which is common to it and the first triangle (see figure 3.2). Then, a triangle is called "+" if we leave it, in traveling along the plan, in the direction indicated by the arrow which is crossed in entering it, and "-" if left in the opposite direction. It can be seen that if, in any given situation, all plusses are exchanged for minuses and viceversa, the array of triangles remains unchanged (though a mirror image case may result), since the choice of the original "+" direction is arbitrary. Notice that the strip of triangles corresponding to the sequence of signs $++++\ldots$ or $---\ldots$ is a straight strip. Several other examples are shown in figure 3.3.

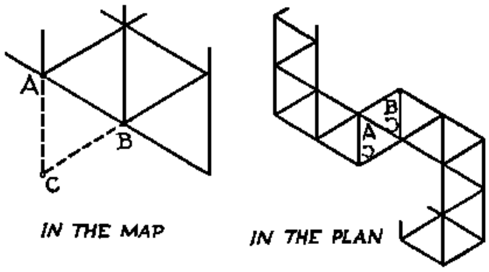
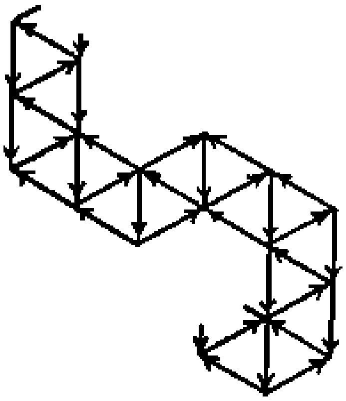



Fig. 3.1



Any plan is part of the triangular lattice constructed as below:

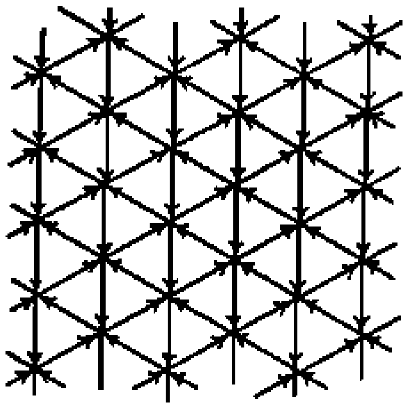
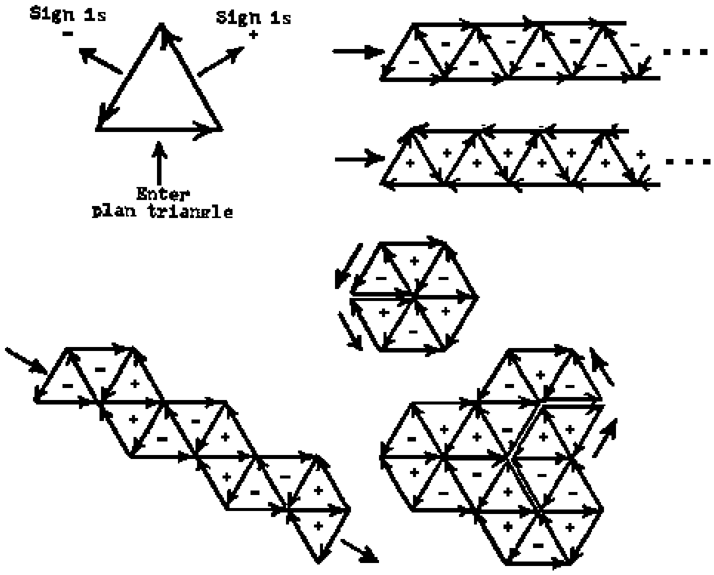


Fig. 3.2

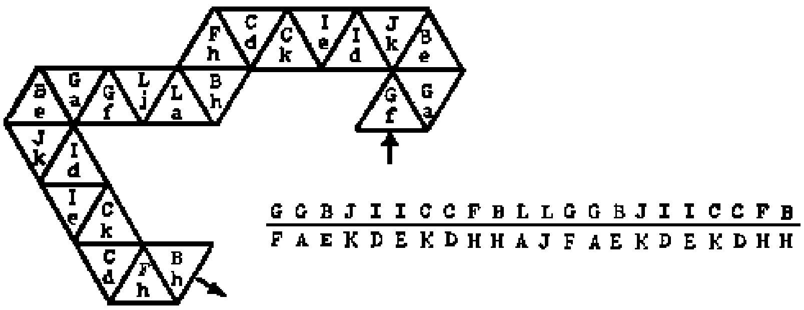



Notice that in reading along the plant backwards
all $+$ signs are exchanged for $-$ and $-$ for $+$.

Fig. 3.3

In adding a side to a flexagon, i.e., in slitting, two minus triangles are substituted for one plus (or two "+" for one "−"). The hexagon, or flexagon of order two, has the sign sequence $+-+-+-$ for its plan. The flexagon of order three, whose plan is a straight strip of nine triangles, has the sign sequence $+++\ +++\ +++$, or, for the unit plan, $+++$ (since all units are identical, usually the sign sequence of only one unit is given). For the flexagon or order four, the (unit) sign sequence is $++--$. Notice that in obtaining each of these sign sequence from the previous example, either a $++$ sequence was substituted for a $-$, or a $--$ sequence replaced a $+$.

Next we must invent a system for recording the "color" or number labeling each side of each leaf. This is done by drawing a line and placing number falling on opposite sides of the plan, in order, on opposite sides of the line. Numbers on a given leaf will appear opposite each other across the line. This system is illustrated in figure 3.4. In substituting in a new side, one number in the split leaf remains stationary and the number opposite it becomes adjacent to it, in the new leaf. The two numbers representing the new side are placed together opposite the old numbers. For example, to find the number sequence corresponding to the flexagon of order five by adding a side to the flexagon of order four we would proceed as in figure 3.5. When one of the original numbers forms a pair of like numbers with another number in an adjacent leaf, the pair should be broken in the number moving process. The reason for this is seen if we observe a splitting operation carefully. The operation occurs at a single thickness, which is attached to the top of the second pat where the new thumbhole is to be created (see figure 3.6). Hence it and one of the next few leaves in the plan, which is the top leaf in the next pat, have one "color" or number in common. When the new thumbhole has been made, the side of the single pat with the common color upon it will now no longer be connected to the top portion of the second pat. When there are no common numbers in leaves adjacent to the given leaf, the given leaf will at least lie <u>between</u> the two leaves that share its colors or numbers with it (the number sequence being considered as cyclic). Then the two numbers on the given leaf should be separated as much as possible from the same numbers on other leaves. This is a generalization of the previous example.




Small letters indicate that letter appears on the back of the leaf.

$$\frac{...A...}{...B...}\ \frac{\text{in splitting}}{\text{produces}} \rightarrow \left\{ \begin{array}{c} \frac{...A\ B...}{...C\ C...} \\ \text{or} \\ \frac{...C\ C...}{...B\ AS...} \end{array} \right.$$

Fig. 3.4

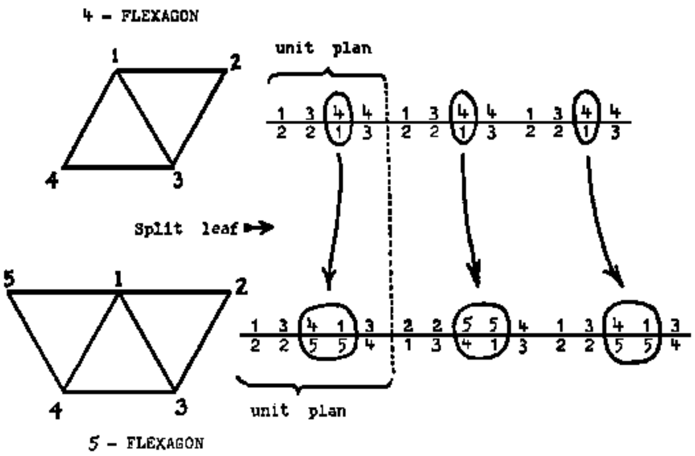



Fig. 3.5

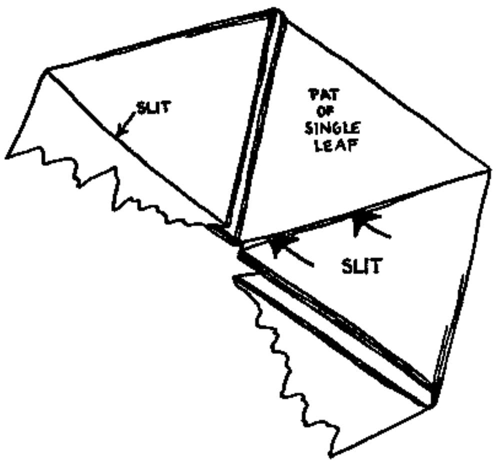
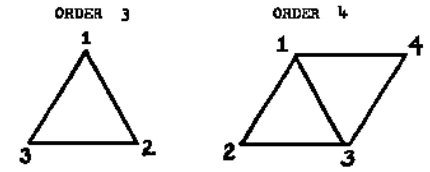



Fig. 3.6

Whenever a leaf is added, the remainder of the sequence to the right must be inverted, to retain the same relative position with the original numbers on the given split leaf. This means that the second unit must have a number sequence inverted in relation to the first if the number of leaves, $N$, is odd. The reason for this is the fact that then $N-2$, the number of folds in the finished flexagon unit, will be odd. Thus odd order flexagons are (one-sided) Möbius bands of $3N-6$ half twists (one of the possible half twists is left out to make each inter-pat hinge).

Combining the sign sequence and the number sequence for the flexagon or order three and for the order four flexagon, we find the results shown in figure 3.7. The plan of a flexagon of order 15 has been constructed for the reader by this method in Appendix B, to illustrate its useful application.




$$\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ \hline 2 & 2 & 1 & 1 \end{array} \qquad \begin{array}{cccc|c} 1 & 3 & 4 & 4 & 1 \\ \hline 2 & 2 & 1 & 3 & 2 \end{array}$$

$$\underbrace{+\ +\ +}_{unit}\ + \qquad \underbrace{+\ +\ -\ -}_{unit}\ +$$

Fig. 3.7

This system can be used to produce any flexagon, but it is nevertheless very complex and bulky. The expenditure of effort can be reduced considerably by using a device known as the Tukey triangle network. This network is actually a map of the hinging of the flexagon.

The traversing of a path in the map was initially said to represent a flexing operation. However, there is another way of looking at this. Each path can be said to represent that position of the flexagon at which the sides at either end of the path in question are the sides exhibited con the front and back of the flexagon in the given position. In this case, flexing is the changing from one path to another and rotation is represented, as before, by the choice of where to go next. Let us now represent each path by its midpoint only, and then let us connect these points in the various ways that the flexing operation can be carried out, i.e., by lines between the midpoints of the sides of each map triangle (see figure 3.8). A sequence of flexes uninterrupted by a rotation is now equivalent to a long straight line in the new network. During a repeated flexing operation only one hinge is used to join pats at other places (inter-pat hinges) than the thumbholes that are used. Hence one line in this new network, called the Tukey triangle network after its inventor, John Tukey, represents the use of one hinge for a number of flexing equal to the number of paths which have been used in the network. It may be convenient at this point to refer to the chart given in figure 2.8.

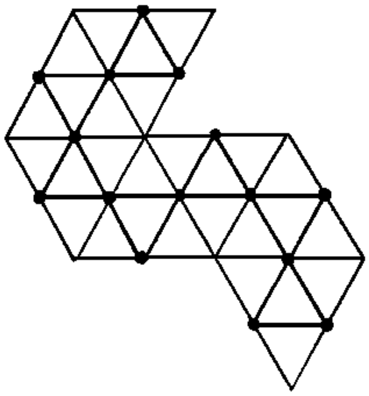


Fig. 3.8

It is interesting to notice that a hinge folds together different directions on opposite sides of its network line in the Turkey triangle network.

This network follows exactly the Tuckerman traverse for any given flexagon. That is, the path in the network obtained by traveling along one line to its end and then traveling along the line meeting it at its end to a third line, and so forth, is identical to the path followed when the Tuckerman traverse is used. The network then has just $N$ lines, one for each hinge in the unit. Since the ends of the various network lines are the only positions at which new sides can be added, and since these are the spots where further flexing cannot take place without a rotation first, these are the only places at which one of the two pats in each unit is a single thickness. Being unable to flex without rotating is an indication of a pat of single thickness - the absence of a thumbhole - in every case. Now if we travel to a single pat, then rotate, then travel along the network in a straight line until we reach another single thickness, we have actually shifted from one hinge along the plan to the next, for the rotation changes the constant hinge from the hinge on one side of the single leaf to that on the other side. Using this new hinge we have traveled to another single thickness, which can only be the next leaf in line along the plan, for it has a hinge in common with the last single leaf. Thus we visit consecutively all $N$ leaves in the plan in a calculable manner. To see this it may be handy to experiment with a flexagon in which consecutive leaves in the plan are labeled with consecutive numbers. When the Tuckerman traverse is performed upon this flexagon the preceding result should become apparent.

Suppose we are given a flexagon, and we want to find its number and sign sequences without dismantling it. Since we know how to encounter consecutive plan triangles, all we need to do is to copy down the two numbers or colors found on either side of each single thickness in the order we meet them. Thus we can identify the numbered leaves in the plan with identifiable faces of the flexagon. Our next problem is to identify these faces with corresponding positions in the map. These positions of course are those at the ends of the hinge lines in the Tukey triangle network. Furthermore, the order in which these faces are encountered is, of course, the order found in following the network. Each leaf of the plan, face of the flexagon, or vertex of the hinge network corresponds to two sides. It is our next problem to devise a method of labeling these sides. For the sake of consistency and convenience, we adopt the following convention: the sides are numbered consecutively from 1 to $N$ about the outer edge of the Tukey triangle network, each side corresponding to a vertex of the map. We can now see that for each side there are two single-leaf flexagon positions, since each map vertex is at the periphery of the map and is therefore bordered along the outside by two paths radiating from it. Furthermore, each of the faces represented by these map boundary lines, which correspond to network vertices, can be labeled by two consecutive numbers, denoting the sides showing at the time that the given face is exposed. Thus we tie the labeling of the map to the labeling of the plan. As yet, however, we do not know which of these two consecutive numbers will be placed on which side of the leaf in the plan. For the sake of convenience we will momentarily employ the convention of using only one of these two numbers if possible; specifically, the lower one. In this way the number for both sides will be determined using only one piece of information; the unnamed side for each face is always the one named plus one.

Notice that during one "roll" (the repeated use of one set of hinges for flexing as long as possible without rotating) the two leaves in each unit which make up the single-leaf pats in the positions at the beginning and end of the roll do not change pats at any time during the roll. However, with each flex all the pats are inverted, with the exception of the portion of one pat which changes pats in the flex. Therefore the relative position (right side up or upside down) of the two connected single leaves remains unchanged. Then if we can derive a statement as to which side of the plan the lower number in the second face will occupy in relation to the side of the plan occupied by the lower number in the first face, this relationship will always be true.

The numbering of the sides is in general like that shown in figure 3.9. Suppose we start at position $a$ and proceed along hinge line $b$ to its end. Then two outcomes are possible: If it takes us an odd number of flexes to reach the new position $c$, the arrows directed along the paths of the map indicate that the lower side number in position $c$ will occupy the same physical surface ( top or bottom) as the lower side number of the original position, $a$. The opposite will be true if there are an even number of flexes in the process; the new lower number will be on the opposite surface (top or bottom) of the flexagon. Also, we must consider that with each flex both single pats are inverted, as we have seen before. Thus although for an odd number of flexes between consecutive single-pat positions it would seem that the lower-numbered sides would be not only on the same surface of the flexagon but on the same side of the plan, this second factor, by inverting the plan in the odd case, acts to cancel the effect of oddness; in both even and odd cases the lower-numbered side of each leaf will lie on the opposite side of the plan from the lower-numbered surface of the preceding (adjoining) leaf in the plan. We finally see that the lower side numbers of the single pat faces will appear on alternate surfaces of the plan, from one leaf to the next. Then if we are given these lower numbers, we can recreate the number sequence of the flexagon rather easily, by placing these numbers on alternate sides of the line and then filling in the missing numbers with the numbers consecutively following those across the line. This filled-in number represents the side in each face that was momentarily neglected.

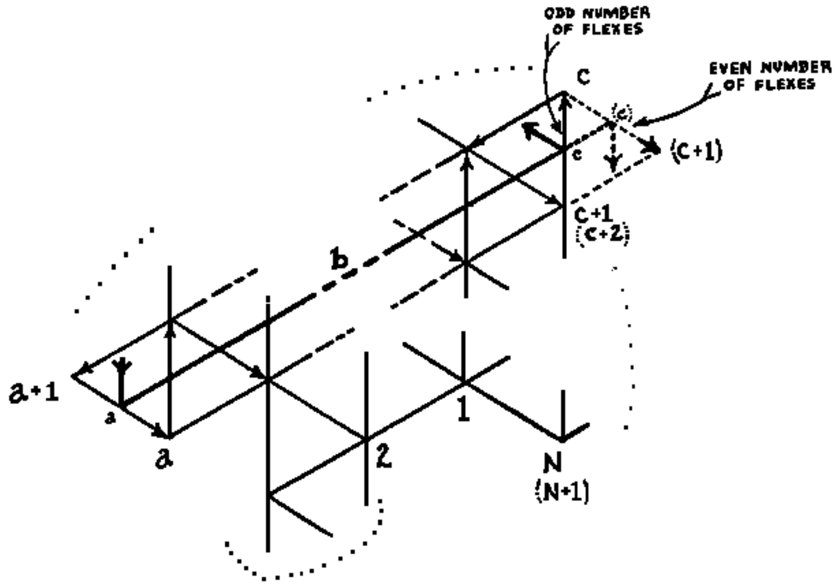



Fig. 3.9

To determine the lower numbers and the order in which they fall, it is only necessary to follow the hinge network and to copy down the lower number at each vertex in the order they are arrived at. This process is simplified if the vertices of the Tukey triangle network are themselves numbered with the lower number of the vertex only. This is easily done, as shown in figure 3.10. Thus, if we are given any map, we can now construct the number sequence of the flexagon plan having this map. Upon closer analysis it will be found that this method of obtaining the number sequence is quite analogous to that developed previously, although it is considerably simpler.

We have yet to consider the problem of the actual shape of the plan; that is, the derivation of the flexagon's sign sequence. To obtain this from the map we again use the Tukey triangles.

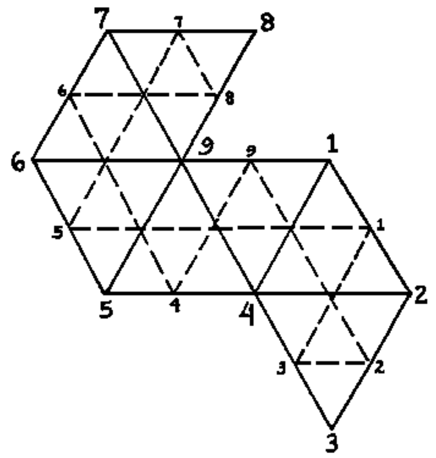



Fig. 3.10

Reference was previously made to the fact that alternate flexings invert any given leaf, provided it be one that is not moved between pats in the process. This means that if one group of hinges is used consistently, as in a roll, the two pats in each unit will rotate 180° about the axis formed by that perpendicular bisector of the constant hinge in each unit which passes through both of the unit's pats (see figure 3.11). Let that angle of each of the leaves which is at the flexagon's center at some given time be labeled "A". Then when a flexing (and thereby a rotation of the pats about the aforementioned axis) occurs a new set of angles, "B", will be found at the flexagon's center. With repeated flexing each of these angles will appear alternately at the center and at the outer edge of the flexagon, as was seen in the last chapter. If we maintain the same constant hinge, there are now only two possible central angles for single pats. Let us assume that at one of the two single-pat positions the angles "A" are at the center of the flexagon. Then the angle between the two hinges bounding the single leaf will be labeled "A". We now flex along a single Tukey triangle network line to the other single pat. If the number of flexes used to reach this position was even, the new single leaf will also have angle "A" toward the center. If the number of flexes was odd, the angle between the two hinges is marked "B". We now recall that these two single leaves are connected by the constant hinge. Then we can determine the hinging of these two triangles in relation to one another, since we know that the first single leaf is hinged at angle "A" and that angle "A" of the first has the same vertex as angle "A" of the second (as angle "B" of the first has the same vertex as angle "B" of the second). This leaves two possibilities for the final hinging of the second leaf, as seen in figure 3.12. We can see from this diagram that the first alternative (even number of flexes in the roll) corresponds to a pair of unlike signs in the sign sequence and that the second possibility (odd number of flexes) corresponds to a pair of like signs in the sign sequence. Thus the sign sequence can be determined by the evenness or oddness of the number of flexes connecting consecutive single-pat positions. We must now find a way of determining this from the Tukey triangle network.

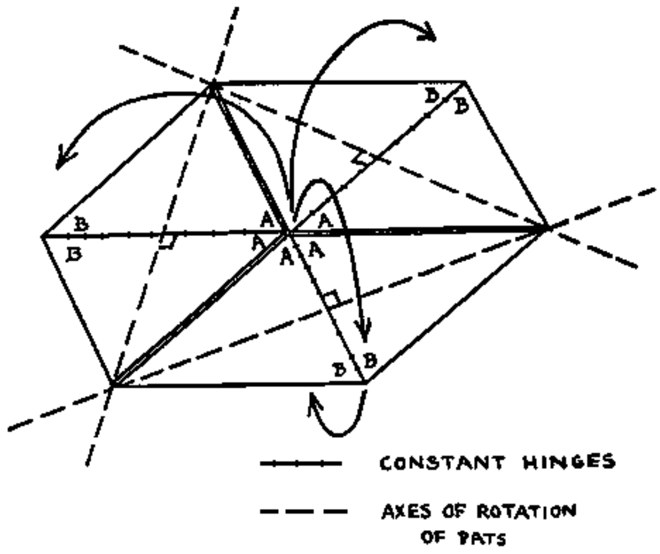



Fig. 3.11

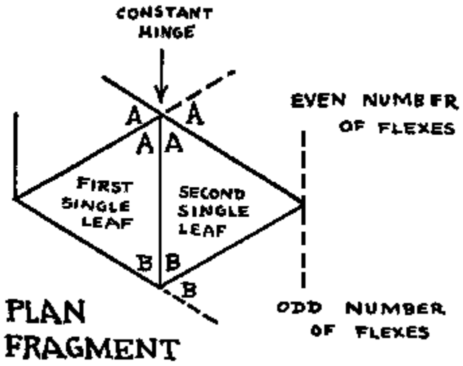



Fig. 3.12

The number of flexes along a given hinge line is equal to the number of map triangles traversed. If the number is even, we leave the hinge line on a line parallel to that from which we entered it. If then we consider the angle between the hinge line by which we approach a vertex and the hinge line by which we leave the vertex, an even number of flexes between the vertices will make the angle at one vertex the negative of the angle at the other (see figure 3.13a). Conversely, odd numbers of flexes between vertices involve equal angles (figure 3.13b). We can now determine the evenness or oddness by letting a clockwise angle between hinge lines represent say a positive leaf, a counterclockwise angle a negative leaf. Each sign may be placed in the network at the vertex which determines its direction; this is the vertex of the appropriate leaf. If a sequence is now made of the signs encountered in passing through the hinge network, in the order they are come upon, this sign sequence will be identical with that produced by the more difficult method developed earlier.

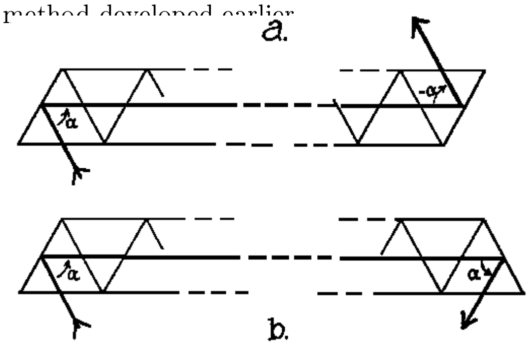



Fig. 3.13

Now that we have worked out this new method, we can do the same problems as before in a much simpler fashion. Choosing as an example the same flexagon of order 15 treated in Appendix B, we first draw the Tukey triangle network and name the vertices with the lower of the two numbers at either side, so that 1 can be added in completing the number sequence (see figure 3.14). Next, traversing the Tukey triangle network, starting at one, and numbering alternate leaves upside down, we get:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 12 | | 13 | | 8 | | 10 | | 14 | | 2 | | 15 | |
| | 4 | | 11 | | 7 | | 6 | | 9 | | 5 | | 13 | | 1 |

We now add 1 to each member to obtain the numbers lying on the other side of the leaves:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 12 | 12 | 13 | 8 | 8 | 7 | 10 | 10 | 14 | 6 | 2 | 4 | 15 | 2 |
| 2 | 4 | 13 | 11 | 14 | 7 | 9 | 6 | 11 | 9 | 15 | 5 | 3 | 3 | 1 | 1 |

We next mark the angular direction between the incoming and outgoing lines at each vertex of the hinge network in the same order as that followed by the each vertex of the hinge network in the same order as that followed by the number sequence (see figure 3.14). If we now associate the number and sign sequences we obtain the sequence:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 12 | 12 | 13 | 8 | 8 | 7 | 10 | 10 | 14 | 6 | 2 | 4 | 15 | 2 |
| 2 | 4 | 13 | 11 | 14 | 7 | 9 | 6 | 11 | 9 | 15 | 5 | 3 | 3 | 1 | 1 |
| − | − | + | + | − | − | − | + | + | + | − | + | − | − | − | − |

If we now remember that terms in this sequence which are separated by $N - 1$ terms, where $N$ is the order of the flexagon, represent identical leaves, we can see that the sequence is cyclic and identical to that obtained in Appendix B.

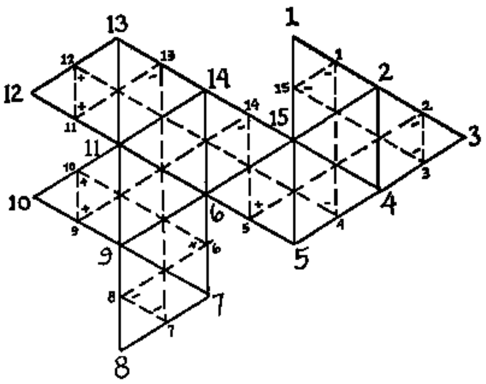


Fig. 3.14

In assembling the flexagon, the plan is repeated twice, cut out, taped together, and labeled according to the number sequence. It is important to notice that if the number of sides is odd, all the number sequence terms are reversed during the first repetition, with numbers falling on opposite sides of the strip from the usual position. The numbers are read off the map in this order on the second time around the Tukey triangle network. In flexagons with an even number of sides, there is no such disturbance. This is the reason for always including a small portion of the first repetition each time we write a sequence: oddness or evenness of the order $N$ must be kept in mind. As an example of this effect, we give the full sequences for the flexagon of order 5 ($N$ odd):

$$\begin{array}{ccccc|ccccc|ccccc}
1 & 3 & 5 & 4 & 4 & 2 & 2 & 1 & 3 & 5 & 1 & 3 & 5 & 4 & 4 \\
\hline
2 & 2 & 1 & 3 & 5 & 1 & 3 & 5 & 4 & 4 & 2 & 2 & 1 & 3 & 5 \\
+ & - & - & - & + & + & - & - & - & + & + & - & - & - & +
\end{array}.$$

To assemble the flexagon in its final form, adjacent like numbers are folded together, except for two numbers comprising a face on the map of the flexagon. These two sides are those which will show when the flexagon is complete. As a last step, the two ends of the plan are taped together, and the flexagon is complete.

This technique of finding the plan of a given flexagon from its map still leads to problems. The first to overcome is the clumsiness of the oriented triangle system. It will be seen that if alternate triangles are shaded, checkerboard style, as in figure 3.15, the arrows about the dark triangles go in one direction and those about the light triangles in the other. Moreover, all the dark triangles stand on a vertex, whereas all the light triangles stand on a base. Using this observation we can now establish a new method of orienting the triangles. In this system arrows are drawn in the plan from the midpoint of each hinge to the midpoint of the next, so that the arrows pass along the length of the plan, one arrow leading to the next. Then arrows at angles of 0°, 120°, and 240° from the horizontal are considered "+", are the triangles containing them, and those at angles of 60°, 180°, and 300° from the horizontal are considered "–" (see figure 3.16).

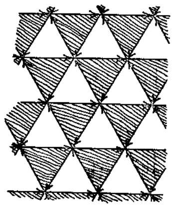


Fig. 3.15

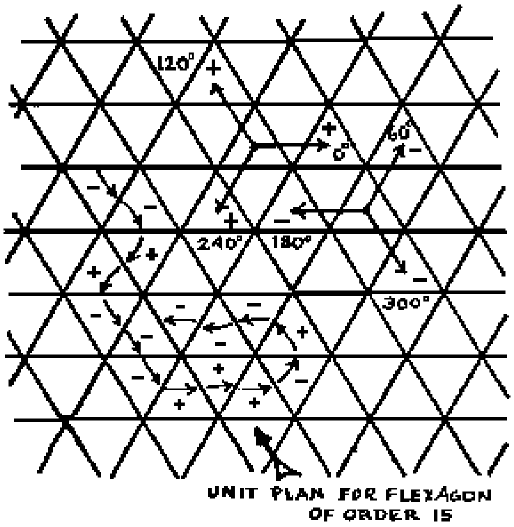



Fig. 3.16

There is a group of problems concerned with translating information given in one manner into another form. The forms used thus far include the map, Tuckerman tree, Tukey triangle network, sign sequence, number sequence, and plan. One thing clearly indicated in all these forms is the order of the flexagon, $N$. It equals the number of vertices in the map, the number of lines in the tree plus 3, the number of exterior vertices in the triangle network, the number of signs or pairs of numbers in the two sequences, and the number of leaves in the unit plan.

Given either the map, the tree, or the Tukey triangle network, the other two are easily formed by construction. The plan is equivalent to the sign and number sequences together. Given the map, the sequences can be found by the methods just described. The main problem, then, is to find the map, given either sequence. In order to establish a preliminary correlation between the number sequence and the map, we may try to use the fact that a side on the map which is touched by only two paths is always represented in consecutive plan leaves. There can be no more than $N/2$ but no fewer than two such sides. However, given the number sequence, we can do much better than this, by using the following process: We first select the lower of the two numbers in each pair of numbers lying opposite each other in the sequence. These numbers fall on alternate sides of the central line. Then, after making a ring of the numbers from 1 to $N$, arranged consecutively, we draw a path connecting the lower numbers in the number sequence, in the order in which they occur in the number sequence. The figure thus formed is the Tukey triangle system (see figure 3.17). This method may be developed further into a strictly numerical method for finding the sign sequence directly from the number sequence, though the formulation of this method is so complicated that it easily pays to pass through the intermediate graphic step of the Tukey triangle network.

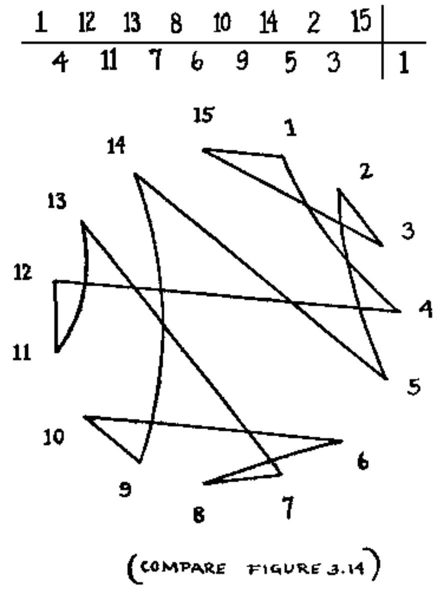



Fig. 3.17

If we are to find the map of a given sign sequence there will generally be several solutions possible. For instance, the sequence + + + + + + + + + + + + produces four distinct flexagon maps. Before trying to find the map corresponding to an arbitrary sign sequence, however, let us try to decide whether a given arbitrary sign sequence actually corresponds to a real flexagon. This problem is solved for the number sequence simply by translating it into map form, but with the sign sequence the solution is not so simple. Rather than the map we here employ a method know as "reduction" of the sign sequence. We can see that the combination + + in the triangle network could be replaced by a single – sign, as shown in figure 3.18a. This would delete one side from the flexagon, and is therefore the opposite of the slitting operation. However, where two signs, adjacent in the sign sequence, correspond to non-adjacent positions in the map, such as shown in figure 3.18b, the reduction to a single – sign has no meaning.

In the case shown, the result is the sequence – + –, which, as is shown, is a degenerate triangle network. Therefore two like signs may be reduced only when a sequence of alternating + and – signs does not result, since such a sequence has no meaning. After the two deleted signs are replaced by one opposite sign, giving a new sign sequence, reduction may be carried out again. We may, of course, delete such a sequence as + + + + + + directly to the sequence – – –, rather than going through the steps – + + + +, – – + +, and – – –. Any replacing of ± ± by ∓ , in other words, can give a true flexagon sign sequence only if the original sequence was also a true flexagon sign sequence. That is, if a sequence may be reduced in some way to a sequence of three like signs, or to any other known flexagon sequence, it represents some real flexagon, for we would then know that it could be produced by slitting. A reducible sequence will represent an identifiable flexagon, since the reduction process is characteristic of the flexagon. Now, if there exist two or more non-equivalent reduction processes for a given sign sequence, there exist an equal numbers of distinct corresponding flexagon maps. These may be reconstructed from the hinge networks of the flexagons which were identified as the result of the reduction process, by the reverse of reduction. which is the addition of new sides in the proper order, according to the path followed by the network. The problem of finding flexagons with a given sign sequence is thus reduced to the problem of finding distinct reduction patterns of the sign sequence. The analysis of the sequence + + + + + + + + + + + + is given as an example in figure 3.19, with the four resulting maps.

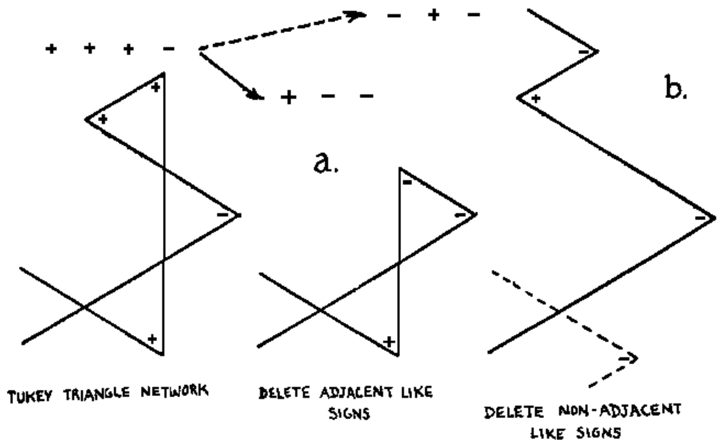



Fig. 3.18

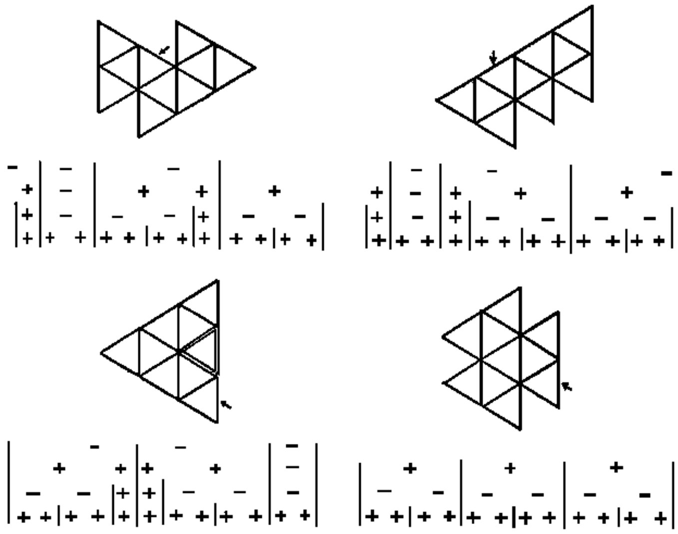


Fig. 3.19

There is also a trial-and-error method for finding the map of a given sign sequence. With a little practice, it may be used quickly and directly with rather good results. It involves the use of a plane marked off with equilateral triangles, with dots placed in each triangle of one of the three families made up of triangles whose vertices are opposite one another, as shown in figure 3.20. The dots identify the triangles in which they lie as potential members of the Tukey triangle network of the unidentified flexagon. The Tukey triangle network is then constructed by experimentation, from the sign sequence, each turn in the network being the vertex of a dotted triangle.

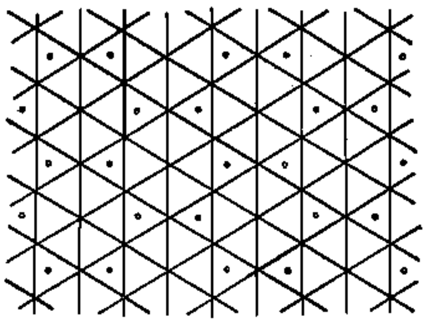


Fig. 3.20

It may have been noticed in the reductions of the 12-sided flexagon's sign sequence above that each of the resulting maps is made up of various combinations of the basic unit shown in figure 3.21, combined in various ways starting from a single initial map triangle. Thus the Tuckerman trees are composed entirely of units such as the one also shown in figure 3.21. Each of the map units, taken singly, corresponds to the addition of four plus signs in the place of one. The four 12-sided flexagons, like all regular flexagons, have maps made up of non-equivalent combinations of this unit, and all combinations of this unit will, in turn, produce regular flexagons.

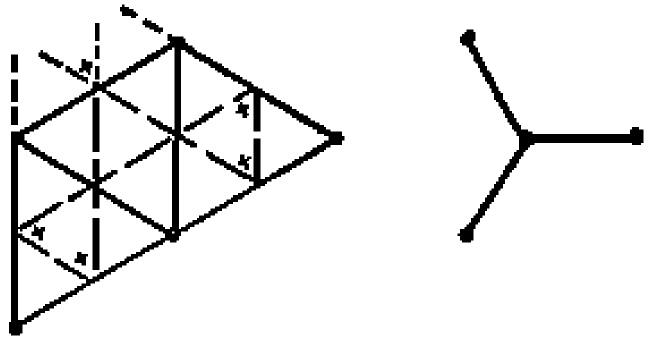


Fig. 3.21

There are a great number of such families of flexagons that are produced using various groupings of basic patterns of signs or map triangles. An interesting example is the family of flexagons whose plans are made up of groups corresponding to the sign sequence $++--$. As regular flexagons are of order $3k$ ($k$ an integer), members of this family will be of order $4k$. Of the members of this family, we will consider only those corresponding to the reduction, in one step, of each $++--$ group to a $+++$ group. That is, we can produce the maps of the flexagons in this family by adding a fourth vertex at every third vertex in the Tukey triangle network of the regular flexagons. One sequence of maps thus produced is shown in figure 3.22. Since every third turn is doubled, all these doublings will occur on the same edge of the map.

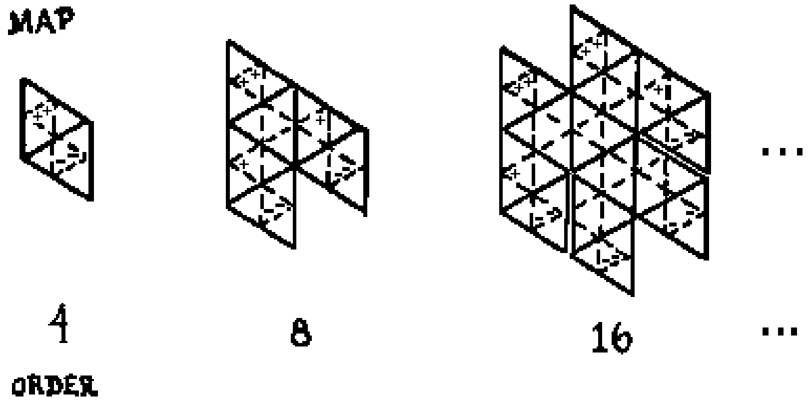



Fig. 3.22

Three families of flexagons, classified according to the shapes of their maps, are worthy of mention. The first of these groups is produced by the end-to-end coupling of the map unit shown in figure 3.23 (with the possible addition of a final single triangle at the end of the chain), generating the long chain-like maps and triangle networks also shown in figure 3.23. In these, the "chain" flexagons, it is possible to visit all the sides without a rotation. In the face at one end of the long hinge line, all the leaves but one are wound about each other. As the long hinge line is followed, the leaves are reeled in from one pat and rolled up on the other. The sign sequence is $+ + - + - + - + \ldots \pm \mp \pm\pm$ (the last sign is $+$ if the order is odd, $-$ if it is even,), so that the unit plan winds around a single point, finally branching off at either end. There are four possible patterns for fastening the three-unit plan together, depending on the value of $N$ (mod 6); these are shown in figure 3.24.

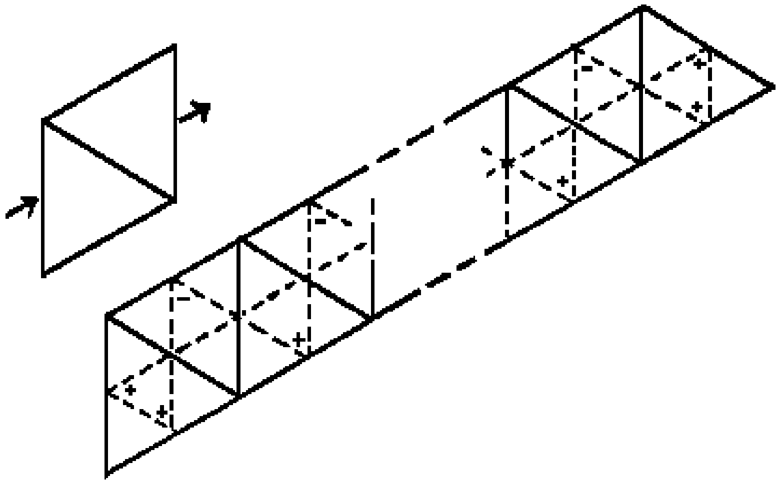


Fig. 3.23

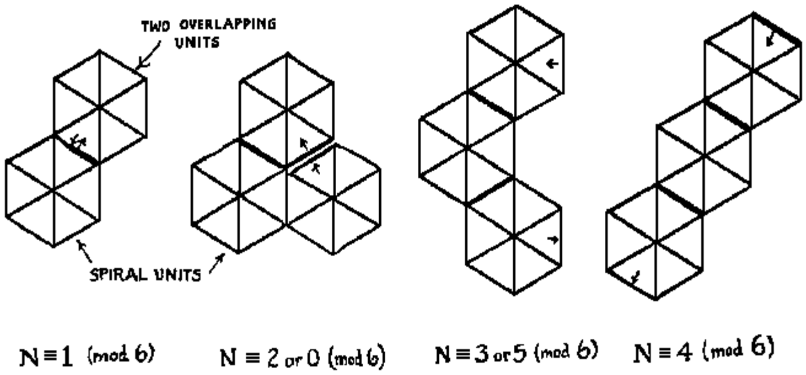



Fig. 3.24

The second family is produced by tying the pairs of map triangles together, end to end, in another fashion. As with the chain flexagons, we may remove half of the last group of two triangles added, to allow the order of the flexagon to be either odd or even. As we see from figure 3.25, the map and Tukey triangle network will, if extended long enough, overlap themselves, but this may again be ignored. In these flexagons, $N-1$ of the $2N-3$ faces (or $2N-2$ out of $4N-6$, if turning the flexagon over is allowed) exhibit the characteristic central side. If we choose to visit the paths radiating from this central side consecutively, we can, but after each flexing the flexagon must be turned over. The general sign sequence for these, the "fan" flexagons, is $+\ +\ \ +\ +\ +\ .\ .\ .\ \pm\pm\ -\ -\ -\ .\ .\ .\ .$ If $N$ is odd, the middle signs will be $+$: if $N$ is even, the middle signs will be $-$, There are now just three ways of fastening the unit plans together (see figure 3.26).

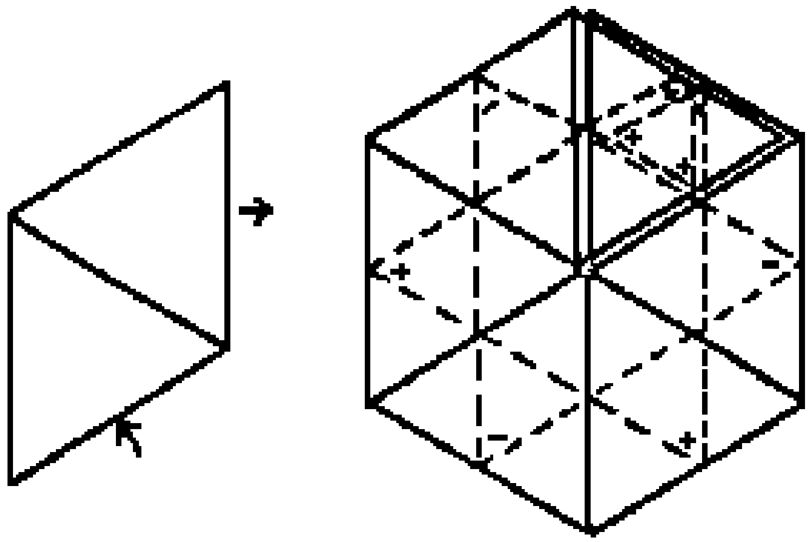



Fig. 3.25

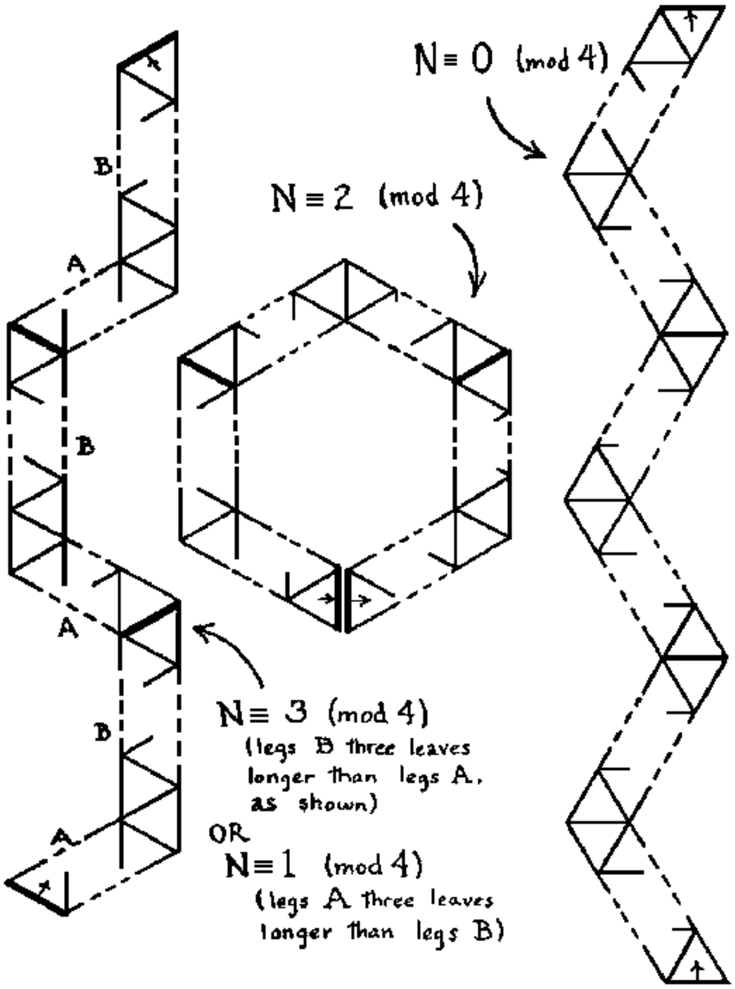



Fig. 3.26

The third of these families is that of the regular flexagons produced by uniform doubling of a straight strip plan; the "star" flexagons.

These three families may be used to distinguish the three flexagons of order six, which are shown in figure 3.27. Flexagons of order less than six may be said to be included in all three families. For order greater than six, all flexagon maps may be described as some combination of the three types, linked together in various arrangements. Thus, after analysis of each linkage in the map, it is possible to describe almost instantly, more or less accurately, the plan of a given map or a map for a given plan. This may be helpful as a check or an aid to precise computation, The method is practicable, though it requires much practice for effective use. As an example of its application, let us look at what has been called the "pinwheel" flexagon. The map shows us that this flexagon is composed of three fans tied end to end as shown in figure 3.28. To find the sign sequence we simply find the sequence for each of the three fans, and then tie their ends together through use of the star flexagon-like central linkage. To make the linkage, we cut out the first group of two plus signs, ++, in each fan, and in their place add in the other two fans, similarly prepared. The general form for the sign sequence of the pinwheel is then:

$+ + + \ldots \pm\pm - - - \ldots + + \ldots \pm\pm - - \ldots + + \ldots \pm\pm - - \ldots .$

This same general pattern may be modified to fit any three branch linkage, and the results are relatively easily applied to the plan.

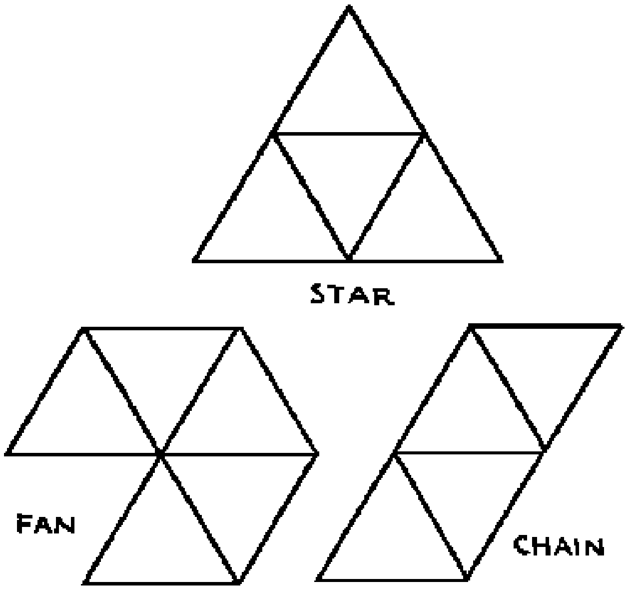



Fig. 3.27

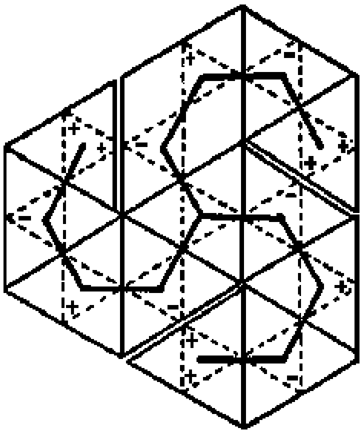


Fig. 3.28

Another interesting group of flexagons is that formed by linking two star flexagons together by means of a chain, as shown in figure 3.29. In the plan, this corresponds to leaving the $+-+-$... whorl in straight strips whose length will correspond to the intricacy of the star fragments attached. The stars on either end of the whorl will each lose one sign.

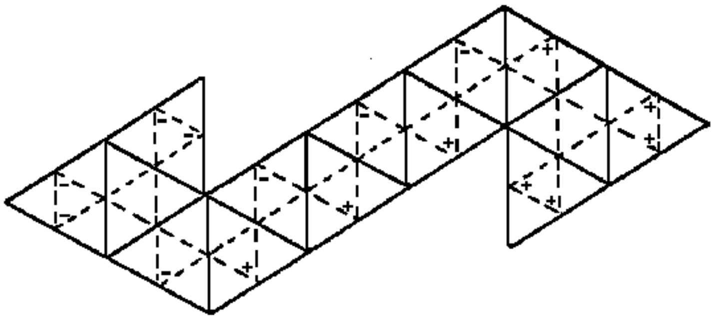


Fig. 3.29

The interlinking of chain, star, and fan flexagons in various ways can lead to interesting results. The fastening of two chains of map triangles end to end at a 120° angle produces, in the sign sequence, a series of alternating signs, followed by two like signs, followed by more alternating signs, followed by two like signs (see figure 3.30). The plan is a pair of overlapping $+-+-\ \ldots$ whorls. The break in the chain is shown by the change in direction of the whorl. If a fan-shaped strand is placed between and joining the two chains, as in figure 3.31, the straight strips connecting the ends of the two whorls in the plan are increased appropriately in length.

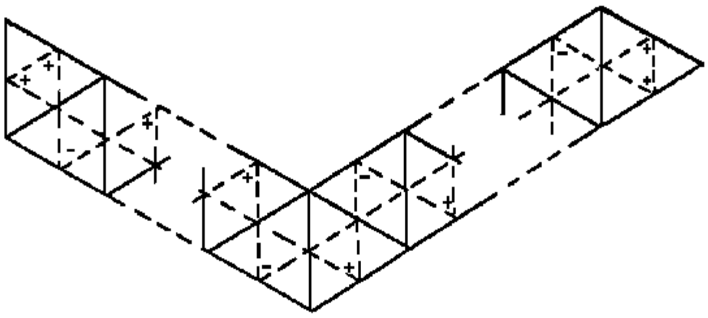



Fig. 3.30

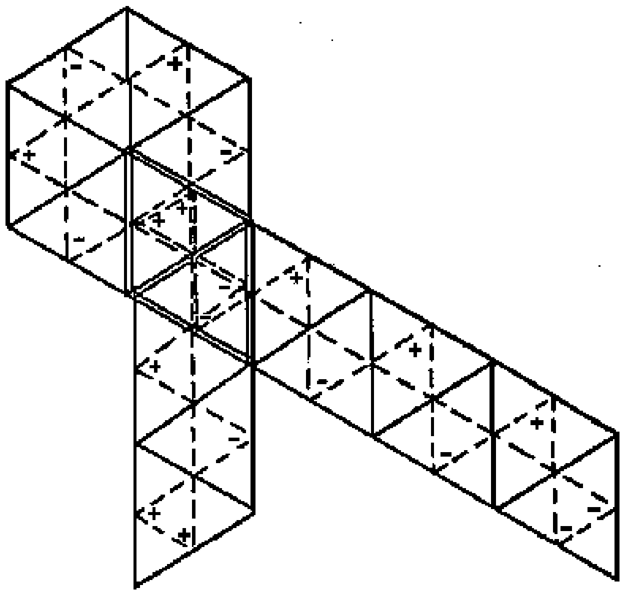


Fig. 3.31

Chains may also be linked together by means of a cross path, as in figure 3.32. In this case, each chain sign sequence will be broken at some point. There will always be four pairs of like signs, which will be distributed according to the breaking points in the chains. The remaining signs will alternate. This means that there will be two whorls, probably of different sizes, connected by a simple crossover (a double sign, which allows us to leave a whorl). If more than two parallel chains are joined, the process becomes more complex.

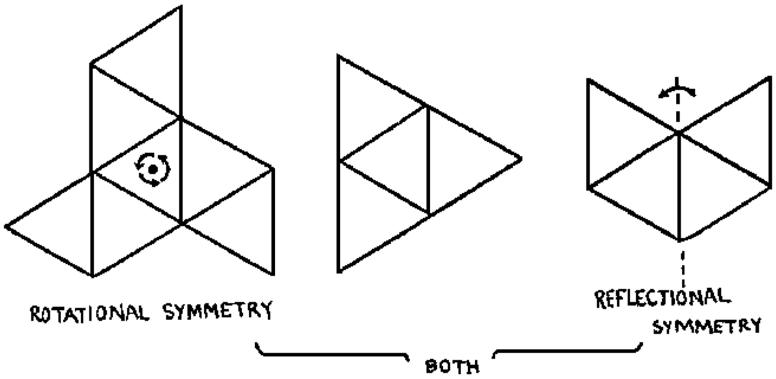



Fig. 3.32

If the plan is folded up backwards, all hinges being folded the wrong direction, the result is interesting, though it looks confusing. A side that had originally been labeled with a given number will instead be labeled with two different numbers, falling on alternate pats, and these two numbers will be the numbers of the sides adjacent to the original side along the outer edge of the map. Thus if the side had originally been numbered $n$, it will now be labeled half $n+1$, half $n-1$.

Certain plans, having greater translational symmetry than is usual, can not only be folded up in the wrong direction, but the folding process may be translated along the plan by less than a unit. They may, so to speak, be folded up using the wrong hinges at the wrong time, but using the same folding pattern. The regular flexagons will clearly possess this property, and may be folded into flexagons having the usual maps but unusual coloring schemes. If, in this process, either the pattern of folding up the flexagon is essentially changed or the plan does not have the necessary symmetry, a flexagon having a different map may result; this is in a sense the way in which different flexagons may be folded from the same plan.

By now the reader has probably begun to wonder how many flexagons can be made of a given order $N$. One clue to the solution of this problem is the fact that each of the various combinations of the $N-2$ triangles in the map produces a different flexagon. We therefore simply construct all possible distinct maps having this number of triangles. Since we have no direct method of finding distinct maps, we are forced to construct all the maps possible and then find which of them are distinct. Before we try to do this, though, we must be sure that we know how to tell whether or not two maps are equivalent. If should be clear that two maps, one of which is the other rotated about its center, are equivalent. So, also, are two maps each of which is the mirror image of the other. If two maps are not related in either of these ways, they represent different flexagons.

It has been pointed out that on a flexagon map each side is represented by a point along the outer edge of the map. This means that there are also as many outer paths as there are sides; i. e., the map is topologically a triangulated $N-$gon. Suppose that we desire to draw a map of a given flexagon. We start with one of the $N$ sides and label an $N-$gon vertex with its color. We could have labeled the same vertex with any of the $N$ colors; these possibilities correspond to the possible flexagons which are equivalent under rotation of the map. Once we have selected the first side, we have two positions in which one of the two sides adjoining it along the outer edge can be placed. If the second map side is placed in one direction from the first , the map will come out a mirror image of the map produced by placing it in the other direction. After these two sides have been placed, however, the remainder of the map is determined. This establishes the criteria for equivalence of maps given in the last paragraph.

It would seem, then, that the problem resolves to merely counting all possible triangulations of an $N-$gon, irrespective of rotations and reflections, and then dividing by $2N$, the number of equivalent maps, thus produced. This method, however, neglects the fact that some maps, due to their symmetry, will not be influenced by rotations or reflections (see those shown in figure 3.33). These maps are not counted so many times as are the asymmetrical maps, during the preliminary estimate, so that dividing by $2N$ will always give a low value. To correct for this, we will have to find out how many symmetrical maps there are of each order.

Another approach would be working out the number of possible trees that would give legitimate Tuckerman trees. The number of possible plans will be related to the number or possible maps, though, as we have said, some (and for large $N$, many) maps have the same plans.

These calculations are related to a long sequence of interesting and well-know combinatorial problems with interesting histories. Aside from flexagons, the general area concerns itself with such diverse things as folding up postage stamps, hydrocarbon isomers, election results, non-associative powers, electrical circuits, etc. We will not go into the details here of any of these problems.

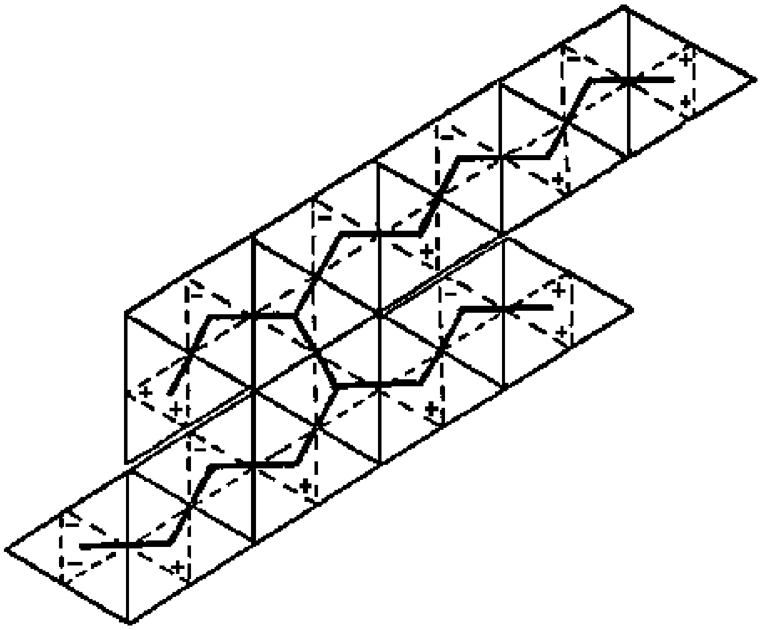


Fig. 3.33

The expressions finally obtained for the number of flexagons of order $N$, using any of the possible methods, are very complicated, with various cases depending on $N$ (mod 6). To give an idea of the numerical results, though, there is only one flexagon apiece for $N = 3, 4$, or 5. We have seen the three of order six. There are four of order seven, 12 of order eight, and 27 of order nine. From 10 on, though, the number begins to climb astronomically: 82 of order 10, 228 of order 11, 733 of order 12, 2282 of order 13, and 7528 of order 14; then 24834, 83898, 285357, and 983244.

The four distinct polygon triangulations for the heptagon (flexagon of order seven) are shown in figure 3.34.

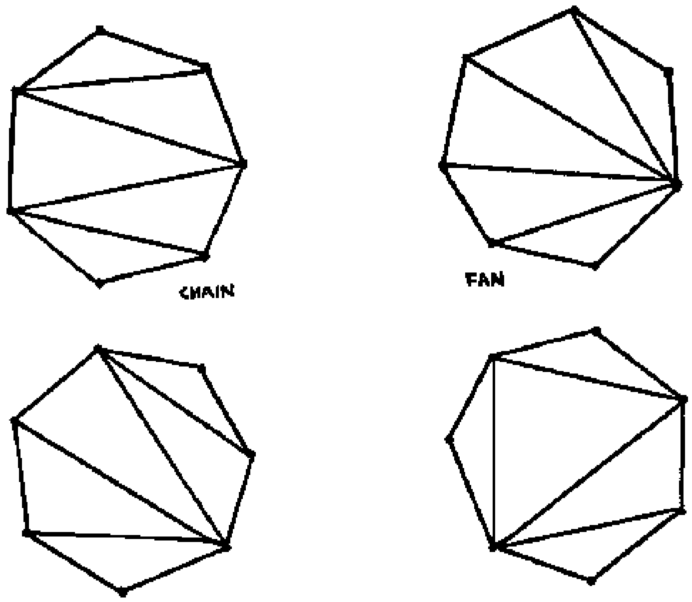



Fig. 3.34

# Chapter 4

# The Pat Structure

So far little has been said of the internal structure of the pat, although it is the basis of almost every significant feature of the flexagon. To treat the pat structure effectively in discussion or theory, the most important requirement is a workable representation. Basically, two kinds of representations are possible. The first is exemplified by the number sequence, which we have already encountered. In this system, generally, an invariant ordering of the outer map faces is established (in the case of the number sequence, these faces are numbered consecutively about the map rim) and then the corresponding ordering of the leaves in the plan which give rise to the corresponding map rim faces is given (this ordering being in effect the basic number sequence, or number sequence consisting of the lower numbers only from the pairs facing one another across the middle line). In the (basic) number sequence the first ordering system of consecutive numbers about the map rim is fixed, so that it need not be specified for each flexagon. However, we are able to fix this order only to within a constant, so to speak; i.e., the flexagons having number sequences

$$a_1\ a_2\ a_3\ a_4\ \ldots \qquad \text{and} \qquad \pm\, a_1 + x \pm a_2 + x \pm a_3 + x \pm a_4 + x\ \ldots$$

(mod $N$) are identical in structure. The important thing is, then, that we control our numbering convention of the map rim faces while describing the pat structure in this manner. But we have already seen the rather complex manner in which such a system fixes the flexagon's pat structure.

In the other basic system of pat representation the leaves in the plan are given some fixed ordering (usually by numbering them consecutively; this again will be effective only to within a constant) and the resultant map rim ordering is recorded. One system using this method is, for reasons that will be given shortly, called the "constant order" numbering system. To give an example of this system, consider first the flexagon shown in figure

4.1. It has the number sequence and plan shown. To use the constant order numbering system, the leaves in the plan are labeled consecutively, the flexagon is folded up, and the corresponding ordering of map rim faces is recorded. The results appear in figure 4.2. Just as the number sequence was essentially a sequence of numbers copied from leaves in the plan, the constant order is a sequence of numbers copied from the map rim, taken consecutively. The constant order is in several respects the inverse or dual of the basic number sequence, as we shall see.

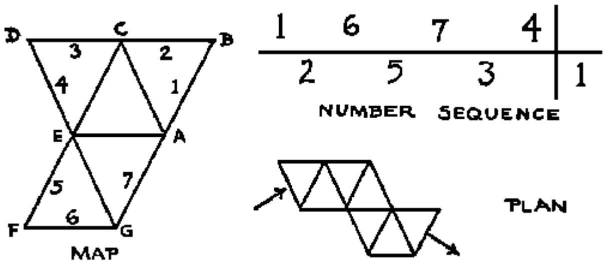



Fig. 4.1

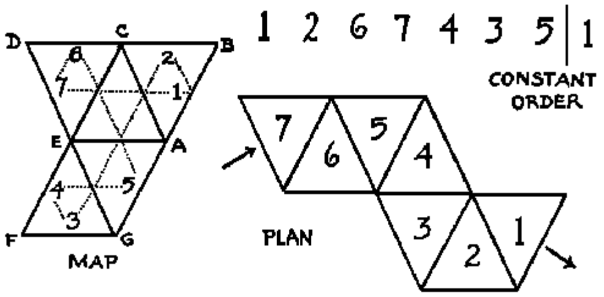



Fig. 4.2

So far, neither of these systems appears to be a representation of the pat structure, on the face of it. However, their relationship to the pat structure becomes apparent when we consider as follows: When a flexagon is assembled, leaf faces that are colored or numbered alike are always placed together facing one another, and from then on, no two differently colored leaf faces are opposed. For, during the ordinary legitimate flexagon operations, such different leaf faces are at no time brought together. Even during flexing, when a leaf face is folded away out of view, it is made to do so by placing it face to face with another similarly colored leaf face. Also, each unit contains two leaves colored each color, with two unequal colors appearing on the other sides of these leaves. A little reflection will shown that any two opposed leaves will not only have the same color on the facing sides, but will have distinct colors on the non-facing sides, and thus, since we know uniquely (within any one unit) the only two leaves which can ever possibly adjoin a given leaf in the folded up flexagon, we can completely describe the flexagon's structure. That is, knowing the leaf-to-leaf structure, we can figure out the order of the leaves when the flexagon is assembled. This order, as we saw above, cannot change, whatever "legal" operations are performed upon the flexagon; the chain of leaf-to-leaf adjacencies is disrupted only when some particular color must "come out of hiding" and be exposed. When this color is again concealed, the constant structure of the leaves is restored.

What, quantitatively speaking, is this constant structure? It is the same as the constant order of the constant order numbering system. To see this, recall that consecutive map rim faces are those having common sides. Thus the two leaves corresponding to adjacent map rim faces will have one color in common and will fall consecutively in the constant structure. It follows that the map rim faces, taken in order about the edge of the map, will fall in the same order as the corresponding leaves, taken in the order in which they lie over one another in the flexagon.

The two numbering systems, each of which is found to be quite useful, are not the only ones, however. The most important variant was developed by C. O. Oakley and R. J. Wisner, who were able to apply it very effectively to the construction or a fruitful analytical flexagon theory.[1] Their system, which is related to the constant order numbering system, treats the pats singly, rather than dealing with the flexagon as a whole. The map is not used at all in their treatment, the pats alone being analysed in the light of their structure.

A pat is designated by a permutation of the first $D$ integers, where $D$ is the number of leaves in the pat, called its "degree". The pat is named by first laying out the pat plan from which it was constructed and labeling the leaves along the strip consecutively. Thus the first plan leaf is labeled "1", the last "D". The pat structure is now obtained from the reassembled pat by reading down through the pat, as though by peeling off single leaves successively. Hence the pat structure may be thought of as a constant order for a single pat, in which the leaf fastened to the "ingoing" hinge is always labeled "1". The "ingoing" and "outgoing" hinges are to be carefully distinguished; this seeming artificiality is important when pats are combined to make flexagons. Then the "ingoing" hinge of one pat is always

[1] C. O. Oakley and R. J. Wisner, FLEXAGONS. American Mathematical Monthly, Vol. LXIV, No 3, March 1957, Pp. 143-154.

the "outgoing" hinge of the next. Also, in recording the pat structure for the two pats in a flexagon face, the pat theory convention is to read off the structure from top to bottom for both pats with the same side on top, unlike the constant order method.

The pat structure of a flexagon, then, will apparently vary radically under flexing and even turning over. The leaves are renumbered, the pat degrees change, etc. For this reason, the structures of the two unlike pats must be kept separately. The notation used is $(a_1\ a_2\ a_3 \ldots a_m,\ b_1\ b_2\ b_3 \ldots b_n)$, where the comma separates the two pats, of degrees $m$ and $n$. Thus, for example, the faces indicated in figure 4.3 by half arrows have the pat structures:

$$\begin{array}{ll} a. & (1, 21) \\ b. & (21, 231) \\ c. & (231, 312) \end{array}$$

The pat next to be divided in flexing is always placed to the right of the comma. We can now, in fact, describe what happens during flexing.

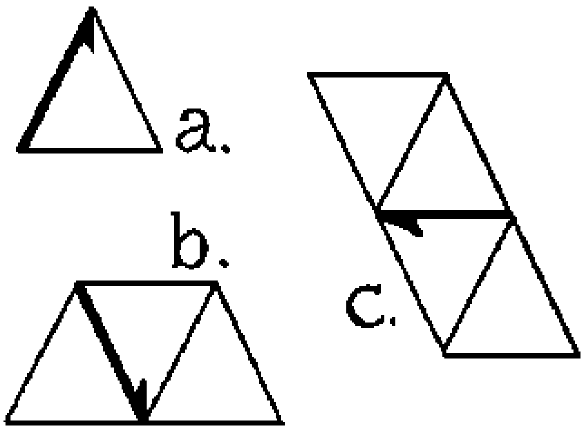



Fig. 4.3

Oakley and Wisner's treatment, which we may for convenience call the "pat theory", is essentially based upon several simple observations concerning flexagons. The most important of these is the identification of the position of the thumbhole in each pat by structural considerations only, i.e., using the pat structure notation only.

The distinguishing feature of a thumbhole is that all of the leaves either above or below it in the pat are connected only among each other. That is, the part of the pat above the thumbhole is connected to the part below by exactly one hinge. This means that a thumbhole will occur in the pat at precisely those places where all the numbers to the left in the pat structure notation are greater than all of those to the right.

The second observation is that, when two pats are adjoined to produce a larger pat, as in flexing (or in generating a larger flexagon, which we have seen is equivalent to flexing), a new thumbhole is produced, naturally, since the two old pats remain in their old constant orders, but the old thumbhole in each pat is suppressed. Thus if two pats, each with a thumbhole, are combined, the resulting pat still has but a single thumbhole. How can this happen? Suppose, using pat structure notation, our old pats were:

$$\begin{array}{lll} a_1 a_2 \ldots a_m & b_1 b_2 \ldots b_n & \text{and} \\ c_1 c_2 \ldots c_p & d_1 d_2 \ldots d_q & , \end{array}$$

where the only thumbholes are between $a_m$ and $b_1$ and between $c$ and $d$. Then all of the $b_j$ are less than the $a_i$ , and $d_j < c_i$. Now if we assume that the first pat above will be on top in the new pat, the only arrangement of the new pat that will have only one thumbhole, between the $a_1 \ldots b_n$ subpat and the $c_1 \ldots d_q$ subpat, and which will maintain the constant orders of both of the old pats, is:

$$b_n + p + q \ \ldots \ b_1 + p + q \ \ a_m + p + q \ \ldots \ a_1 + p + q \ \ d_q \ \ldots \ d_1 \ c_p \ \ldots \ c_1.$$

This is, in fact, precisely the experimental result when two pats are combined to produce a new one: the two old pats are inverted first, then joined <u>in the correct order</u>. Notice that leaves are necessarily renamed during the course of the joining, since there are now $p + q$ more leaves than before in the $a_1 \ldots b_n$ pat. In this way advantage in taken of the ambiguity of a constant in the order two systems of pat structure notation. Note also that so long as the two things joined were pats with one thumbhole each, the new thing will clearly be such a pat too. The inverting of subpats before joining them corresponds to the complicated inverting of the number sequence that occurred during primitive flexagon construction in the last section, when it was said that we try to have the new hinges connect leaves which are as distant as possible.

Now we can see our way clear to another pat theory conclusion: allowing for pats of thickness one ( no thumbholes), all other pats may be built by the above process. This gives us a recursive definition of a pat, and its characterization by a permutation of integers. A pat is either a single thickness or the combination of lesser inverted pats.

Using this algorithm it is easy enough to construct all possible pats of low degrees, as in Table 4.1. Notice that the pat of degree $D$ is by no means unique. There are in fact $\frac{(2D-2)!}{D!(D-1)!}$ pats of degree $D$, as Oakley and Wisner demonstrate.

| Degree of Pat | Pat Structure | Degrees of Component Subpats | |
|---|---|---|---|
| 1 | 1 | – | – |
| 2 | 2,1 | 1 | 1 |
| 3 | 23,1 | 2 | 1 |
| 3 | 3,12 | 1 | 2 |
| 4 | 324,1 | 3 | 1 |
| 4 | 243,1 | 3 | 1 |
| 4 | 34,12 | 2 | 2 |
| 4 | 4,213 | 1 | 3 |
| 4 | 4,132 | 1 | 3 |
| 5 | 2534,1 | 4 | 1 |
| 5 | 2453,1 | 4 | 1 |
| 5 | 3254,1 | 4 | 1 |
| 5 | 4235,1 | 4 | 1 |
| 5 | 3425,1 | 4 | 1 |
| 5 | 435,12 | 3 | 2 |
| 5 | 354,12 | 3 | 2 |
| 5 | 45,213 | 2 | 3 |
| 5 | 45,132 | 2 | 3 |
| 5 | 5,1423 | 1 | 4 |
| 5 | 5,1342 | 1 | 4 |
| 5 | 5,2143 | 1 | 4 |
| 5 | 5,3124 | 1 | 4 |
| 5 | 5,2314 | 1 | 4 |

TABLE 4.1

To get a clearer idea of the possible pat structures, we could consider a pat large and complex enough that none of the subpats or subsubpats or subsub...subpats are of a single thickness. In such a pat the only important feature of pat construction is that each pat is composed of two inverted subpats. Then the situation may be diagrammed using a permutation graph (see figure 14.2). The vertical axis may be thought of as height of the subsub...subpats above the bottom of the pat, the horizontal axis as distance of each subsub...subpat from the ingoing hinge. This kind of pat, in which doubling is uniform throughout the subpat structure, will be found in star flexagons. Other flexagons may have single leaves where the graph shows subpats, thus suggesting the idea that all flexagons arise from star flexagons by various deletion processes.

But if we accept the above definition of a pat, what is a flexagon? Pat theory's concise and obvious answer is that a flexagon is an ordered pair of pats. The two pats are ordered so that it will be meaningful to rotate the flexagon. Precisely, rotation interchanges the two pats. Since we can find thumbholes, flexing can be concisely defined as a specific manipulation of the integers comprising an ordered pair of pat-structure-notation pats: flexing carries flexagons of the form $(a_1 \ldots a_m, b_1 \ldots b_n \ c_1 \ldots c_p)$ into the flexagons $(c_1 + m \ldots c_p + m \ a_m \ldots a_1, \ b_n - p \ldots b_1 - p)$, when there is a thumbhole between $b_n$ and $c_1$. Note that the last (lowest) part of the second pat becomes the new first (highest) part of the first pat. The other subpats are merely inverted.

The trouble with the pat theory's definition of a flexagon from our point of view is that when we perform one of the legitimate flexagon operations we get a completely new and not obviously related ordered pair of different pats. Thus the physical flexagon really corresponds to an equivalence class of $4N - 6$ pat theory flexagons, where $N$ is as always the order of the flexagon(s), or, equivalently, the sum of the degrees of the two pats.

However inconvenient this may appear in terms of building and operating specific physical flexagons, this system clearly reduces the flexagons to a set of permutations with seemingly workable constraints. Hence the pat theory provides a valuable tool in calculating the number of flexagons possible for a given order. The only real complication is that the $4N - 6$ pat structures per flexagon are not necessarily distinct; in fact, we can see just when duplications of a single pat structure will occur by examining the symmetry of the map. Thus the flexagon shown in figure 4.4a will have 10 rather that the expected $4N - 6 = 30$ distinct pat structures, that in figure 4.4b 18 instead of 34. These duplications must be counted out of the final tally. Fortunately, the two symmetries shown are the only ones, and they always occur separately, never together in the same flexagon.

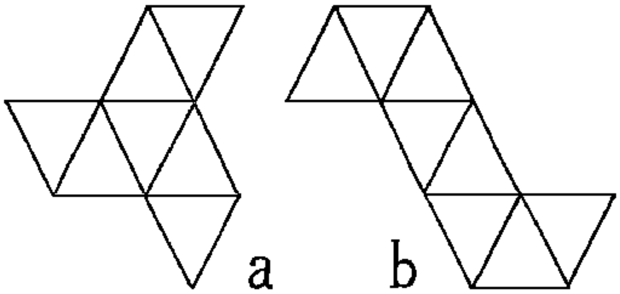



Fig. 4.4

One final difference between the pat theory flexagon and the ordinary physical flexagon involves the direction of winding up a flexagon. An ordinary paper flexagon may be wound up in two directions, depending upon which way the first fold is made, or, alternatively, if this has been determined already by the flexagons coloration, which direction of turning in the Tukey triangle network (in finding the sign sequence) is assigned a + value. These two ways do give slightly different flexagons, one being the mirror image of the other. Ordinarily this distinction is neglected or, if a convention has been established, corresponds to making a mirror image of the map. Often, even then, symmetry of the map may make these two mirror image flexagons identical, as with that shown in figure 4.4b. However, the pat theory naturally detects this distinction and, in counting the number of possible flexagons, will include both right- and left-handed versions. These may be accounted for later, in which case the usual figures for the number of possible flexagons are finally obtained.

To further illustrate the power of the pat theory, Oakley and Wisner calculate the number of possible regular flexagons in a similar fashion. They must here add the restriction that the sum of the degrees of the two pats be a multiple of three, but with neither of the degrees themselves ever a multiple of three. It is then necessary for one pat to have degree $3i+1$, the other $3j-1$, where $i$ and $j$ are any integers. The fact that regular pats may not be order divisible by three may be seen empirically; the reason should become clear later [2] . Again we are able to formulate a simple recursive definition of the (regular) pat, with $(3i+1)$-type pats always arising from the adjunction of two $(3j-1)$-type pats, and viceversa, except for the pat of degree one.

These recursive pat construction techniques do provide a fascinating (if not terribly informative ) method of actually building flexagons. As for the

[2]Incoming and outgoing hinges of the pat will overlap, so that a complete pat is not formed, yet further winding is frustrated

actual calculations for finding numbers of flexagons, they are too involved to appear here. Suffice it to say that, using the various recursive definitions, a number of generating functions are defined and their relations solved to give preliminary results for the number of possible pats of given degree. The number of flexagons of order $N$ then follows, this being refined until the various symmetries are eliminated. The same process is applied to the regular-flexagon problem.

The resulting values are unwieldily as stated in formulas, with several different cases. The total number of flexagons rises roughly as an exponential of 4; a fairly good approximation, to within about 10% from $N = 9$ through N=15, is the expression $4^{0.8(N-6)}$. The number of regular flexagons rises roughly as an exponential of $(4^{4/3})/3 \doteq 2.12$ except, of course, that all regular flexagons are of orders divisible by three. A rough approximation for regular flexagons is given by the expression $2.12^{3/4(N-9)}$, for orders less than 25. The actual values are given in table 4.2 for flexagons of order less than 19. Also included in the table for comparison are values of $\frac{(2D-2)!}{D!(D-1)!}$, the number of pats of degree $D$. This value is in fact closely related to the value for the number of flexagons, being somewhat less than $4D-6$ times the number of flexagons of order $D$. The figure $4D-6$ also occurs in the ratio $\frac{4D-6}{d}$ between successive terms in this sequence. The terms in this sequence also arise as the righthand-most terms in the interesting triangle:

$$\begin{array}{ccccc} 1 & \underline{1} & & & \\ 1 & 2 & \underline{2} & & \\ 1 & 3 & 5 & \underline{5} & \\ 1 & 4 & 9 & 14 & \underline{14} \\ \vdots & & & & \ddots \end{array}$$

in which each term is the sum of the numbers above it and to its left. The number of regular pats is also included in table 4.2, and the regular pats of degree $D \leq 7$ are shown.

| N=D | Number of Flexagons of Order N | Number of Pats of Degree D | Number of Flexagons | Regular Pats |
|---|---|---|---|---|
| 1 | | 1 | | 1 |
| 2 | 1 | 1 | | 1 |
| 3 | 1 | 2 | 1 | |
| 4 | 1 | 5 | | 1 |
| 5 | 1 | 14 | | 2 |
| 6 | 3 | 42 | 1 | |
| 7 | 4 | 132 | | 4 |
| 8 | 12 | 429 | | 9 |
| 9 | 27 | 1,430 | 1 | |
| 10 | 82 | 4,862 | | 22 |
| 11 | 228 | 16,796 | | 52 |
| 12 | 733 | 58,786 | 4 | |
| 13 | 2,282 | 208,012 | | 140 |
| 14 | 7,528 | 742,900 | | 340 |
| 15 | 24,834 | 2,674,440 | 14 | |
| 16 | 83,898 | 9,694,845 | | 969 |
| 17 | 285,357 | 35,357,670 | | 2394 |
| 18 | 983,244 | 129,644,790 | 74 | |

Table 4.2

Note: the number of flexagons given here is the reduced value for the number of models mentioned in Remark C, p. 154, of the Oakley-Wisner article. The expression for odd N, corresponding to that given in Remark C for even $N$, is $W^*_{2m+1} = (V^*_{2m+1} + v_m)/2$.
I have also established that the corresponding results for regular flexagons are the reduced values $U^{**}_N$:
$U^{**}_{2M} = \frac{1}{2}(U^*_{2M} + \frac{1}{2}U_M)$ , and $U^{**}_{2M+1} = \frac{1}{2}(U^*_{2M+1} + \frac{1}{2}u_M)$ .

Regular Pats of Degree $\leq 7$:
1 2,1 34,12 5,2143 3254,1 67,34125 67,14523 36745,12 56347,12

Returning our attention to the order two flexagon representations, we observe that although the number sequence is equal in value with the constant order in counting the number of flexagons (see the portion of chapter XIV on Duals), when it comes to operating the flexagon using some algorithm like that given for the pat theory, the number sequence is perhaps) less easily interpreted than the constant order, which bears a strong resemblance to the pat theory notation. This is not, however, to say that a suitable algorithm for flexing the number sequence cannot be devised; it would in fact be very similar to that for the constant order. At any rate it should be convenient to know how to convert easily from a constant order to the number sequence of the same flexagon or viceversa.

It will be remembered that the existence of the constant order was demonstrated using the fact that like-numbered leaves always meet in the folded-up flexagon. Hence we may number the "leaves" in the number sequence from 1 to $N$ and see which pairs of leaves lie together. For example, in the sequence

$$\begin{array}{rcccccccc|c}
 & 1 & 5 & 5 & 4 & 7 & 1 & 6 & 3 & 1 \\
\cline{2-10}
 & 2 & 4 & 6 & 3 & 8 & 8 & 7 & 2 & 2 \\
\#: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1
\end{array} ,$$

where $N = 8$, leaf #1 lies next to leaf #8, since they have the number 2 in common and thus must be folded together when the flexagon is wound up. Leaf #8 in turn shares the number 3 with leaf #4, etc. The final sequence will be 1 8 4 2 3 7 5 6, the constant order for this flexagon. The reverse process gives the number sequence: we assume that #1 and #8 have 2 in common, #8 and #4 have 3 in common, etc. up to $N$ and then 1, which is shared by #6 and #1. Note, by the way, that if the number sequence numberings are read down through the pat, they will be found to be arranged consecutively. In the first process above, it was the lower numbered side of the leaf that was searched for each time, and the higher-numbered side was found on its back. This is why the lower numbers alone (in the example above, 1 4 5 3 7 8 6 2), or basic number sequence, is all that we really require. These are the numbers obtained from the Tukey triangle network. Then we can also obtain the constant order directly from the triangle network, as follows: we pass through the network in the usual way, numbering each vertex as we come to it, consecutively from 1 to $N$. We then pass about the outer rim of the network and take down these numbers in the order we find them there, starting at #1. This will be the constant order. The process is illustrated in figure 4.2. To reverse this process and construct the Tukey triangle network from the constant order, we arrange the constant order in a circle and join consecutive integers (mod $N$) with line segments. An analogous process creates the network from the basic number sequence. In fact, it turns out that all of the above routines for converting from number sequence to constant order <u>or</u> back are equivalent processes.

We way now feel free to discuss the effects of the various operations upon the constant order. First we establish the notion of a face in constant order notation. The constant order may be considered as something like a pat structure for the whole folded-together unit. Then when the flexagon is opened out at a face, to read the constant order we are forced to read through one of the pats upside down. We can uniquely specify any position of the flexagon and at the same time give its structure by indicating how the constant order is split into pats and further indicating which of the two pats is to be read upside down or considered inverted. The latter tells us which surface of the flexagon is up; it corresponds to the pat theory convention of writing all pats from top to bottom rather than bottom to top. The inverted pat is identified by underlining it; the two pats are separated by a comma. Furthermore, the right-hand pat will be the one in position to be split in the next flexing, just as with the pat theory notation. Again using the faces shown in figure 4.3 to illustrate, we have:

| | | | |
|---|---|---|---|
| a. | $(\underline{3}, 21)$ | Constant order: | 132 |
| b. | $(43, \underline{521})$ | Constant order: | 14352 |
| c. | $(126, \underline{435})$ | Constant order: | 126435 |

The initial choice of an underlined pat is arbitrary for each flexagon, and the constant order may vary by a constant or a multiple of $-1$, modulo $N$, so that the same pat structure may have many equivalent representations (more accurately, each flexagon may have many different representations). Face b. above, for example, is identical in structure with any of the follo-wing: $(15, \underline{243}), (\underline{21}, 354), (34, \underline{251}), (\underline{12}, 534)$, etc. Once one of these equivalent representations has been chosen for the face at hand, an algorithm for flexing it can be established. This algorithm should preserve the constant order (which is not preserved under the equivalences above, except where the map is symmetrical). It will split the right-hand pat at the thumbhole, which may be located by finding the unique place in the pat where all but two leaves to the right or left (at most) are fastened to each other only. That is, a pat or subpat will always consist of at permutation of consecutive integers, modulo $N$. For Example, in the flexagon face $(\underline{3}, 4125)$ the thumbhole occurs between leaves 4 and 1, since 125 is a permutation of consecutive integers (mod 5). The ordinary constant order may be split into two pats by dividing it so that both pieces satisfy this criterion. Finally, in flexing the whole flexagon is inverted; if we read initially up through one pat and down through the order, after flexing we read down through the first and up through the second. That is, to keep the correct leaves on "top" of the flexagon, the "inverted" pat must become uninverted and viceversa. The reason for this is that in flexing, the two subpats that remain in their old pats (one of which is, in fact, the old left pat) are actually turned over, whereas the remaining right subpat changes pats with out being inverted.

Within these limitations, there are still two cases that arise, depending upon which pat is underlined: flexing takes the flexagon
$(a_1a_2\ldots a_m, \underline{b_1b_2\ldots b_n\ c_1c_2\ldots c_p})$ into $(\underline{a_1a_2\ldots a_m\ b_1b_2\ldots b_n}, c_1c_2\ldots c_p)$
but takes the flexagon

$(\underline{a_1a_2\ldots a_m}, b_1b_2\ldots b_n\ \ c_1c_2\ldots c_p)$ into $(c_1c_2\ldots c_p\ \ a_1a_2\ldots a_m, \underline{b_1b_2\ldots b_n})$, where the thumbhole lies between $b_n$ and $c_1$. Each of these agrees with the pat theory algorithm; they differ so that turning the flexagon over (which carries $(a_1a_2\ldots a_m, \underline{b_1b_2\ldots b_n})$ into $(\underline{a_1a_2\ldots a_m}, b_1b_2\ldots b_n)$ and viceversa), flexing it, then turning it back over will not be equivalent to a simple flex. This result should be expected; in fact, this complex operation precisely nullifies the result of rotating, flexing, and rotating once more. This may be checked using the operations defined above. Rotating, of course, carries the flexagon $(\underline{a_1a_2\ldots a_m}, b_1b_2\ldots b_n)$ into $(b_1b_2\ldots b_n, \underline{a_1a_2\ldots a_m})$ , and vice versa.

This gives us all the information necessary to operate a flexagon in either pat theoretic or constant order notation alone. To execute the Tuckerman traverse, for example, we would proceed as illustrated below for the flexagon shown in figures 4.1 and 4.2:

Starting position: Face 1, side A on top.

| Face and Operation | Pat Theory Notation | Constant Order Notation |
|---|---|---|
| A-B Flex | ( 1 , 4 2 3 6 5 1 ) | $( 1 , \underline{2\ 6\ 7\ 4\ 3\ 5} )$ |
| C-A Flex | ( 2 1 , 4 5 2 1 3 ) | $( \underline{1\ 2} , 6\ 7\ 4\ 3\ 5 )$ |
| E-C Flex | ( 4 3 5 1 2 , 2 1 ) | $( 4\ 3\ 5\ 1\ 2 , \underline{6\ 7} )$ |
| D-E Rotate | ( 6 2 1 5 3 4 , 1 ) | $( \underline{4\ 3\ 5\ 1\ 2\ 6} , 7 )$ |
| D-E Flex | ( 1 , 6 2 1 5 3 4 ) | $( 7 , \underline{4\ 3\ 5\ 1\ 2\ 6} )$ |
| C-D Rotate | ( 3 2 6 4 5 1 , 1 ) | $( \underline{7\ 4\ 3\ 5\ 1\ 2} , 6 )$ |
| C-D Flex | ( 1 , 3 2 6 4 5 1 ) | $( 6 , \underline{7\ 4\ 3\ 5\ 1\ 2} )$ |
| E-C Flex | ( 2 1 , 4 3 5 1 2 ) | $( \underline{6\ 7} , 4\ 3\ 5\ 1\ 2 )$ |
| A-E Flex | ( 3 4 1 2 , 3 1 2 ) | $( 1\ 2\ 6\ 7 , \underline{4\ 3\ 5} )$ |
| G-A Rotate | ( 5 6 2 1 4 3 , 1 ) | $( \underline{1\ 2\ 6\ 7\ 4\ 3} , 5 )$ |
| G-A Flex | ( 1 , 5 6 2 1 4 3 ) | $( 5 , \underline{1\ 2\ 6\ 7\ 4\ 3} )$ |
| E-G Flex | ( 3 2 5 4 1 , 2 1 ) | $( \underline{5\ 1\ 2\ 5\ 7} , 4\ 3 )$ |
| F-E Rotate | ( 6 1 4 5 2 3 , 1 ) | $( 3\ 5\ 1\ 2\ 6\ 7 , \underline{4} )$ |
| F-E Flex | ( 1 , 6 1 4 5 2 3 ) | $( \underline{4} , 3\ 5\ 1\ 2\ 6\ 7 )$ |
| G-F Rotate | ( 2 5 6 3 4 1 , 1 ) | $( 5\ 1\ 2\ 6\ 7\ 4 , \underline{3} )$ |
| G-F Flex | ( 1 , 2 5 6 3 4 1 ) | $( \underline{3} , 5\ 1\ 2\ 6\ 7\ 4 )$ |
| E-G Flex | ( 2 1 , 3 2 5 4 1 ) | $( 4\ 3 , \underline{5\ 1\ 2\ 6\ 7} )$ |
| A-E Flex | ( 3 1 2 , 3 4 1 2 ) | $( \underline{4\ 3\ 5} , 1\ 2\ 6\ 7 )$ |
| C-A Flex | ( 4 5 2 1 3 , 2 1 ) | $( 6\ 7\ 4\ 3\ 5 , \underline{1\ 2} )$ |
| B-C Rotate | ( 6 3 1 2 5 4 , 1 ) | $( \underline{6\ 7\ 4\ 3\ 5\ 1} , 2 )$ |
| B-C Flex | ( 1 , 6 3 1 2 5 4 ) | $( 2 , \underline{6\ 7\ 4\ 3\ 5\ 1} )$ |
| A-B Rotate | ( 4 2 3 6 5 1 , 1 ) | $( \underline{2\ 6\ 7\ 4\ 3\ 5} , 1 )$ |

A-B = Starting position

Notice that the sides associated with each face are easily found from the constant order notation; the top side is that associated with both top leaves, the bottom side is associated with both bottom leaves. The pat structure is derived from the constant order by inverting underlined pats and adding a constant where necessary. To remember the constant order flexing algorithm, we need only keep in mind that it is always the lower right subpat that changes pats. Also, note that in the Tuckerman traverse the single leaves that occur as pats in the constant order notation always appear in consecutive (or as above, inverted consecutive) order.

It is quite difficult, given two pat structures, to find whether they belong to the same flexagon. However, with the constant orders given, the ambiguity of a constant that may arise is easily taken care of by comparing the sequences of $N$ terms found by calculating the differences between adjacent terms of the constant orders. There are two ways of taking these differences for each of the constant orders. If the sequences for the two constant orders, which must be considered cyclic, are equal, the flexagons are the same. On the other hand, to calculate whether two constant order faces have the same structure, the simplest thing is to convert to pat theory notation.

Most of the developments of this chapter will be generalized quite easily to cover all of the new flexagons that we will encounter.

# Chapter 5

# New Angles

The theories concerned with the flexagon committee's "flexible hexagons" seemed at first to be a fairly complete description of the flexagon family. Each assumes a more or less clear-cut definition of the flexagon, from which it proceeds to deduce certain properties. However, it has been found that, by altering the definitions used slightly, whole new "dimensions" of flexagons can be discovered.

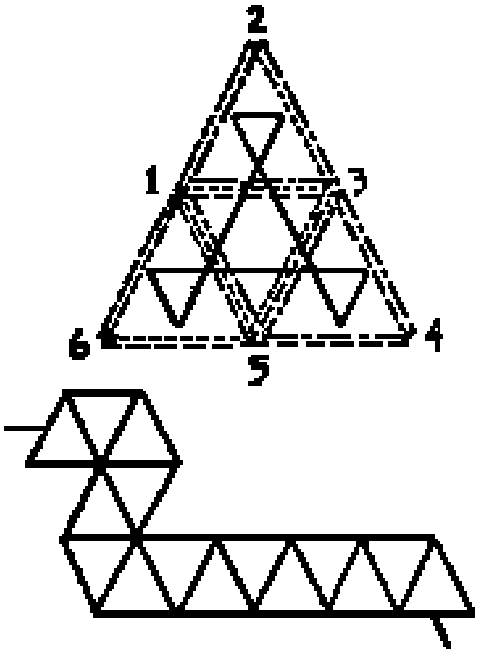
It may be best to clear up the haze surrounding the definitions of flexagons now before they are extended any further. It should be clear that the conditions of any definition will be determined by the object itself. It will be impossible to give an absolute definition of flexagons since it is a matter of taste what one wishes to include in this category. As we proceed, we will come to count more and more strange figures under the general heading "flexagons". Most of these are not included by definitions already considered; their only justification is the interesting results they produce. Of course, reason dictates that only figures related to flexagons or including flexagons as a special case should be considered.

The first extension to be added in this section is hinted at by the pat structure definition: A flexagon is an ordered pair of pats. This pair is repeated twice to form the actual paper flexagon. Why then can we not build a flexagon with the pair of pats in duplicate? Or in quadruplicate? It is soon found that there is no reason at all, flexagons can be made with any number of units. More than four units cause appreciable overlapping of pats and produce relatively unstable models, but are theoretically practicable. The four-unit flexagon does not lie flat, but this makes it all the more fascinating, One-unit flexagons must be operated in the imagination, or else cut $N$ different times to reveal all the faces.

The two-unit "cup" flexagon is built by joining the pats to form a tetrahedral angle. Since the vertex cannot be "pushed through", placing the inner surface on the outside, the outer side remains constant. Flexing is always accomplished by pinching together pairs of adjacent pats and occurs between the constant outer side and the variable inner side. The map, therefore, presents the appearance of spokes emanating from the central point representing the constant side, since no outer faces or paths can be used. Because arrows adjacent along the spokes point in different directions, the flexagon must be turned over after each flexing to use the next spoke. The obvious way to take advantage of this is to build a two-unit fan flexagon, which will have no unused sides.

The construction and operation of this fan flexagon suggest a further possibility. During the entire operation of the flexagon, the stacks of leaves remain on the outer side, slowly shifting between alternate pats. These pats, due to the fashion in which they have been built, can be constructed to incredible length. Until now, the physical limitation of paper thickness had made it impossible to construct flexagons of orders greater than 48 or so. However, utilizing the principle of the cup fan flexagon, we can now build 2- (or 3-, if flexing is restricted to the "cup" pattern) unit flexagons of unlimited size. A three-unit flexagon of order 658 has been built by the authors; all the sides can be exposed but only $N-1$ paths may be used, instead of the usual $2N-3$.

Another possibility, related to the cup flexagons, is the mixed flexagon. Mixed flexagons are based on the observation that the units used need not be identical. If they are different, each will have its own map, and the map of each will partly coincide with those of the others. If a11 three coincide at any face, this face will open out and lie flat. For faces where only two of the maps coincide, the third unit will not open out, and the flexagon will look like a cup flexagon. If only one map occupies a face, it will not open out at all. To build mixed flexagons, the three maps must be drawn with differently colored pencils, overlying in the desired way. In copying down the sequences, all proceeds as usual except that after each trip about the Tukey triangle network, one must change maps. The tukey triangle network is drawn over all the maps, but only the portion corresponding to the map being used is followed at a given time. A simple mixed flexagon is shown in fig. 5-1. Mixed flexagons may be extended considerably by the use of more than 3 units.




| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 3 | 1 | 5 | 3 | 4 | 4 | 6 | 6 | 2 | 2 | 4 | 4 | 5 |
| 3 | 1 | 5 | 6 | 6 | 1 | 5 | 3 | 1 | 5 | 3 | 1 | 5 | 3 | 1 |
| + | + | - | + | + | - | + | + | + | + | + | + | + | + | - |

Figure 5.1

Just as the pat structure definition does not mention the number of units in the flexagon, so also it neglects to say that the leaves must be equilateral triangles. In dispelling this requirement, we open up another new dimension.

The relative position of the vertices of the leaves within any given flexagon is constant. Therefore, varying the angles of the triangles used as leaves should make no difference in the operation of the flexagon. For instance, a $45^{\mathrm{O}}$ - $45^{\mathrm{O}}$ - $90^{\mathrm{O}}$ flexagon of order 6 may be constructed quite easily, by the doubling-over technique, from a strip such as the one shown. (fig. 5.2) We have seen, by means of “dots”, that various angles occupy the center of the flexagon at various times. Varied angles provide an excellent form of “dot”.

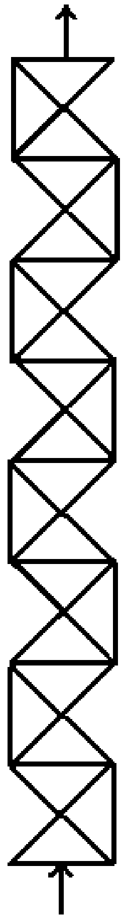


Figure 5.2

When the $45^{\circ}$ angles of the $45^{\circ}$ - $45^{\circ}$ - $90^{\circ}$ flexagon are at the center, four units must be used, in order to prevent cup-flexagon-like operation. When the $90^{\circ}$ angles are pointed toward the center, there is, therefore, a surplus of $360^{\circ}$. A number of things may be done about this. The most obvious is to ignore it and flex on. In this particular case, however, opposite units may be laid flat and, with a half twist, the two middle units may be opened out flat. The final product is a bicolored rectangle (fig. 5.3).

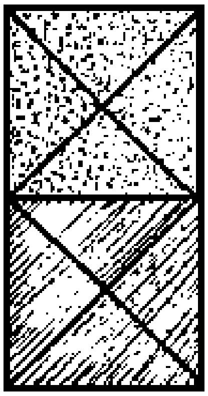


Figure 5.3

This is typical of a second type of distortion of the flexagon. A flexagon with more than $360^{\mathrm{O}}$ about its center may always be forced to lie flat by the following procedure: Simply fold a unit together, then fold together the next two pats on either side of the first two, etc., just as was done to distort the structure of the 3-equilateral flexagon,.

With care a flexagon of any number of sides, any number of units, and any shape of triangle may be built. However, beyond order 3 in flexagons of many units the structure is generally so easily distorted that the flexagons are not worth the trouble of construction.

The irregular strip used in making a scalene or isosceles flexagon of order 3 corresponds to the straight strip in equilateral flexagons. A process has been developed to simplify the making of what will be called by definition a "straight" strip; i.e., the strip of leaves corresponding to a straight chain of equilateral triangles. Draw any triangles and label the vertices $A$, $B$, and $C$. Then construct

$$\begin{array}{lcl} \triangle A'BC & \cong & \triangle ABC \\ \triangle A'B'C & \cong & \triangle A'BC \\ \triangle A'B'C' & \cong & \triangle A'B'C \\ \triangle A''B'C' & \cong & \triangle A'B'C' \end{array}$$

etc. as in figure 5.4. This is continued until enough leaves have been produced. To find by construction the number of leaves required for all faces to have at least $360^{\mathrm{O}}$ about their centers, encircle the vertex of the smallest angle and measure off around the circle to find the minimum even number of arcs, equal to the arc of the smallest angle, which will cover the circumference. 3/2 of this number is the required number of leaves. One of the most interesting of these flexagons, because of the distortions possible with it, is the $30^{\circ}$ - $60^{\circ}$ - $90^{\circ}$ flexagon of order three. The five unit star-shaped flexagons are also amusing.

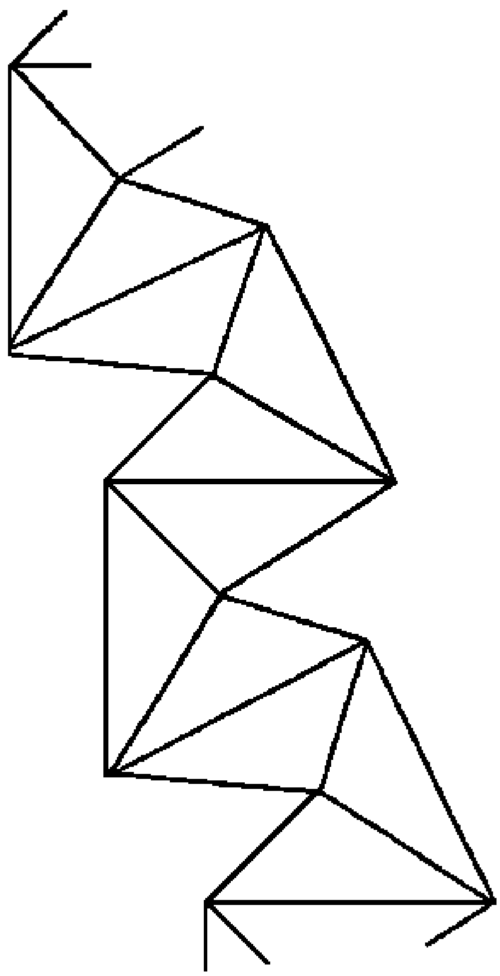


Figure 5.4

Notice that as any one of the angles in the leaves approaches zero, the number of units required to make all faces have at least $360^{\circ}$ about their centers becomes larger and larger. It soon becomes most practical to abandon some of the faces by using some arbitrary small number of units, but at least 2. Suppose, then, that we let the arms of one angle swing all the way around to a zero degree angle. This should not disturb us, with, say, 2 units, any more than a $1^{\circ}$ angle would. The problem now is how to operate with leaves having parallel edges infinite in length. This is easily solved by chopping off the $0^{\circ}$ vertex at infinity, just as we chopped off vertices of ordinary flexagons to prevent binding during flexing. The infinite vertex may as well be chopped off so as to leave each leaf square, as in figure 5.5. In building flexagons like this out of squares, one must remember that one edge of each square leaf is theoretically a cutting off, and cannot be used for a hinge.

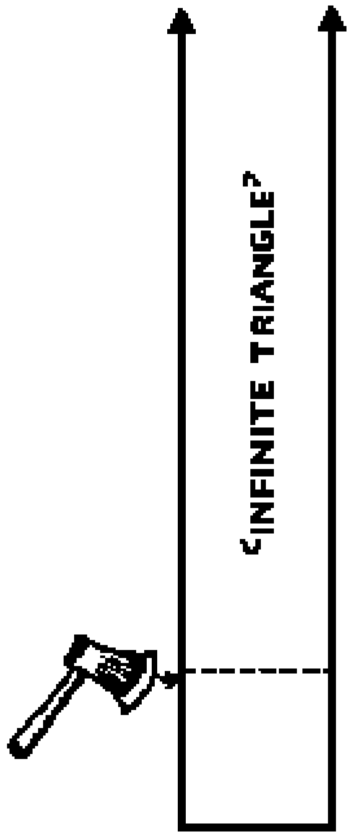



Figure 5.5

When an infinite vertex tries to come to the center of the flexagon, its parallel hinges will prevent it. It will form instead an infinitely long cylinder, or, in the case of our 2-unit flexagon made of squares, an open-ended cube (see figure 5.6). Needless to say, such faces will not "push through", or reverse inward and outward sides. Other faces of such a flexagon will appear as in figure 5.7.

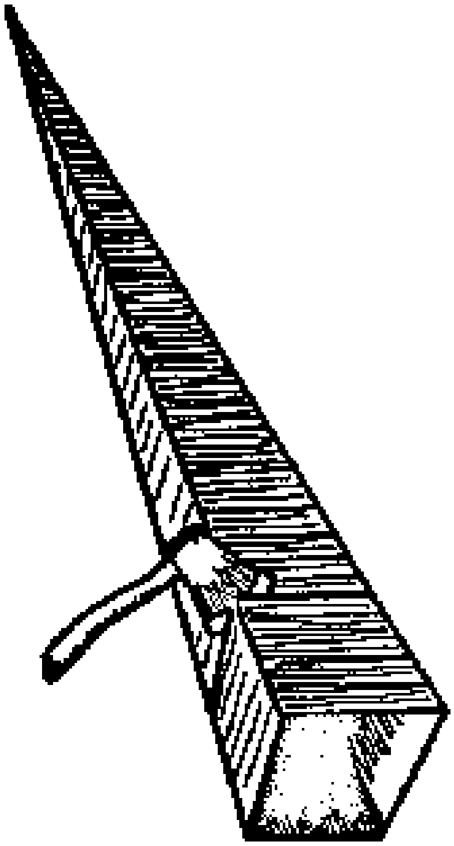


Figure 5.6

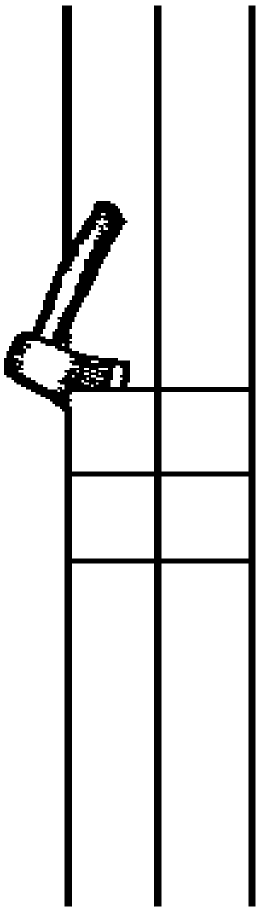


Figure 5.7

There is no reason to stop at zero. Leaves with negative angles (duly chopped off) are completely acceptable, although the flexagons they make may at first seem confusing. These flexagons will actually be made from trapezoidal leaves. When the negative angle in not toward the center the flexagon will look like figure 5.8, and when the negative angle in "toward the center", it will actually be pointing away. In fact, since it points away, the flexagon may be made to lie flat, upside down, during such faces, as in figure 5.9. Thus these faces can be "pushed through".

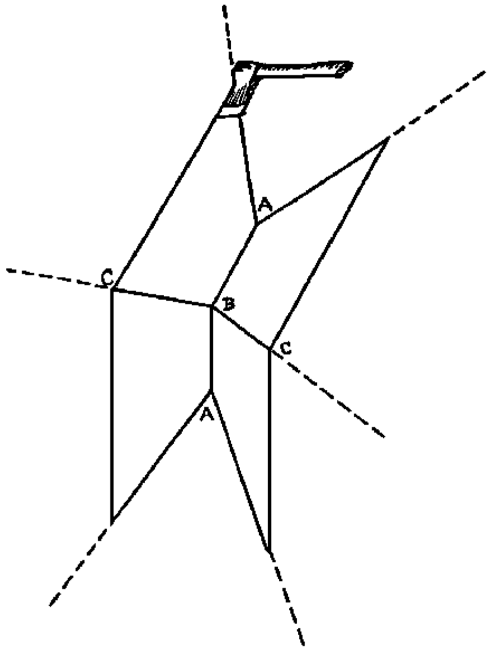



Figure 5.8

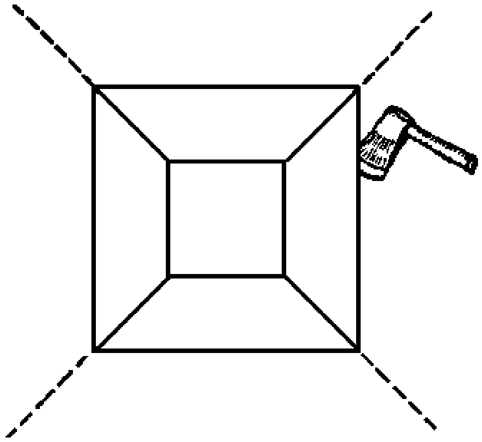
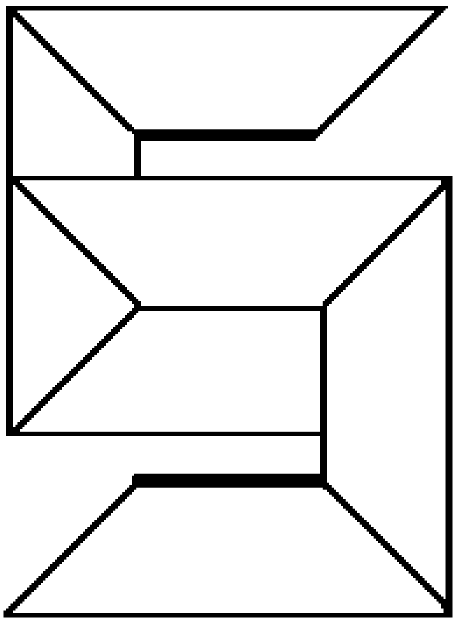


Figure 5.9

A little experimentation will show how this can be the case, improbable as it sounds. An interesting example is the regular flexagon of order 6, built from the leaves in the unit plan shown in figure 5.10. It must be remembered, in flexing such a flexagon, that, when the negative angle is "toward the center", the flexagon is upside down: the "upper" side shows underneath, the "lower" one on the top of the flexagon. Still, in flexing, the "lower" side must be folded together, so that the "upper" side will remain during the next face; i.e., we must flex backwards.



UNIT PLAN

Figure 5.10

It will be noted in some faces of this flexagon, as in all flexagons with more than $360^{\mathrm{O}}$ about the center, that these faces are rotated only with difficulty, and could not be rotated if made of rigid leaves. Thus, to rotate the hinges $AB$ of the face shown in figure 5.8 to the high position of hinges $BC$, the leaves must be bent considerably.

It is not hard to see how leaves with negative angles could be chopped up into pentagons, hexagons, and so forth (see figures 5.11 and 5.12). We have already used hexagonal leaves, made by cutting off corners of triangular leaves (see fig. 5-13). Thus we have already begun to relax the restriction of leaf shapes to triangles. However, so far neither the pat theory definitions, the map, the tree, nor the triangle network has needed alteration. Only the original descriptive definition, "flexible hexagon", needs changing. Now that we have seen how polygons of all kinds can be used to build flexagons, we will break the last remaining restriction in this direction, by allowing the polygons to have hinges on more than 3 sides. Doing this will give us another larger field of flexagons in which, at last, not only will leaves not necessarily be triangular, but map polygons will not necessarily be triangular, and neither will hinge network polygons. Moreover, flexagons will not necessarily be pairs of pats, as heretofore defined.

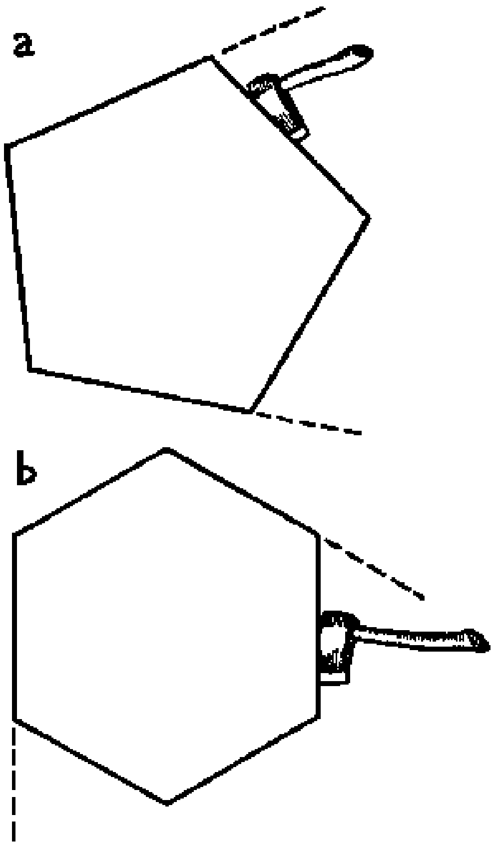



Figure 5.11

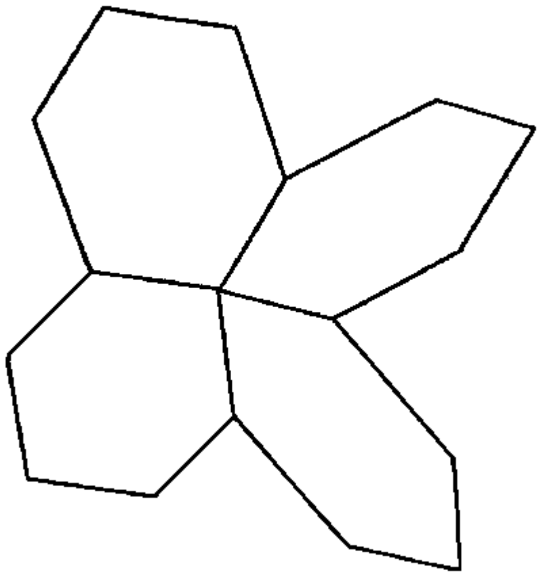


Figure 5.12

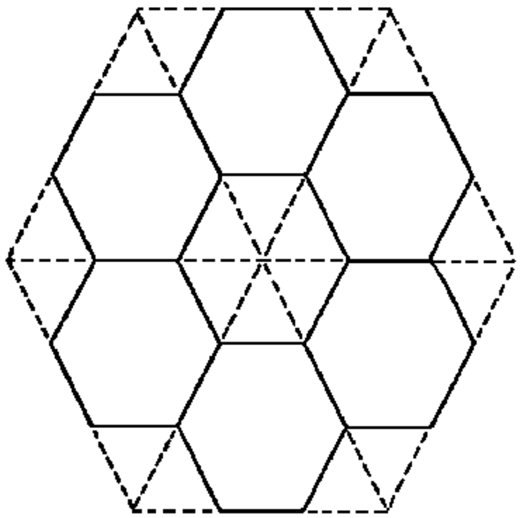


Figure 5.13

# Chapter 6

# G-Flexagons

A pat is made up of <u>two</u> lesser inverted pats. The only reason for our limiting the number of lesser pats to two is that with any more, we run out of different places to hinge them together and are forced to use one spot twice. This causes a "flap" to appear, and the flexagon begins to fall apart. In fig. 6.1, for example, we have three subpats in one pat, in the pat structure 1 2 3. Since the triangles have room for only three non-overlying hinges, one of the four hinges will be superimposed (as at AB), creating the loose flap shown which is not usually permitted in flexagons [1]. If, however, there were four positions for hinges, it would be possible to distribute the four hinges without causing flaps. Obviously, to have four positions we must use leaves with at least four edges.

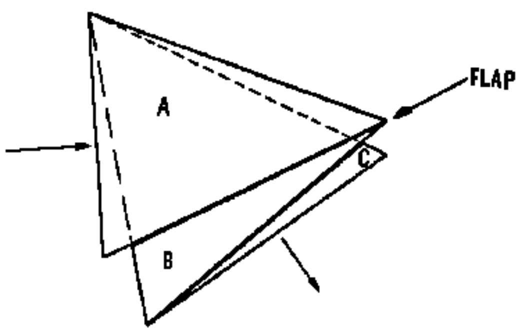



Figure 6.1

[1]However, Dr. F. G. Mannsell and Miss Joan Crampin (see bibliography) considered flaps, and presently (sect ____) we will study flexagons having superimposed hinges.

Just as we passed from three-edged leaves and pats made of two lesser pats to four-edged leaves and pats made up of three lesser pats, so we can pass from n-gonal leaves and pats with $(n-1)$ subpats to $(n+1)$-gonal leaves and pats made up of $n$ subpats.

Let us apply our theory to the construction of a flexagon constructed out of squares, and see how it works.

There are four possible positions around a given square at which we can attach another square by a hinge. If we label these positions clockwise 1–2–3–0, as in figure 6.2a, we can designate exactly by which side a given square is hinged to another. What we want to do is to hinge four squares together in such a way that a given square is attached to two adjoining squares by only two hinges and in such a way that each of the four hinge positions is used only once. At the moment, we will consider the constant order 1 2 3 4. We will hinge leaf number 1 to leaf number 2 with the hinge in position 1, or the 1–hinge; 2 to 3 with the 2–hinge, and 3 to 4 with the 3–hinge. We will leave the 0 position open, as it will hinge leaf 4 to leaf 1 in the next unit (see figure 6.2b. What we have just constructed is only one unit of a flexagon. The unit may be opened up so that leaf 1 is in the left hand pat and leaves 2 through 4 are in the right hand pat. All that is necessary in order to complete the flexagon is to add one more identical unit, hinging the leaf numbered 1 of one unit to leaf 4 of the other. If desired, an $n$ unit flexagon may be built, but those with more than two units will not lie flat.

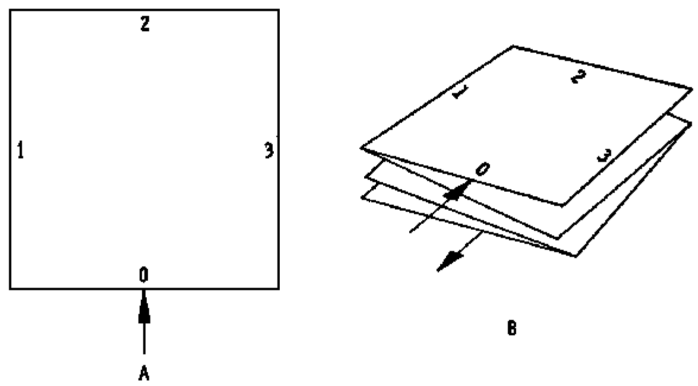



Figure 6.2

Thus far, the numbers we have given the leaves are pat structure numbers. It will be convenient to number the leaves in such a way that the number on the back of a leaf is one more or less than the number on front, just as in the triflexagons. If we number the four leaves facing up "1" and then fold the two pats of each unit together so that the unit structure is 1; 2 3 4, then the leaves may be numbered 4–1 (4 on top, 1 underneath) 1–2, 2–3, 3–4; in order top to bottom. The resulting plan is shown in figure 6.3; This flexagon may be flexed by grasping the two units on the hinge or axis between them (called the "flexing axis"), with the thumbs. The outside edges of the flexagon which are parallel to the flexing axis are then pushed down, thus pinching two adjacent pats together. The flexagon may then be opened with the thumbs to the new side (see figure 6.4). After flexing, the flexagon must be rotated $90^{\circ}$ and flexed along a new flexing axis. If we start at side 1, we find that we can flex to 2, 3, 4 and back to 1 again in that order. By turning the flexagon over, the order can be reversed (4–3–2–1). We now see that indeed, just as was predicted, we can make a flexagon that requires more than three operations to complete the cycle [2].

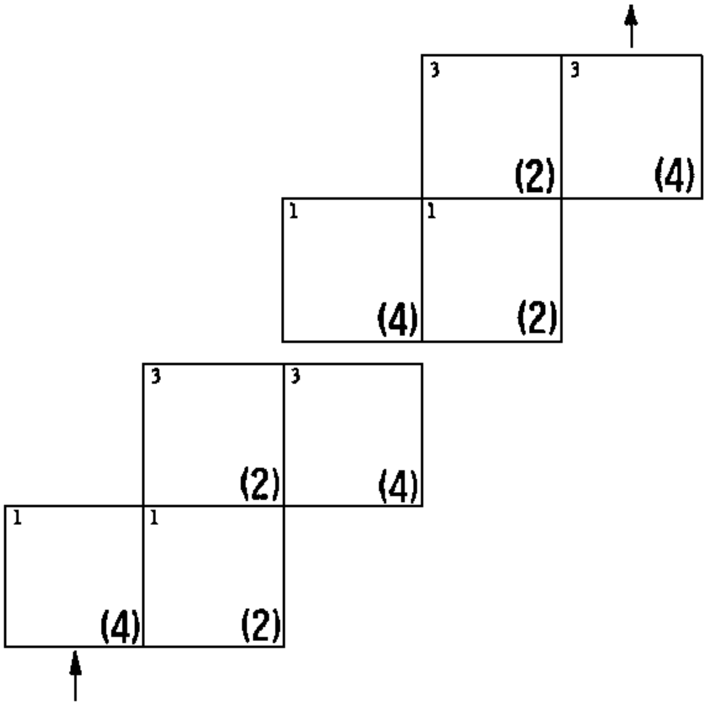



Figure 6.3

[2]A cycle is defined as a series of flexes from a given side back to that side without retracing any path or turning the flexagon over.

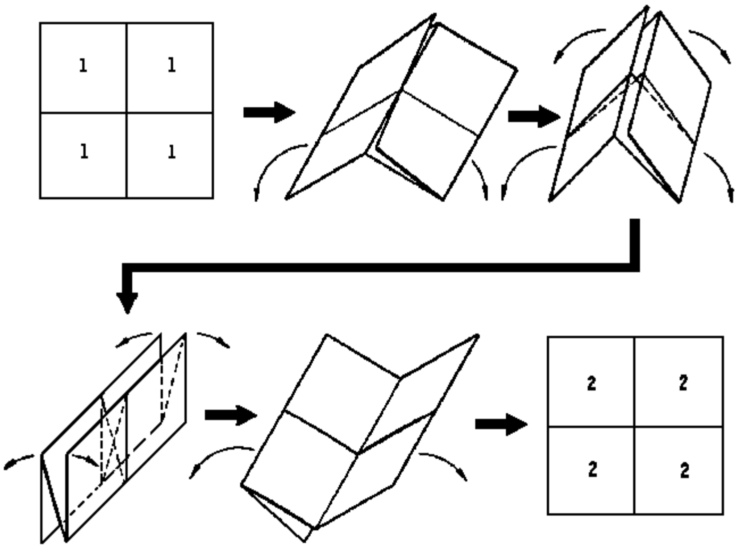



Figure 6.4

Similarly, we may use a polygon of $n$ sides and, by hinging to it $n$ congruent polygons consecutively in the hinge positions $1, 2, 3, ---, (n-1)$, 0 and numbering as described for the square flexagon $(n-1, 1-2, \ldots etc.)$, we may obtain a unit of a flexagon requiring $n$ flexes to travel from a given face back to itself again without retracing the path. The map of this flexagon may be represented by a polygon on $n$ sides, as in figure 6.5. A flexagon which requires $n$ flexings to return to a given side will be called a "cycle n" flexagon, or just a plain n–flexagon, it being convenient to read "n-" as the Greek prefix, tri–, tetra–, penta–, ..., octa–, for the 3 to 8 cycle flexagons. The flexagons higher than cycle four will not lie flat since the sum of the angles around the center for the two units combined will be greater than $360^{\circ}$. The plans for penta– and hexa– flexagons are shown in figure 6.6.

It should be noted also that the flexagons just discussed flex in such a way that the leaves are taken from the upper left and lower right pats (flexing axis held vertically) and are deposited on the upper right and lower left pats, respectively. It would be possible to make a flexagon which takes leaves from the upper right and lower left and deposits them on the upper left and lower right pats, respectively if the strip of polygons were wound the other way—that is, hinged in the order 3 2 1 0 instead of 1 2 3 0 [3].

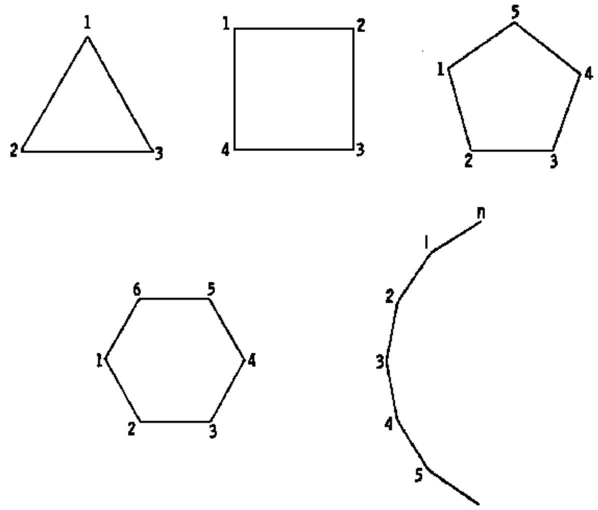



Figure 6.5

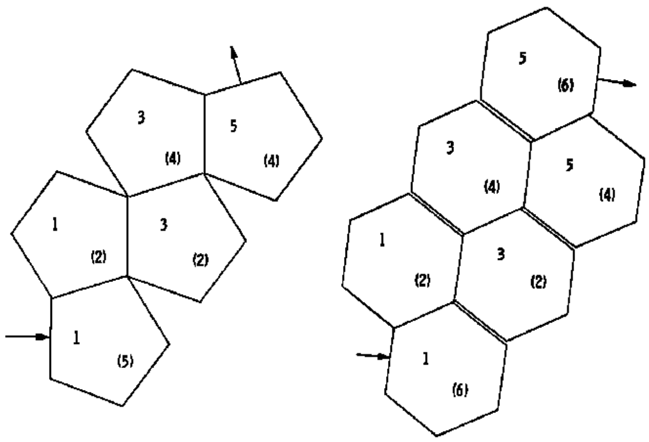



Figure 6.6

[3]From now on we will consider only the top unit when discussing which pat the flexagon peels from in flexing. Thus one hinged in the order 1 2 3 0 is called a left-flexing flexagon, while one hinged in the order 3 2 1 0 is a right-flexing flexagon

**Construction of a $G$–Flexagon of General Order $N$.**

If, after flexing a $G$–flexagon of the kind described above once, the same flexing axis is maintained, the flexagon will not flex since there is but a single leaf in the left pat. This pat is joined to the upper leaf of the right pat by a hinge in position $(G-1)$, where $G$ is the cycle of the flexagon. This single leaf in the left pat may, however, be slit in the manner used with the tri–flexagons. That is, another leaf may be hinged by a 1–hinge underneath the single leaf [4].

This allows a flex which will expose a new side. Note that the new hinge must be in the "one" position, otherwise, the flexagon still will not flex. In general, if the new side is added at side $a$, having come from side $(a+1)$, the new side will become $(a+1)_2$ and the sides $(a+1)_1, (a+2)_1, (a+3)_1 \ldots G$ (where $G$ is the old order) must be renumbered (the subscript refers to which of two cycles the side belongs). If we now rotate the flexagon and attempt to flex, the flexagon instead opens up using a set of hinges other than the 1–hinges and becomes a closed strip of four polygons in which the hinges do not meet in the center. This phenomenon is called tubulation. When an order five tetra–flexagon tubulates, it resembles a cube with two opposite faces missing (see figure 6.7a). A flexagon which is tubulating has the side which was tubulated from, $(a+1)$, on the outside and another side, $(a+2)$, on the inside. This relationship is shown on the map with a dotted line drawn between the two sides as in figure 6.7b and c.

If the flexagon is turned over and flexed back to $(a+2)$ it may then either flex to $(a+3)$ or tubulate to $(a+1)$, depending on whether the flexagon was rotated or not before flexing again. The operation of tubulating is very much like that of flexing, for it is seen that tubulating removes leaves from the left pat and deposits them on the right pat. Indeed if the effective hinge, that is, the hinge (not the zero-hinge) of each unit which is being used for the operation, were in a one–position, the operation would be a flex. If we were to cut the effective hinge used in this tubulation, which in this case is in a $(G-2)$ position, turn the flexagon inside out and tape the hinge back together, we would have side $(a+2)$ on the outside and side $(a+1)$ on the inside and could then close the tubulation, open to side $a$, and flex back to side $(a+1)$. This particular sequence requires only three operations (flexings and tubulations) to return to a given side without retracing a path, and thus may be looked upon as an attempt by the flexagon to become a tri–flexagon. However, we do not want to have to cut the hinge every time we run into a tubulation.

We want to have a complete cycle of ordinary flexes. We may do this

[4] An $n$–hinge is one which occupies position $n$ when a unit is folded together in such a way that the zero hinge connects the two units of the flexagon.

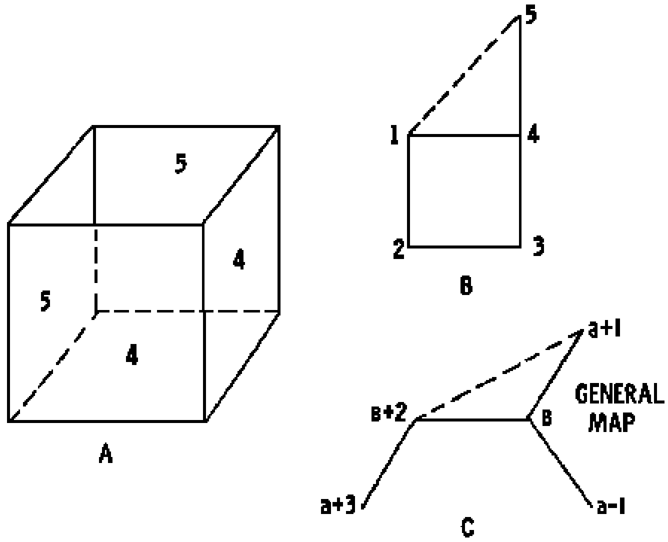



Figure 6.7

in the same manner as before, by slitting one of the leaves which is present at the face $(a+1)$, $a$. But in this case we find, oddly enough, that there are three possible ways of doing so. First of all, having come to side $(a+1)$ from side $a$, but after rotating and before the flexagon is tubulated, there is a single leaf, which can be slit, in the right hand pat. Also, tubulation removes all but a single leaf from the left hand pat. The third possibility is the middle subpat. This may be slit, in much the same manner as the single leaf. Slitting the right hand or middle leaves produces a cycle with a mixture of right and left flexes and will be considered in chapter 9. Any flexagon made so that it always flexes left (or right if it is wound the other way) will be called a proper flexagon. Any flexagon which does not flex consistently left or right and whose subpats are not hinged consecutively 1 2 3 ... $G$ is an improper flexagon. Since all our previous flexings have been from the left hand pat, at the present we shall consider the slitting of only the single left hand leaf arrived at after tubulating. If, after slitting this leaf and numbering the new side we rotate and try to flex again, we find that again the flexagon tubulates, this time using a $(G-3)$ [5] hinge. Now we again have a number of choices for slitting the leaves, it is possible not only

[5]This tubulation, if cut and turned inside out, would produce a four–cycle. In general, if the effective hinge of a tubulation is $(G-r)$, the cycle attempted is $(r+1)$.

to slit either the left or the right leaves, but to slit any one of the subpats in between them, hinging the new side in a number 1 position. The subpats are for all intents and purposes single leaves in this case. However, in order to be consistent, keeping the flexagon proper and flexing left, we will choose to slit the left hand leaf. Each successive time we slit, flex, and rotate, the hinge position of the tubulation's effective hinge decreases by one. After $(G-2)$ slittings, the effective hinge will be a 1–hinge, and the flexagon will flex normally through the new cycle. The new sides may be numbered in succession counter–clockwise about the map $(a+1),(a+2)\ldots(a+G-2)$ where $G$ is the cycle. The other sides through side $G$ may be renumbered, starting with $(a+G-1)$. By the method just explained, a cycle $G$–flexagon of any order may be constructed. The map will be made up of polygons with $G$ sides, which will be joined to one another by single edges (see figure 6.8).

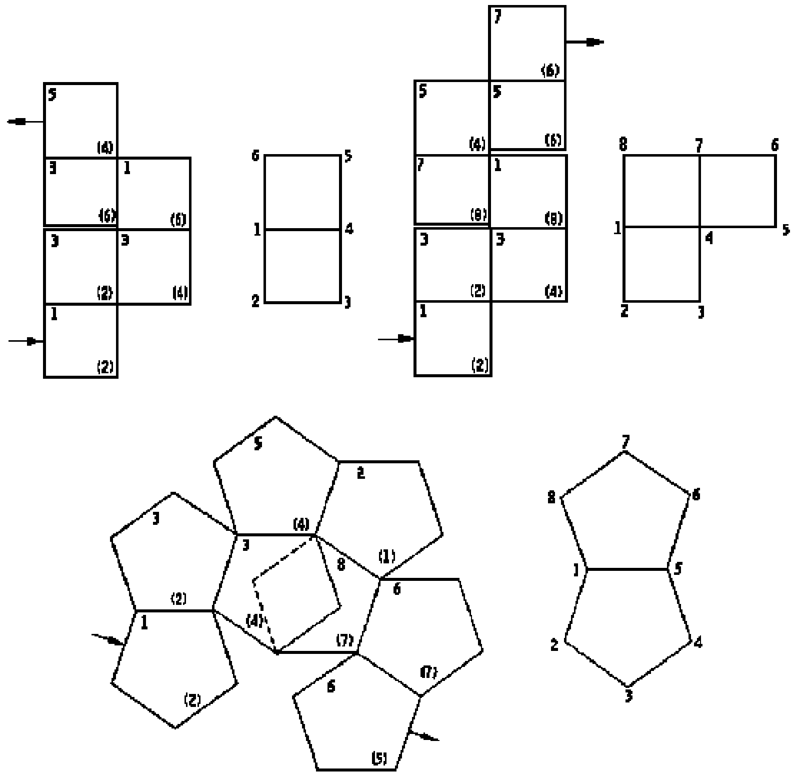


Figure 6.8

The ability to tubulate is extremely important and is deserving of extra study. If we add a new side by slitting the single left hand leaf in a tubulating flexagon and hinging the new leaf in a number one position, we find that we have not destroyed the tubulation but rather we have hidden it by making it easier for the flexagon to flex using a 1–hinge. This can be seen by clipping closed the side which would normally turn up next in any given flexing operation. If this if done the tubulation again appears. Tubulations which have been concealed by the addition of new sides are called "hidden" tubulations, while those which are a normal part of a series of flexes are called "exposed" tubulations. In any flexagon, there are hidden tubulations from any given side to every other non adjacent side of a given cycle. These hidden tubulations may be shown in the map as in figure 6.9a but since they clutter up the drawing, they are often omitted. Each hidden tubulation from a given side to each non adjacent side uses a different hinge. In all, there are $(G-3)$ possible tubulations using $(G-3)$ hinges originating at a given side. The hinges in positions 1 and $(G-1)$, are used by forward and backward flexes respectively (see figure 6.10a). Furthermore these hidden tubulations may be considered as short cuts for if the tubulation were cut, turned inside out, and normal flexing resumed, one or more sides would be omitted from the cycle (see figure 6.9b).

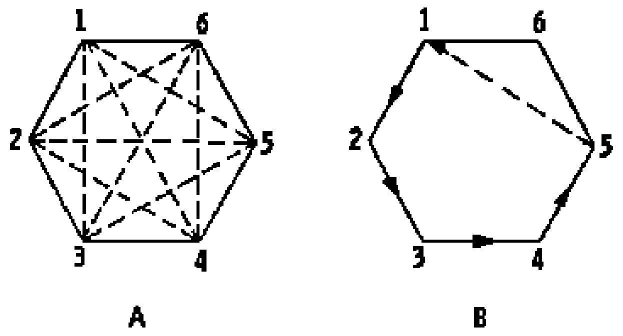



Figure 6.9

It is convenient to call all tubulations flexes and to give them a number which corresponds to the position of their effective hinges. Thus a normal flex which uses a 1–hinge will be called a "1–flex" while a tubulation which uses a 2–hinge will be called a "2–flex" and so on. The tubulation which we first encountered in slitting leaves to add a new side (figure 6.7c) was a $(G-2)$–flex. The backward flex, which use a $(G-1)$–hinge, is a "$(G-1)$–flex" (see figure 6.10b). The proper octaflexagon is a striking proof that a tubulation should be considered a flex. The angle between the input and output hinges in a 2–flex is $90^{\circ}$ and in a two unit flexagon the sum of the angles about the center is $360^{\circ}$, so the tubulating flexagon will lie flat. Furthermore, since the tubulating flexagon does lie flat, it is not necessary to force the tubulation. Two separate yet complete cycles of 2–flexes can be made, the operation resembling very closely that of 1–flexes in a tetraflexagon (see figure 6.11).

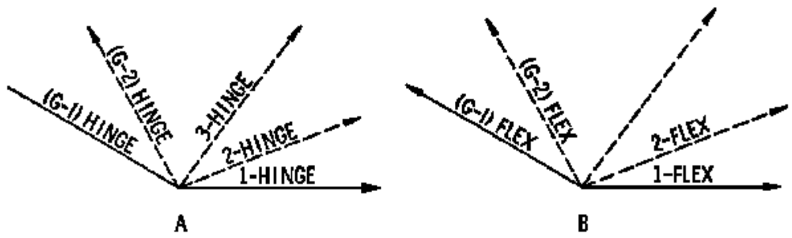



Figure 6.10

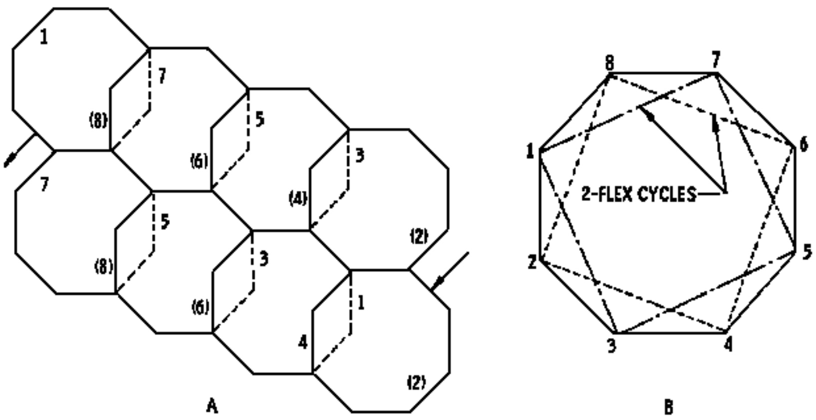



Figure 6.11

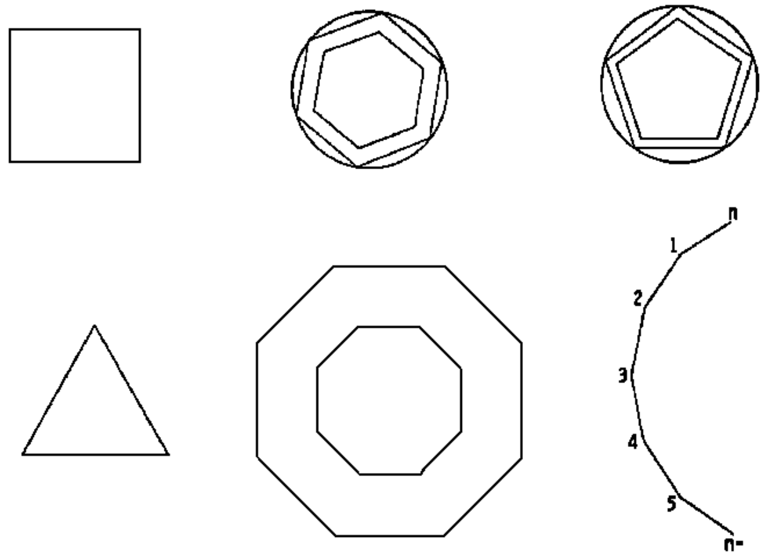



Figure 6.12

# Chapter 7

# Proper Flexagons

All proper flexagons, no matter what the cycle, have certain very important characteristics which may now be used to define the family of proper flexagons. The first important characteristic is found in the hinging order of all proper flexagons with complete $1-$flex cycles. Take, for example, a complete single cycle flexagon. The leaves of this flexagon are always hinged to one another in the hinge order $1\ 2\ 3 \ldots (G-1)\ 0$. This sequence is called the hinge sequence. This particular hinge sequence in which the numbers run consecutively from 1 to 0 mod. $G$ is characteristic of single cycle proper flexagons. The constant order is the other important characteristic. In single cycle $G-$flexagons it is always consecutive: $1\ 2\ 3 \ldots (G-1)\ g$. If more cycles are added to the single cycle proper flexagon, some single leaves of the original flexagon become subpats. The subpats will contain all the sides of the new cycle, and since the new cycles are proper, the pat structures and hinge sequences of the subpats must be consecutive. It is true that the constant order and the hinge sequences of proper flexagons with more than one cycle are not consecutive, but the subpats will always be arranged consecutively from top to bottom, and the hinging of those subpats will always be consecutive.

It is important to notice that a $G-$flexagon with $G$ greater than 3 has more than one thumbhole, a thumbhole occurring when there is a positive hinge difference between two successive leaves in the pat structure. If we look at the pat structure of the "simple" proper pat, that is the pat from a single cycle flexagon (as opposed to a compound pat from a multi-cycle flexagon), we find that it may be either 1 or $1\ 2\ 3 \ldots m$ where $m$ is the degree of the pat. The first pat has no thumbholes but in the second pat there will be thumbholes between leaves 1 and 2, 2 and 3, $\ldots m-1$ and $m$, or $m-1$ thumbholes per pat. If we designate these thumbholes from top to bottom 1 2 3 ... $(m-1)$ and then observe their operation, it will be seen that thumbhole 1 has a number 1 hinge associated with it and thus is used in a 1−flex, while each succeeding thumbhole is used for a correspondingly higher order flex. After a flex and rotation, the thumbholes should be renumbered, but it should be observed that what was originally a number 1 thumbhole has now opened out to display a new side while the old number 2 thumbhole has been rotated in such a way that its hinge is in a number 1 position. The thumbhole at the very bottom has been generated by folding together the side which was on the back of the flexagon before flexing. This last thumbhole has a hinge in a $(G-1)$ position and may be used for flexing backwards. This discovery, that a given pat has $(m-1)$ thumbholes, makes it necessary for us to revise our definition of a pat. A pat, therefore, may be defined (in a circular way) as a series of subpats, each of which is a pat in itself; alternate subpats being inverted. The total number of subpats in a given unit is equal to the cycle. In proper flexagons, there are $(G-1)$ subpats in one pat of the unfolded flexagon and one subpat in the remaining pat.

Each single leaf of a simple proper flexagon unit is associated with two hinges and each hinge with two leaves. We may show this by writing down the constant order and indicating the hinges between two leaves as follows:

$$\begin{array}{cccccc} & 1 & 2 & 3 & 0 & \\ \underline{1}; & 2 & 3 & 4 & - & \underline{1}; \end{array}$$

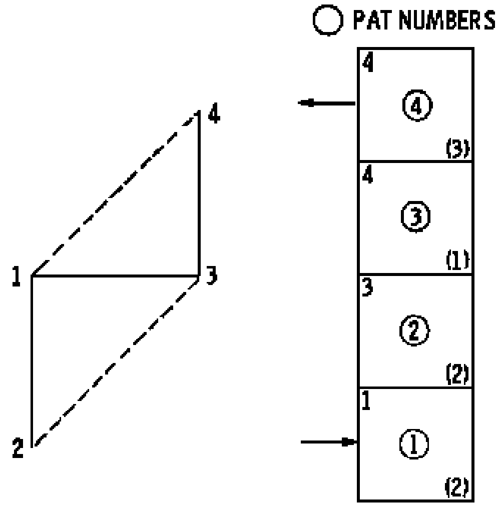



Figure 7.1

This is the constant order and hinge structure for a one-cycle proper tetraflexagon. But what would happen if we were to try to write the unit and hinge structures for a tetraflexagon with the map as in figure 7.1. Its constant order is 1 2; $\underline{4\ 3}$ while its hinge sequence is 1 3 1 0. Combined this would make:

$$\begin{array}{cccc} & 1 & 3 & 1\ \ 0 \\ 1 & 2; & 4 & 3 - 1; \end{array}$$

This is a rather inconvenient method of notation, so instead, we will only associate one hinge with one leaf. Arbitrarily we will identify the first leaf with the first hinge of the hinge sequence. The leaf with pat number 2 will be associated with the second hinge, etc. The example above would then become:

$$\begin{array}{cccc} 1 & 3 & 0 & 1 \\ 1 & 2; & \underline{4} & \underline{3}. \end{array}$$

Previously, we assigned hinge positions around a polygon in a certain direction of ascending values. This direction was clockwise and may be indicated around the face of a polygon by vectors drawn in the same direction along the edges of each side. A certain side will be designated as the zero position and it will be agreed that the side toward which the zero-side vector points to will be the "1" position. Then we may systematically follow the vectors and number each position with the number of vectors which are between it and the zero position (see figure 7.2a). These vectors may be drawn on both sides of all the leaves in the unit in such a way that when the unit is folded together, the vectors all point in the same direction. This "orientation" of the polygons of a unit will give us a frame of reference when the flexagon is unfolded. Before unfolding the flexagon, it would be well to note the relationship existing between the two hinges associated with a given leaf. If the hinge associated with the previous leaf (with respect to increasing pat numbering) was in an $a-$position and the one we are concerned with is in a $(a+x)-$position, the position this $(a+x)-$hinge holds with respect to the $a-$position hinge is $x$, the difference between the two hinges. This difference between two hinges attached to the same leaf is the "hinge difference" across that leaf. The advantage of the hinge difference is that it is independent of the arbitrarily assigned zero point for the hinge sequence. The hinge sequence for a one cycle proper flexagon is $123\ldots G-1$, $0123\ldots$. The hinge difference across the first leaf is $1-0=1$, that across the second $2-1=1$, etc. Since the hinge sequence is consecutive the hinge differences will all be one. These hinge differences nay be written in the following manner:

| | | | | | | |
|---|---|---|---|---|---|---|
| Hinge difference | 1 | 1 | 1 | 1 | | 1 |
| Hinge sequence | 1 | 2 | 3 | 4 | ... | 0 |

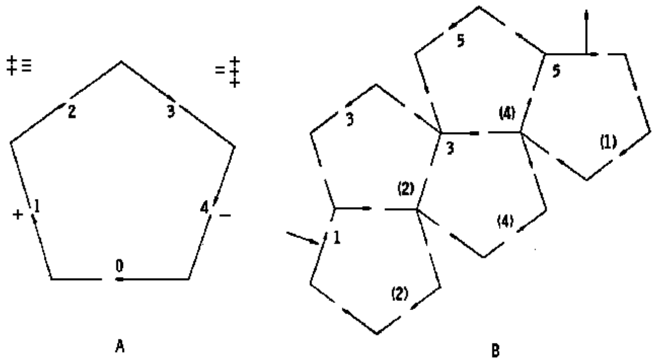



Figure 7.2

The first "1" is the difference between an understood 0 hinge and the hinge in position one. The last "I" is the difference between $(G-1)$ and (0 mod. $G$) or just $G$. If we now open up the flexagon, we see this constant difference of 1 quite plainly. A given hinge is always one side removed from another hinge attached to the same leaf (see figure 7.2b). One will notice, however, that the direction is alternately left and right. This is because alternate leaves in the pat were inverted, as a result of the folding process. The vectors help keep track of things, for they point in a consistent manner around the unit polygons of the folded flexagon. The hinge of a certain leaf in the unfolded flexagon will always bear the same relationship to the vectors of that leaf as the hinge of any other leaf. That is, if we decide to travel along the plan in a given direction, the vectors we cross in going from one leaf to another will always point consistently toward or away from the next hinge. Whichever way they do point, they can always be made to point the other way by starting at the other end of the plan.

It is convenient for certain purposes which will become apparent to call the direction toward which the vector points + and the direction away from it −. Thus a 2−hinge is +2 or $\overset{+}{+}$ while a $(G-1)$−hinge is − and a $(G-2)$−hinge is $(\overset{-}{-})$ (see figure 7.2a). The sign sequence for a proper flexagon of one cycle, then is $+++\ldots$ or $---\ldots$. Here we see that + and − may be exchanged for one another just as in the triflexagons.

**The Polygon System**

There are $(G-1)$ possible hinges for a leaf with G sides. [1] With proper flexagon leaves, however, there are only two possibilities if the flexagon is

[1] Such a leaf is said to be of "class G", as is the corresponding flexagon.

"complete", i. e. if it contains no incomplete cycles. To prove this, let us look at a proper complete single cycle $G-$flexagon. This flexagon, as we have seen has two possible sign sequences, $+++ \ldots ++$ or $- - - \ldots - -$. Now, if we add another cycle to this flexagon, we will be making a subpat out of what was previously a single leaf. If we added the new cycle between sides $(a)$ and $(a+1)$, the single leaf we must slit will have $(a)$ on one side and $(a+1)$ on the other. The $G$ possible places for attaching new cycles on the map correspond to the $G$ single leaves in a unit of a one cycle $G-$flexagon. If we do build subpats out of one or more of the single leaves of a single cycle $G-$flexagon, each subpat must have the same characteristics as the large pat in order for the flexagon to remain proper. Thus we know that the hinge difference between successive leaves in the subpat must consistently be either + or $-1$ and that the pat structure must be consecutive (these are the two requirements for a proper pat). We also know that the hinge difference across the whole subpat must be $+1$ when viewed from the top, with the vectors pointing clockwise since the leaf from which the subpat was generated had a $+1$ hinge difference. Since there will be $G-2$ sides and hence $G-2$ leaves added to the one leaf already present (as a former member of the large pat) there will be a total of $G-1$ leaves in the subpat. If the hinge differences must be consistently either $+1$ or $-1$ for each of the leaves, and if there are $G-1$ leaves, which must have a total hinge difference of $+1$, the individual hinge difference must be $-1$, since $0 - (G-1)\mathrm{mod}.G = +1$. If the individual hinge difference were $+1$, the total difference would be $0 + (G-1) \bmod G \overset{-}{\_} -1, \overset{-}{\_} +1$. Therefore, the hinge differences between the leaves of the subpat are negative with respect to those of the large pat. Similarly, if another subpat were to replace a leaf of this subpat, the individual hinge differences would be $+1$, since the hinge difference across a leaf of the first subpat is $-1$. The pat structure of any subpat will be inverted with respect to the next larger pat or subpat of which it is a member, just as was so with triflexagons. Therefore, when the leaves of a large pat are numbered $(m) - 1, 1 - 2, \ldots (m-1) - (m)$ in ascending order from top to bottom, the subpats will be numbered in ascending order from bottom to top. Now, when such a compound pat is unwound, the progression of the number sequence from smaller to larger numbers in that portion of the plan which corresponds to the subpat will be just opposite the progression in that part of the plan in which the subpats were single leaves. This means that if we have a given complete one cycle $G-$flexagon whose number and sign sequences are:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| + | + | + | + | | + | + | | + | + |
| (1) | 3 | (3) | 5 | ... | (a) | a+2 | ... | (G-1) | 1 |
| 2 | (2) | 4 | (4) | ... | a+1 | (a+1) | ... | G | (G) |

(where $G$ is even) and we wish to insert another cycle adding $(G-2)$ sides between sides $(a)$ and $(a+1)$, the number and sign sequences will become:

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| + | + | + | | + | − | − | | − | − | + | + | + | | | + |
| (1) | 3 | (3) | ... | a | (a+G-2) | a+G-2 | ... | a+2 | (a) | a+G | (a+G) | a+G+2 | ... | (2G-3) | 1 |
| 2 | (2) | 4 | ... | (a-1) | a+G-l | (a+G-3) | ... | (a+1) | a+l | (a+G-1) | a+G+1 | (a+G+1) | ... | 2G-2 | (2G-2) |

For instance, if we wanted to construct a complete order l4 octaflexagon in which the second cycle is added between 5 and 6 (figure 7.3a), the sign and number sequence for the plan would be:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| + | + | + | + | − | − | − | − | − | − | − | + | + | + |
| (1) | 3 | (3) | 5 | (11) | 11 | (9) | 9 | (7) | 7 | (5) | 13 | (13) | 1 |
| 2 | (2) | 4 | (4) | 12 | (10) | 10 | (8) | 8 | (6) | 6 | (12) | l4 | (l4) |
| | | | (a-1) | (a+G-2) | | | | | (a+l) | (a) | (a+G-l) | | (2G-2) |

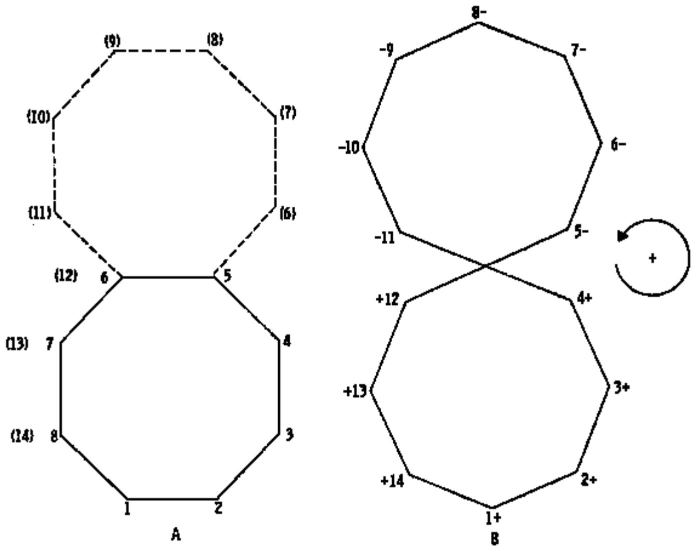



Figure 7.3

If we want to add another subpat to one of the leaves of the new subpat, its numbers would be reversed with respect to the already reversed numbers of the first subpat: they would increase from left to right and the signs would be +.

To figure out the plan by this method would be a long and tedious job, but it happens that one can use a shortcut device which is similar to the Tukey triangle system in the triflexagons. This shortcut makes use of a "polygon system" which is simply a generalization of the triangle system. The network is arrived at by drawing lines between the midpoints of the polygons making up the map and assigning numbers to the vertices thus formed (see figure 7.4a). The polygon system is followed in the same manner as the Tukey triangle system: The numbers of the vertices are written down in order to give the basic number sequence. The number sequence for the plan is arrived at by writing down the numbers of the basic number sequence alternately above and below a line and then adding 1 to each, placing the result on the other side of the line as shown:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| + | + | + | + | + | + | − | − | − | + | + | − |
| 1 | 3 | 3 | 11 | 11 | 1 | 7 | 7 | 5 | 9 | 9 | 55 |
| 2 | 2 | 4 | 10 | 12 | 12 | 8 | 6 | 6 | 8 | 10 | 4 |

A certain positive direction in the polygon system may be assigned. When a bend in the polygon system is made in that direction, the vertex at the bend is labeled +; when the bend is in the opposite direction, the vertex is −. Using this method, the sign and number sequences of any proper complete $G$−flexagon can be found, and its plan constructed in a manner similar to triflexagon plans: A system of oriented polygons is used. Choose one such polygon, and choose the side through which to enter that polygon. A + of the sign sequence means that the polygon must be left through the side toward which the vector of the side entered points; a − means the polygon must be left through the side away from the vector. The process is repeated using the next sign in the sign sequence, until the desired number of units have been manufactured (see figure 7.4b).

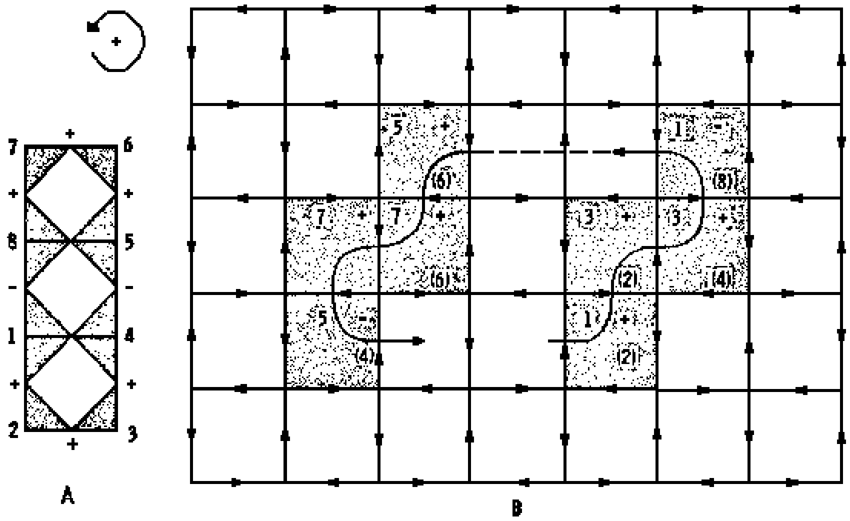



Figure 7.4

A Tuckerman tree may be drawn to represent the map polygons, and the polygon system may be drawn about it; The Tuckerman tree for a given $G-$flexagon is not unique however, since a $2G-$flexagon can be built with the same tree. The polygon system will work for proper complete flexagons only, because it allows no more than two possible choices for hinging, whereas a leaf of class $n$ offers $(n-1)$ possibilities. Only in the case of triflexagons will the polygon system account for all possibilities.

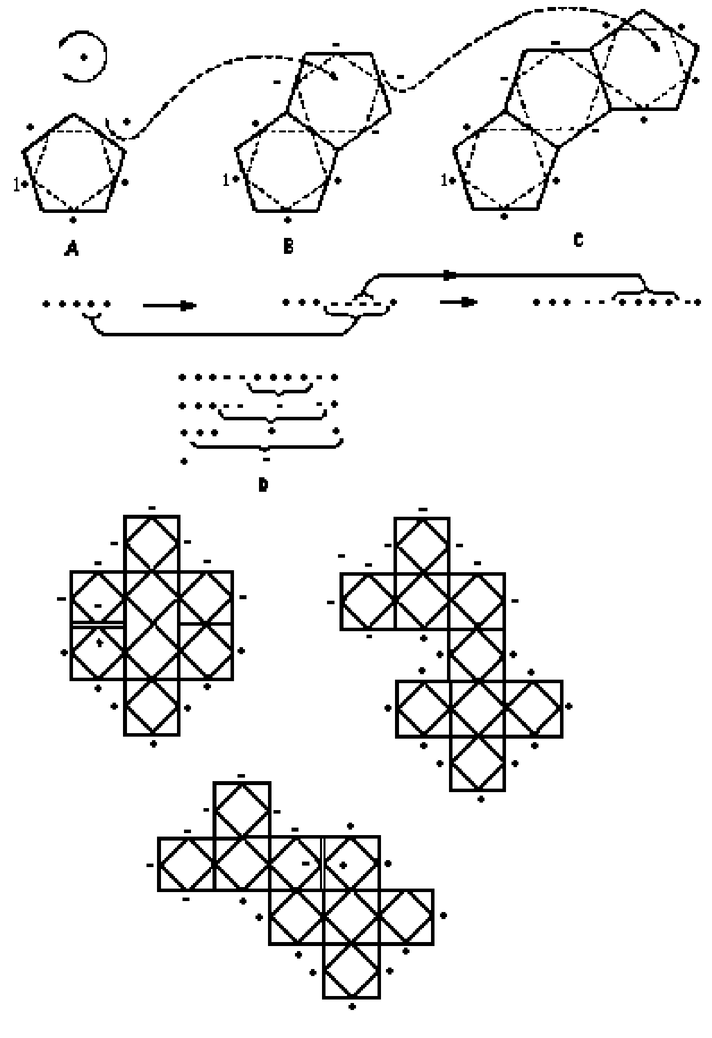
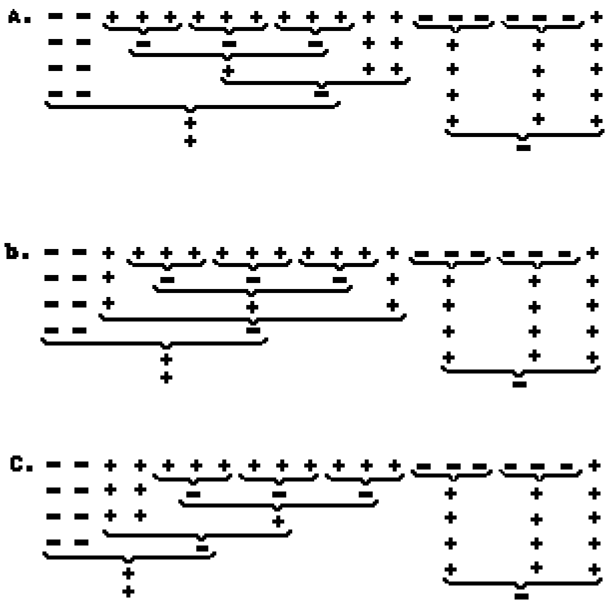



Figure 7.5

A complete proper $G$−flexagon of one cycle has a sign sequence composed entirely of $+$ signs. If we add another cycle to this single cycle, one of the $+$ signs is withdrawn from the sign sequence and in its place are put $(G-1)$ minus signs (see figure 7.5b). If we wish to continue, we may withdraw one of the minus signs and substitute $(G-1)$ plus signs (see figure 7.5c). Similarly, we may systematically reduce a flexagon with a number of cycles to one with a single cycle by exchanging $(G-1)$ adjacent signs of one kind for one of the opposite signs (see figure 7.5d). A given proper cycle always reduces by this method to $+-$. Since the $+$ in the $+-$ represents a 1−hinge and – represents $(G-1)$−hinge, we can add the two hinges: $1+(G-1)=G=0$ mod. $G$. Thus we have established that the sum of the signs of a given sign sequence must be congruent with 0 mod. $G$. In triflexagons, $G=3$ so the summation of the sign sequence equals 0 mod. 3, which has already been shown. If a given sign sequence is congruent to 0 mod. $G$, we may find out how many different flexagons can be made from it by reducing groups of $(G-1)$ of a given sign to one of the opposite sign.

Example:

The tetraflexagon sign sequence $--+++++++++++------+$ may be reduced to $+-$ in a number of different ways:



These are just three possibilities. There are more.

# Chapter 8

# The Flexing Operation and Tubulating Proper Flexagons

The flexing operation, we have seen, requires a thumbhole with a 1−hinge. In the case of the proper flexagon, this is the number one thumbhole. This thumbhole splits the left hand pat into two groups, one composed of a single subpat and the other composed of $G-2$ subpats. The complete flexing operation inverts the top-most subpat of the left pat, leaving it on the left side, while it deposits the remaining subpats on the inverted right pat (see figure 8.1). The number one thumbhole has been opened out to display the next side. The two new pats are now joined by what was originally the 1−hinge. When the flexagon is rotated, the hinges must be renumbered. The original 1−hinge will become the new 0−hinge, and each of the other hinges will have values one lower than before the flex and rotation. In the normal flexing operation of a proper flexagon, which has a consecutive subpat hinge sequence and a consecutive subpat structure, the order of turning up sides must also be consecutive (as has been shown empirically). This is because each successive thumbhole is associated with a correspondingly numbered hinge, (i.e. the 1−hinge with the number one thumbhole, etc.). Each flexing operation subtracts one from the value of each hinge, thus bringing the thumbhole with which any particular hinge is associated closer to the position for being opened up next. A subpat hinge which in in position $R$ will be opened after $R$ flexes and rotations.

If we flex along a given cycle of a flexagon, we notice that we always progress in the same direction, either clockwise or counterclockwise, along

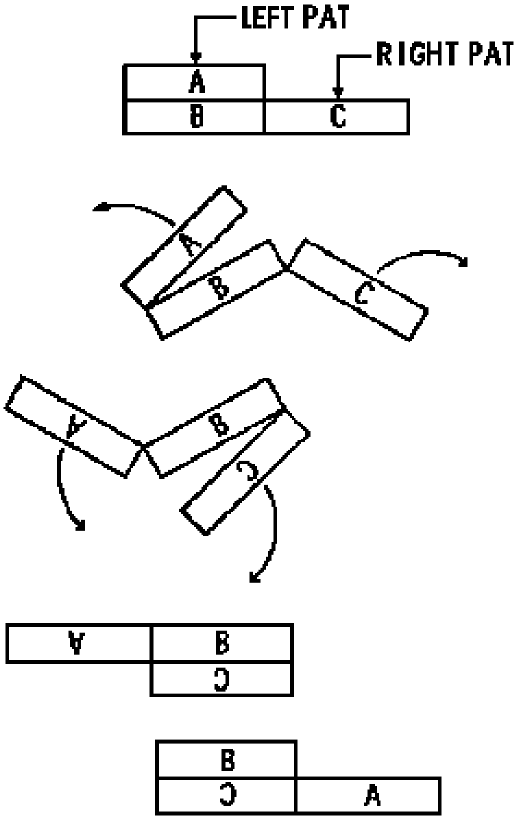


Figure 8.1

the path of the map. If we draw vectors along the edges of the map (indicating in which direction we are progressing), the vectors for a given cycle will always point consistently clockwise or counterclockwise. Whichever way they do point, their direction may be reversed by turning the flexagon over. If a second cycle is added and vectors are drawn on the map, this new cycle will have one vector in common with the first cycle, but the direction of all the vectors around the center of the map polygon representing this cycle will be just opposite from that of the vectors in the first cycle. In fact, the vectors of any map cycle which has one edge in common with any other given cycle will point in the opposite direction with respect to that cycle. This means that all of the polygons in the map of a given flexagon will be oriented; the sense of the orientation may be changed by turning the flexagon over, since in so doing all vectors are reversed.

The reason for the reversed pat structure of a subpat with respect to its large pat can now be explained. Consider the history of a proper subpat of a proper flexagon. The pat structure of this subpat will remain unchanged throughout the flexing operation (assuming the flexagon flexes left) provided it is in the left pat and provided it is below thumbhole 1, which has a hinge in the "1" position. This is because a 1−flex moves all of the subpats which are below thumbhole 1 from the left pat to the right pat, unchanged in any way (i.e. uninverted). However, if this subpat is on the top of the left pat, and the flexagon is flexed, the subpat will remain in the left pat but will be inverted. A rotation and flex will reinvert it and place $G-2$ subpats on top of it, leaving it undisturbed for $G-2$ flexes thereafter. However, if we decide not to rotate, but to flex along a new cycle, this subpat alone will remain in the left hand pat. As the flexagon must always flex left (we built it that way), the first flexing operation will open up the first thumbhole to a side, $(a-l)$, the last side up being (a) if the flexing was proceeding in an ascending order around the map (see figure 8.2a). For instance, in an order 6 tetraflexagon as shown in figure 8.2b, if we flex from 1 to 2 to 3 to 4 to 1, and then decide to change over to a new cycle, we must flex next to side 6. Although the numbering of the map is still counterclockwise, the flexing vectors have changed direction, and the flexing must proceed against the numbering. In order for the side following side (a) to be side $(a-1)$, the $(a-1)$ thumbhole must be the lowest thumbhole in the subpat containing the new cycle, for that subpat will be inverted, making the $(a-1)$ thumbhole the top most one when the subpat is left along in the left pat. Similarly, in flexing about the second cycle (assuming there are no other cycles attached to the second one) $(a-2)$ will follow $(a-1)$ and will in turn be followed by $(a-3)$, and so on until $(a-G-1)$ is reached. Since side $(a-1)$ used the first thumbhole in the subpat, sides $(a-2)$ through $(a-G-1)$ must use the others in order, and when the subpat constant order is inverted by the subpat's incorporation into the large pat, $(a-G-1)$ will be the side nearest the top. Since each thumbhole in a subpat can be associated with a single leaf, the thumbhole order becomes reversed also. Since the basic number sequence must increase consecutively when read down the pat, the pat structure for the subpat must be inverted with respect to the large pat.

Since we are on the subject of flexing operation, let us consider flexing operations other than the 1−flex; i.e., the tubulations. For all proper flexagons, the $2-, 3-, 4-\ldots, (G-1)-$flexes remove all but $2, 3, 4, \ldots (G-1)$ leaves respectively from the left pat and deposit them from the right pat; As has been stated before, a tubulation acts like a flex. For instance, in a

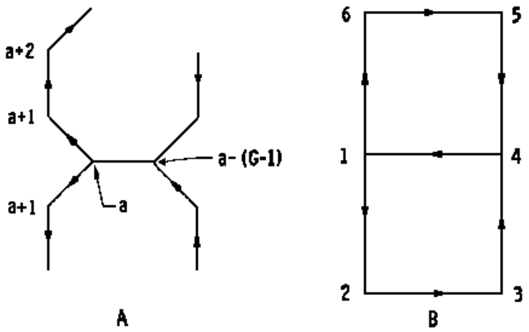



Figure 8.2

tetraflexagon of order 3 (see figure 8.3a) which has a tubulation from 1 to 3, we can 1 flex from face $2-1$ to face $3-2$ and when we tubulate, we can cut the hinge and lay the flexagon out in the form of a straight strip of squares with one on the top and three on the bottom. This, then, is face $1-3$ (see figure 8.3b). When we want to flex from face $1-3$ back to $2-1$, we fold the three's so that they face each other and tape the cut hinge back together. The flexagon will now open up to side 2. We should notice, however, that this process of turning the tubulation in side out has also exchanged the position of the two pats of a unit with respect to each other. This is equivalent to a rotation so in this operation, we have both flexed and rotated.

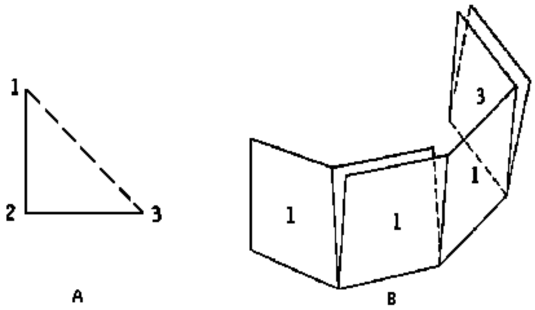



Figure 8.3

## Flexing Characteristics

In connection with the flexing operation, the relationships between the right and left flexing flexagons should be discussed (The right or left flexing aspect of a flexagon will be called its flexing characteristic.). We have seen that the difference between right and left flexing flexagons lies in the way the flexagons are wound up. What changes, then, can be made in the plan of a flexagon with one characteristic to convert it into a flexagon of the other characteristic. That is, how can a plan be made to wind up the other way? Perhaps the most obvious is interchanging the number sequence numbers on the back and front of a plan. Thus, if

$$\begin{array}{ccccc|c} + & + & + & & + & + \\ 1 & 3 & 3 & \dots & 1 & 1 \\ \hline 2 & 2 & 4 & \dots & \mathrm{N} & 2 \end{array}$$

represents a left flexing flexagon, then

$$\begin{array}{ccccc|c} + & + & + & & + & + \\ 2 & 2 & 4 & \dots & \mathrm{N} & 2 \\ \hline 1 & 3 & 3 & \dots & 1 & 1 \end{array}$$

represents a right flexing flexagon, all else being equal. Another way of changing the flexing characteristic is by changing all signs in the sign sequence. If the tetraflexagon

$$\begin{array}{cccccc|c} + & + & + & - & - & - & + \\ 1 & 3 & 3 & 1 & 5 & 5 & 1 \\ \hline 2 & 2 & 4 & 6 & 6 & 4 & 2 \end{array}$$

is a left flexing flexing flexagon, then the flexagon

$$\begin{array}{cccccc|c} - & - & - & + & + & + & - \\ 1 & 3 & 3 & 1 & 5 & 5 & 1 \\ \hline 2 & 2 & 4 & 6 & 6 & 4 & 2 \end{array}$$

is a right flexing flexagon. Now, if we look at these plans, it becomes apparent that they are no more than mirror images of each other. In fact, if we watch the flexing operation of a left flexing flexagon in a mirror, we see the operation of a right flexing flexagon. A right flexing flexagon is the mirror image of a left flexing flexagon.

## Tubulating Proper Flexagons

Any tubulating flexagon (that is, one with exposed tubulations, also called an incomplete flexagon) may be made by deleting a given number of sides from a certain complete flexagon of order $N + (F_i - 1)$ where $N$ is the order of the tubulating flexagon and $F_i$ is the order of the flex involved in a tubulation. What we want to know now is what effect on the plan this tubulation has. Let us take for an example a complete proper hexaflexagon of order 6 (see figure 8.4a). If we wish to delete a side, say side 5, we fold the two 5's in the plan together. When we do this, we find that the hinge difference across the double leaf is $+2$ (or $-4$) (See figure 8.4b). Now if we want to delete side 6, fold the 6's together, and the total hinge difference across the triple leaf is $\pm 3$ (see figure 8.4c). If we were to take two hexagonal leaves, each with a hinge difference of $+2$, (see figure 8.4 d,e), the total hinge difference when the leaves are folded together would be $+4$ or $-2$. In general, whenever we fold together two leaves with hinge differences respectively of $+a$ and $+b$, the total hinge difference will be $(a+b)$ or $(a+b-G)$. To produce an exposed $n-$flex, we must delete $(n-1)$ leaves from a complete proper pat, but since each of these leaves has a hinge difference of 1, the total hinge difference across the leaf resulting from the deletions must be $1 \times n$. Therefore, in the plan, an $n-$flex requires a leaf with a hinge difference of $n$.

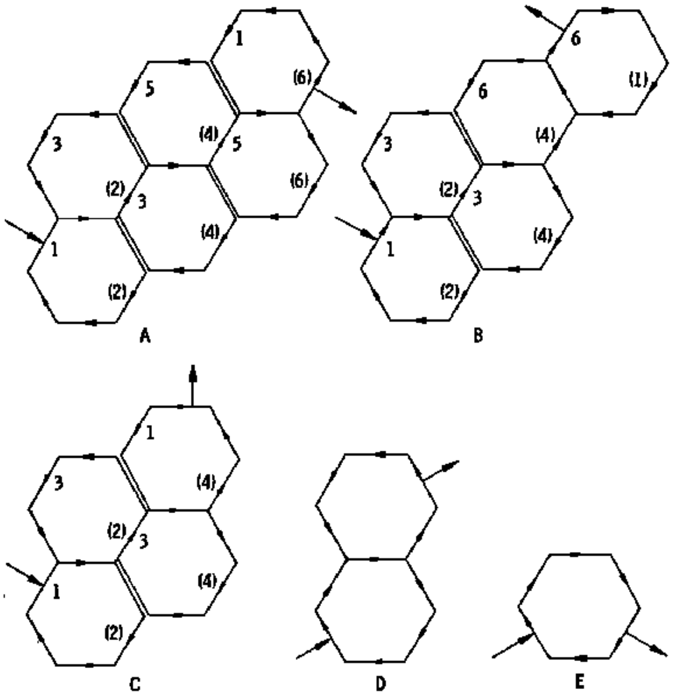



Figure 8.4

Example:

Find the plan for the flexagon with the map as shown in figure 8.5a. The method of attack for this problem is to view the tubulating flexagon as a product of the deletion of sides from the flexagon with the map as shown in figure 8.5b. The sign and number sequence for this flexagon will be:

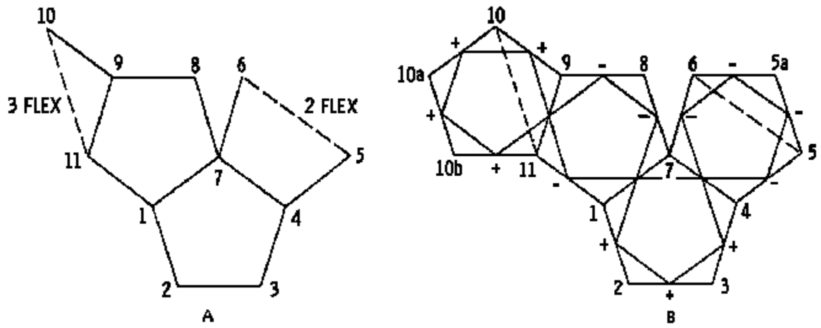



Figure 8.5

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| + | + | + | - | - | - | - | - | - | + | + | + | - | - | + |
| (1) | 3 | (3) | 7 | (5a) | 5a | (4) | 1 | (9) | 10a | (l0a) | 11 | (8) | 8 | (1) |
| 2 | (2) | 4 | (6) | 6 | (5) | 5 | (11) | 10 | (10) | 10b | (10b) | 9 | (7) | 2 |

Now if we delete side $5a$, the hinge difference across the new leaf will be $(-)+(-)$ or $-2$. The process of deleting this side will turn upside down that part of the plan from 5 on:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| + | + | + | - | $\substack{-\\-}$ | - | - | + | + | + | + | - | - | + |
| (1) | 3 | (3) | 7 | (5) | 5 | (11) | 10 | (10) | 10b | (l0b) | 9 | (7) | (2) |
| 2 | (2) | 4 | (6) | 6 | (4) | 1 | (9) | 10a | (10a) | 11 | (8) | 8 | (1) |

The sides $10a$ and $10b$ will be deleted next. The hinge difference across this new leaf will be $(+)+(+)+(+) = \substack{+\\+\\+}$ or $\substack{-\\-\\-}$ , and since there are two deletions in this operation, the part of the plan beyond 11 will be flipped over and then flipped back:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| + | + | + | - | $\substack{-\\-}$ | - | - | + | $\substack{-\\-}$ | - | - | + |
| (1) | 3 | (3) | 7 | (5) | 5 | (11) | 10 | (10) | 9 | (7) | (2) |
| 2 | (2) | 4 | (6) | 6 | (4) | 1 | (9) | 11 | (8) | 8 | (1) |

A shortcut that eliminates all of this complex figuring is to draw a polygon system including all tubulations (see figure 8.6). The points in the polygon system which touch tubulations are given values according to

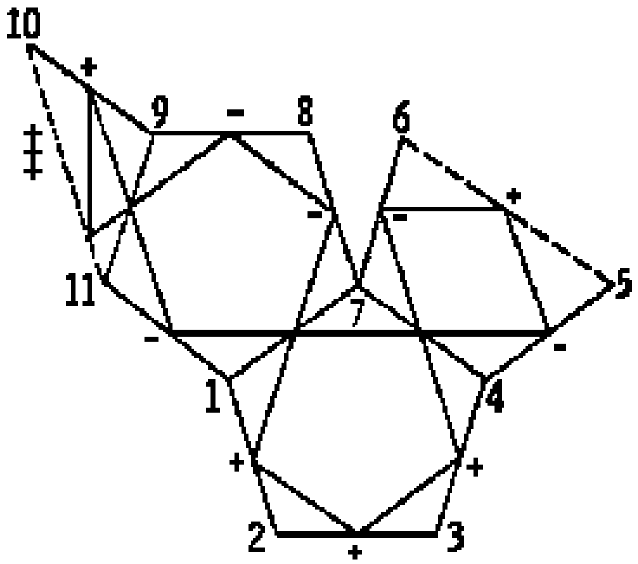



Figure 8.6

the order of the flex the tubulation is, and according to whether the turn is positive or negative. This method produces identical results with the previously mentioned method.

# Chapter 9

# Improper Flexagons

A few things should be noted here about hidden tubulations. First of all, it is impossible to add another side to a hidden tubulation of a proper flexagon, for even the 2−flex leaves more than one leaf in the left hand pat. This means that if the leaf were slit in order to open up the new side, one of the original sides would be suppressed and could not be turned up. Second, as has already been shown, a hidden tubulation, if flexed through, shortcuts another side, omitting it from the normal cycle. Many of these tubulations must be forced in order to shortcut any sides.

This is because the sum of the hinge differences about the center does not equal $360^{\mathrm{O}}$. However, if we were to make the hinge a number 1−hinge, we would still shortcut one or more sides, depending on the value of the original tubulation, but we would not have to force this shortcut. If we are to make an $r$−flex into a 1−flex, we must exchange the two hinges 1 and $r$, or else we would have two hinges of a kind in the same large pat. For example, consider a proper complete one cycle pentaflexagon on which we have reached side 1 after having come from side 5. Instead of 1 flexing to side 2, we want to 1−flex to side 3. We first must tubulate to side 3 (using a number 2−hinge), then exchange the hinge between leaves 1 and 2 in the large pat for that between leaves 2 and 3. We can do this by cutting the two hinges and taping leaves 1 and 2 together with a 2−hinge and leaves 2 and 3 with a 1−hinge. Although the flex we can use to travel to side 3 is a 1−flex, the flex leaves two leaves in the left-hand pat. When rotation is completed this pat will have a hinge in a 1−position, because the 1−flex will have subtracted "1" from the hinge values of all hinges, including the 2−hinge between leaves 1 and 2. The orientation of a right pat is opposite that of a left pat, so that the new 1−hinge is to the right of the 0−hinge (see figure 9.1).

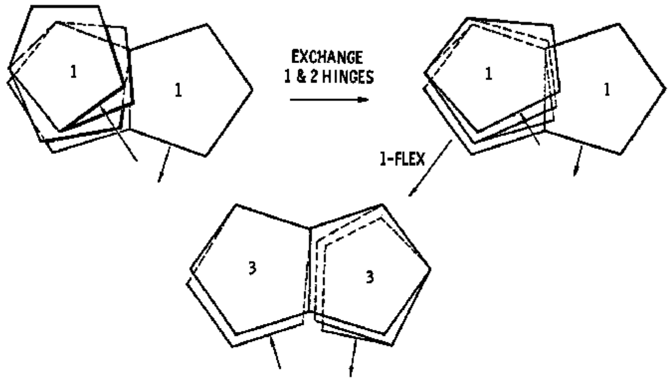



Figure 9.1

Now, when we 1−flex again, the flexing operation takes one leaf off the right hand pat, leaving a single leaf in that pat and exposing face 2, the face which we originally cut off. After rotation, the hinge for side 4 is in a number 1 position and the flexagon flexes from the right pat again, leaving three leaves in that pat. The next flex, which opens up side 5, is a left-flex, and puts all the leaves except one in the right pat. The flex after that is also, of necessity, a left flex to side 1, and so starting the cycle over again. The map of this flexagon shown in figure 9.2a; If the shortcut we wanted to make had been from 1 to 4, we would instead exchange the hinges 1 and 3 without altering in any way hinge 2. The flexagon would then open up to side 4, while leaving three leaves in the left pat. After rotation, the original number 2 hinge (attached to side 3), which would have been unchanged by the exchange of hinges, would be in a 1 position, so the flexing (right) would open up to side 3. At side 3, the hinge for side 2 will be in the 1 position, there having been 2 flexings since it was in a number 3 position. After side 2, the hinging was unchanged, so the next flexes will be 2, 5, 1 and the cycle will repeat (see figure 9.2b). The thing that is important to notice is that after an exchange of the two hinges in the proper flexagon, the succeeding flexes go back to the sides which were originally shortcut, before proceeding with the rest of the cycle. If, in this last flexagon, we decide we want to make another shortcut from side 4 to side 2, omitting side 3 temporarily, we can interchange their respective hinges (number 1 and number 2), 1 flex to 2, then go back to 3 and from there to 5 to 1 (see figure 9.2c). Thus, we may make any kind of shortcut we want by simply exchanging two hinges. This interchange of hinges in no way alters the pat structure of the flexagon–only the hinge sequence. It should be noticed that there are still hidden tubulations from any side to all other non-adjacent sides. Those along the outside of the map polygons are generally indicated with dotted lines, for, as we shall see, they are quite important.

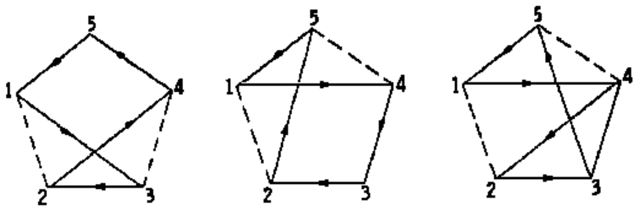


Figure 9.2

Given the hinge sequence and the pat structure, it is possible to find the plan. Let us examine the plan of a proper flexagon whose hinges have been interchanged. Take as an example the flexagon in figure 9.2a. The proper flexagon from which this one was made had this constant order and hinge sequence:

| | |
|---|---|
| 0 1 2 3 4 0 | Hinge sequence |
| 5-1 2 3 4; 5-1 | Pat structure |

The sides of that flexagon were associated with the leaves of the pat in this manner:

| | |
|---|---|
| 5-1 2 3 4; 5-1 | Pat structure |
| 1 2 3 4 5 1 | Basic number sequence |

The hinges of sides 2 and 3 were then exchanged. This means that we can find the new hinge sequence by exchanging the numbers in the hinge sequence corresponding to these sides:

| | |
|---|---|
| | Exchanged hinges |
| 0 2 1 3 4 0 | Hinge sequence |
| 5-1 2 3 4; 5-1 | Pat structure |
| 1 2 3 4 5 1 | Basic number sequence |

or, since the hinge between leaf 1 and 2 is identified with leaf 1, by convention, and since leaf 1 is also identified with the first number of the basic number sequence, we may write this:

| | |
|---|---|
| 0 2 1 3 4 0 | Hinge sequence |
| 5 1 2 3 4; 5 | Pat structure |
| 5 1 2 3 4 5 | Basic number sequence |

We may now obtain the hinge difference and then the sign sequence from this:

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 2 | 1 | 3 | 4 | 0 | Hinge sequence |
| | 2 | 4 | 2 | 1 | 1 | Hinge difference |
| 5 | 1 | 2 | 3 | 4 | 5 | Basic number sequence |

which gives:

$$\begin{array}{ccccc|c} + & - & + & + & + & + \\ + & & + & & & + \\ (1) & 3 & (3) & 5 & (5) & 2 \\ \hline 2 & (2) & 4 & (4) & 1 & (1) \end{array}$$

from which we get the plan.

This flexagon differs from the proper flexagon in that the hinge sequence is not consecutive for one cycle. Any flexagon which is not basically consecutive in some part of its hinge structure and is not a product of deleting sides from such a flexagon is an improper flexagon. It is quite possible for an improper flexagon to have a pat structure identical with either proper complete or proper incomplete (tubulating) flexagons. There are no improper triflexagons, and just one improper tetraflexagon of one cycle, but there are three improper pentaflexagons of one cycle, eleven improper hexaflexagons of one cycle and forty-two improper heptaflexagons of one cycle (see figure 9.3).

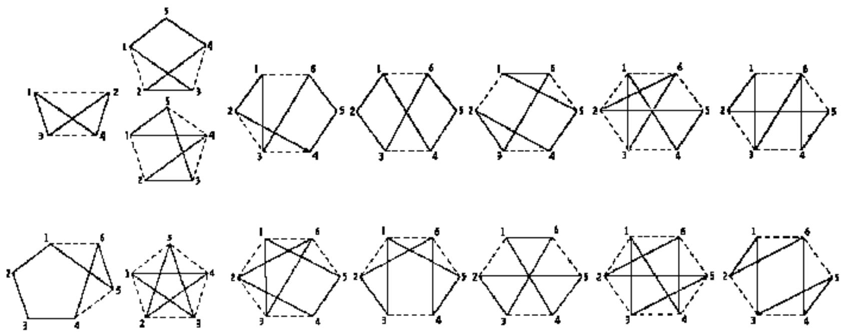


Figure 9.3

We have been able to make proper flexagons with any desired map by simply slitting single leaves of a flexagon of lower order. How can this be done with improper flexagons? Let us take two pentaflexagons one proper and one improper (see figure 9.4 a & b) and delete sides 5 and 4 respectively by paper clipping the leaves together. The maps of these two deleted flexagons will be identical except for a minor difference in numbering (sec figure 9.4 c & d). Now, if we wish to reopen the missing side we have a number of choices (four to be exact), as we saw in Chapter VI. We will consider two of them, one being slitting the left most leaf, and the other slitting the right most leaf. If the left hand leaf is slit, the flexing operation can remove all but one of the leaves from the left pat and deposit them uninverted on the inverted right pat. In the next flex, the thumbhole associated with side 1 will open up, taking all the leaves off the left pat, since it is the uppermost thumbhole. If the right leaf were slit, the following flex would take the bottom leaf off the right pat and place it on the <u>inverted</u> left pat. The rotation will necessitate a right flex, involving the hinge which was originally in a $2-$position, i.e. the hinge associated with side 1. However, the order of the thumbholes has been reversed by the pat inversion of previous right flex, so instead of being the top thumbhole, the side 1 thumbhole is going to be on the bottom, and the next right-flex is only going to add one leaf to the left pat, and will invert the remainder of the right pat. The thumbhole adjacent to the one associated with side 1 is associated with side 2 and also was originally hinged in a 3-position. After two flexes it will be hinged with a $1-$hinge, and having been inverted twice, it will be the top thumbhole. The flexing operation exposing side 2 will remove the remainder of the leaves from the left pat and place them on the right pat, preparing for another left flex. Thus, in constructing a flexagon, whenever we have two choices in slitting to get a new side, slitting the single left leaf exposed by the tubulation will produce a proper portion of a cycle, while slitting a single leaf in the other pat will give an improper portion of a cycle.

There are certain basic characteristics of the improper flexagon. It has the same pat structure, the same map (except for order of flexes), and hence the same number sequence as the corresponding proper flexagon. It will be found, of course, that the proper flexagon with the same pat structure as an improper flexagon has an identical polygon system. Since an improper flexagon is not constantly a left or right flexing flexagon, the thumbholes do not necessarily have to be in one pat. The number of thumbholes in a folded unit, however, is always equal to $(G-1)$.

### The Order $N$ Improper Flexagons

In places where a right pat was slit instead of a left pat, it is impossible to add more (for a new cycle) sides without suppressing others, for the hinge joining two pats resulting from a $2-$flex joins the <u>topmost</u> leaves in each pat and since the added leaf must be at the bottom of the pat there would

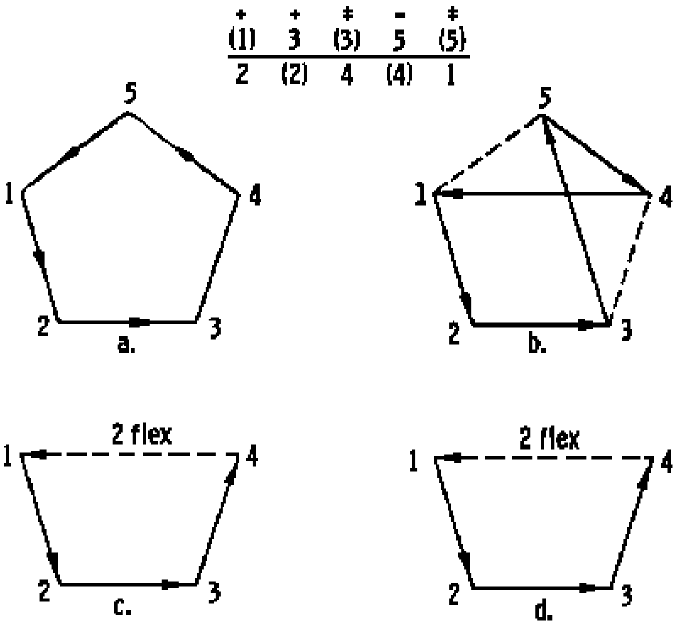



Figure 9.4

have to be a 1−hinge connecting the top and bottom leaves of the pat. This arrangement would suppress all in between sides. All of the places in the cycle of an improper flexagon at which there are no degree 1 pats represent shortcuts, made by leaving at least two leaves in the left pat. Thus, none of the diagonals of the map, which are shortcuts or just plain "cuts" [1] can be added to in the conventional manner. That is, after 1−flexing along an out-side edge or 0−cut there will be a single leaf in either the left or right pat. This leaf may be slit and a new side revealed. There is a choice in slitting for all the slits after the first until $(G-2)$ slits have been made. Thus, to any improper cycle it is possible to add at a 1−flex 0−cut another cycle of any nature, so long as it too has a 1−flex 0−cut in common with the first cycle (see figure 9.5).

Because of the nature of a shortcut, any 0−cut tubulation also has a single leaf in either the left or right pat. We have seen that we can force the tubulation, rotate, and open up to the next side along the outside edge

[1] A "cut" is a shortcut flex which omits one or more sides. It is a classified by degree, representing the number of sides they cut out. A 0−cut is a flex along the outside of the map polygon. In a proper flexagon, the $n$−cut is always an $(n+1)$−flex.

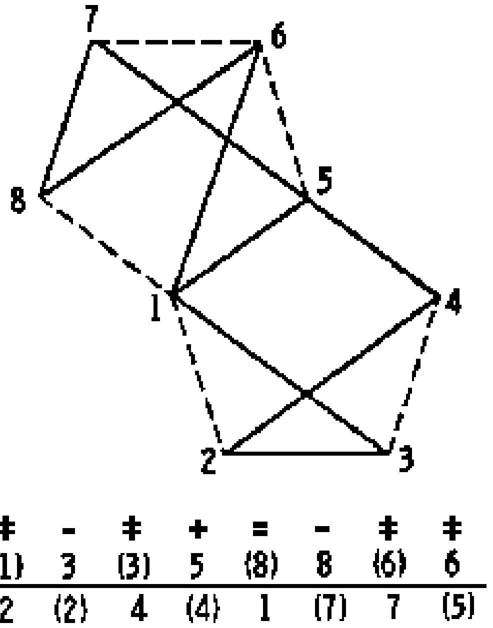



Figure 9.5

or 0−cut cycle [2]. What if we decide not to rotate but to slit the single leaf instead? This opens up a new side which can only be reached through the tubulation. Take, for example the single cycle improper pentaflexagon shown in figure 9.6a. If we flex from 6 to 5, we may tubulate to 3 by removing all but one leaf from the right hand pat (after rotation). We can now force this tubulation, then, instead of rotating and opening up to side 2, by a $-1$ flex or side 6 with a $+1$ flex, we may slit the single right leaf and open up to a new side, 4 (see figure 9.6b). From 4, it happens that we can rotate and 1−flex to 5. From 5, tubulate to 3, and if we again force the tubulation and rotate, we can continue to the three-cycle $5 - 3 - 4$. If instead of rotating after forcing the tubulation, the same flexing axis is maintained, we can flex to side 6.

There is, however, a much easier way to turn up the new side. If we flex from 2 to 3, we can tubulate to 5, but if we force this tubulation we

[2]A 0−cut cycle ia one composed completely of 0−cuts, no matter what the value of flexes involved. Thus, on the map it is equivalent to following along the outside edge of the map polygon.

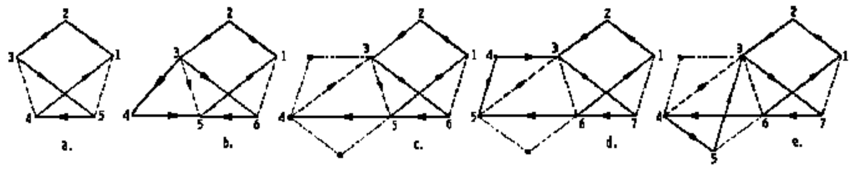


Figure 9.6

cannot open up to 4 with a 1−flex (although this may be accomplished with a backwards flex). We can, however, open up to 4 without forcing the tubulation but instead after tubulating, rotate the tubulation, (by exchanging right for left pats) and open up with a forwards flex to 4. Similarly, after we flex from 4 to 5, we may tubulate to 3, but without forcing, we can rotate the tubulation and open up to side 1. This method is much easier than the method involving forcing the tubulation. The operation just described is called "tubulating through". In constructing this type of flexagon in the first place, it is much easier to tubulate say from 3 to 5, rotate the tubulation, and then slit the single leaf in the right hand pat to open up to 4.

Returning to figure 9.6a, if we flex to 5 from 6, we could tubulate to 3. If instead of forcing the tubulation, we were to rotate it and slit the single left leaf, we would open up to a side, 4, which would 2−flex instead of 1−flex to 3 (see figure 9.6c). After rotation, we may open up this tubulation by slitting one of two leaves depending whether we wish a 0−cut flex or a shortcut flex. Then, from this side, we may 1−flex to 3 (see figure 9.6 d & e). Now let us reconsider the flexagon shown in figure 9.6b. If we flex from side 4 to side 5, we find that after rotation there is a single leaf in the left hand pat. Also, this single leaf is connected to the lowest leaf in the right pat, which means that either leaf may be slit without suppressing another side (call this choice "A"). Whichever one we slit, we may flex to a new side which then tubulates to side 3. This tubulation may be opened in three ways, (choice "B") in the manner we have used for all tubulations. Whichever way we do it, we can flex back to side 3.

This completes a separate improper cycle joined to the original by a tubulation. Choice "B" decides which of three improper cycles will be added, and choice "A" determines how it will be oriented with respect to the original cycle. If we agree to slit the left most leaf in choice "B", slitting the left leaf in choice "A" will give the map in figure 9.7a, while slitting the lowest right leaf will produce the map in Figure 9.7b.

This method of adding more cycles attached by tubulations may be

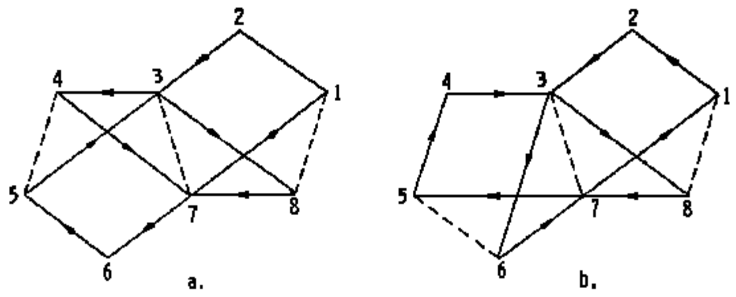



Figure 9.7

extended to include the $G-$flexagon. We may tubulate through from any tubulation which is on the $0-$cut cycle by tubulating, rotating, and slitting the single leaf. After we have completed a series of $1-$flexes (and if necessary, slittings) which leads back to the side across the tubulation, we have a choice of two possible leaves to slit to start the series of flexes which will complete the cycle. The orientation of the new cycle will be determined by this slitting (in symmetrical cycles, such as in tetraflexagons, this difference in orientation will appear to be a reversal of vectors - see figure 9.8). When all cycles are completed, the flexagon must be renumbered by folding the unit together and numbering the leaves consecutively down the unit $1-2,\ 2-3,\ \ldots,\ (N-1)-n,\ N-1$.

### The Determination of the Plan from the Map for Improper Flexagons

Each and every $0-$cut is characterized by the presence of a single leaf or sub-pat in one of the two pats. This single sub-pat will have a hinge difference across it equal to the order of the $0-$cut flex, be it a $1-$flex or a tubulation, since the order of a flex is defined as being equal to the value of the hinge difference across any pat involved in the flexing operation. Since in following the $0-$cut cycle each sub-pat is left alone in a pat in the order that the sub-pat turns up in the plan, the plan for one cycle of an improper flexagon may be constructed by following the $0-$cut cycle while noting what order flex is required to reach each face from the previous one. The results may be checked by orienting the leaf polygons and noting exactly what the hinge difference across each single sub-pat really is. Care must be taken, however, that the orientation is such that a forward $1-$flex uses a number 1 hinge, not a $(G-1)$-hinge.

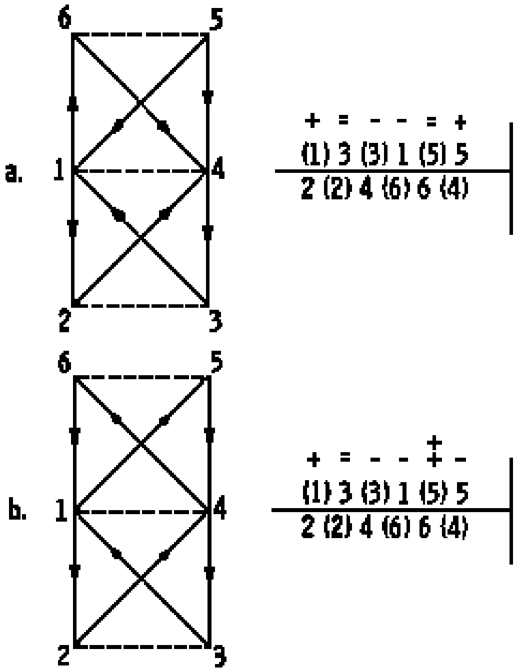



Figure 9.8

Let us see how this works with the pentaflexagon pictured in figure 9.9. Let us start at face 1-4. The first 0$-$cut flex is between side 1 and side 2. This flex removes all leaves except leaf 1, 2 from the left hand pat. We note that this is a 1$-$flex, for it uses hinges which are in a 1 position, and further, that the hinge difference across the single leaf 1, 2 is +1. The next 0$-$cut flex is from 2 to 3. This also is a 1$-$flex, and we note that leaf 2, 3 has a hinge difference across it of +1. The 0$-$cut between 3 and 4 is a tubulation with flex value of 2. Leaf 3, 4 has a hinge value of +2. To reach side 5, we must back flex, or 4$-$flex ($G-1$ flex). Leaf 4, 5 is -1 (but note that side 5 faces away). From here, we may 2$-$flex to 1, noting that leaf 5, 1 is +2. From this information we may obtain the number and sign sequences, and then the plan:

$$\begin{array}{rrrrr} +1 & +1 & +2 & \text{-1} & +2 \\ (1) & 3 & (3) & 5 & 5 \\ \hline 2 & (2) & 4 & (4) & 1 \end{array}$$

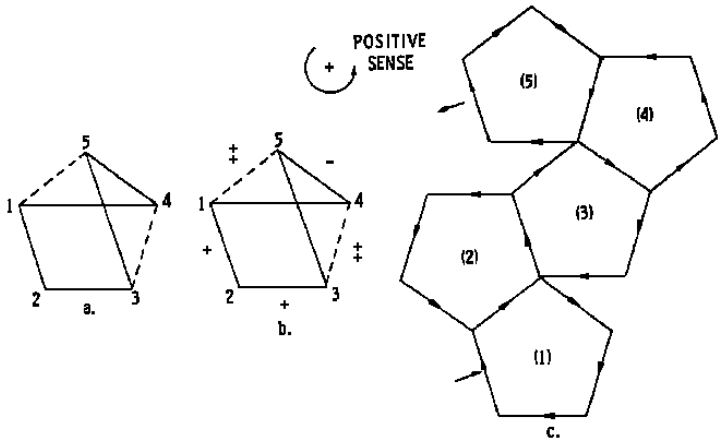



Figure 9.9

It should be noted that before proceeding with the determination of the plan, it was necessary first to decide which way around the 0−cut cycle we were to travel, and second, just how to orient the leaf polygon. The orientation was determined by choosing arbitrarily a forward $\pm 1$−flex and assigning to it a positive sense. This may be represented on the map as in figure 9.9b. In such a flexagon, any flex represented by a solid vector is a 1−flex. If the path is traversed in reverse, that is, against the vector, a $(G-1)$ − hinge is required. Thus, although traveling from 5 to 4 requires a 1−flex, traveling from 4 to 5 requires a 4−flex or a $(-1)$−flex. Here, then, is a method for determining the signs for the hinge differences across all leaves with a $+1$ or $-1$ hinge: A direction about the map is designated as +, and one of the 0−cut 1−flexes is assigned a vector in this + direction. The 1−flex cycle is then followed, and vectors are drawn on all lines so that one vector enters and one vector leaves every vertex of the map polygon. Those 1−flex vectors that are of the 0−cut cycle may be given $+1$ or $-1$ values depending on whether they point with or against the preassigned + direction. If the values of any tubulations on the 0−cut cycle are known, they may also be assigned vectors in accordance with their + or − sense (see figure 9.9b). It should be noted that a $+n$−flex is congruent to a $-(G-n)$−flex mod. $G$.

Here we have the true meaning of the polygon system. For any given proper flexagon, the orientation of the map polygon is opposite for adjacent cycles, and thus the signs in the plan are reversed. The convenience of turning in a + or − direction when traveling around the map polygons works with proper flexagons but not with improper ones, in which signs other than $\pm 1$ are used.

With the knowledge we now have we may find the plan of any given proper and many of the improper tetraflexagons. By assigning a positive direction and by fixing one vector, all of the other 0−cut flexes may be given a direction for any proper tetraflexagon and for any improper tetraflexagon, in which each cycle has only 1−flexes in common with the neighboring cycles (those flexagons in which the cycles are attached by tubulations will be discussed later). All tubulations in tetraflexagons and pentaflexagons will signify hinge differences of +2 or -2, but in all tetraflexagons these are indistinguishable as far as the plan is concerned. Thus, given a tetraflexagon as shown in figure 9.10a, we assign a + direction (counterclockwise), draw vectors, and then construct the polygon system (see figure 9.10b.). As the polygon system touches each path in turn, we look at the type of flex; if it is a tubulation (2−flex) we can simply write $\substack{+\\+}$ (+2), whereas if it is a 1−flex we compare the vector with the preassigned positive direction. If the vector goes with the + direction the vector is +1 and if it goes against the + direction the vector is −1. In improper flexagons with cycles higher than four we have a problem, for we have no way of reading the value of the tubulations in the 0−cut cycle nor can we tell the sign associated with the tubulation at a glance, directly from the map. In order to calculate the value of the tubulations, we must employ a new principle. This principle states that the sum of the hinge differences encountered in flexing from a given point to another given point is constant regardless of the paths followed. To demonstrate this let us look again at the flexagon in figure 9.9. If we are at side 3 with 2 on the back and we want to flex to side 5, there are a number of paths we can use. Let us consider the two paths $3-5$ and $3-4-5$. If we 1−flex from 3 to 5 and don't rotate, we find that the hinge difference associated with the flex is +1 and that this hinge difference is arrived at by adding the hinge differences in the leaves which comprise the left pat. It is the hinge difference across these two leaves in which we are interested. This hinge difference, being +1 means that the hinge difference across the leaf 3,4 plus that across 4,5 must be equal to +1. In this case $3,4 = +2$ and $4,5 = -1$ and the sum is +1. Now if we go to side 5 by the paths $3-4-5$, we would expect to first encounter a hinge difference across leaf 3,4 and then a hinge difference across leaf 4,5. Since these are unchanged for a single leaf, we would expect to encounter first a +2 flex and then a −1−flex. The sum of these two flexes is +1, as predicted.

In general, if a shortcut flex between two sides removes all but $M$ leaves from one of the pats, then in traversing the 0−cut cycle between these two

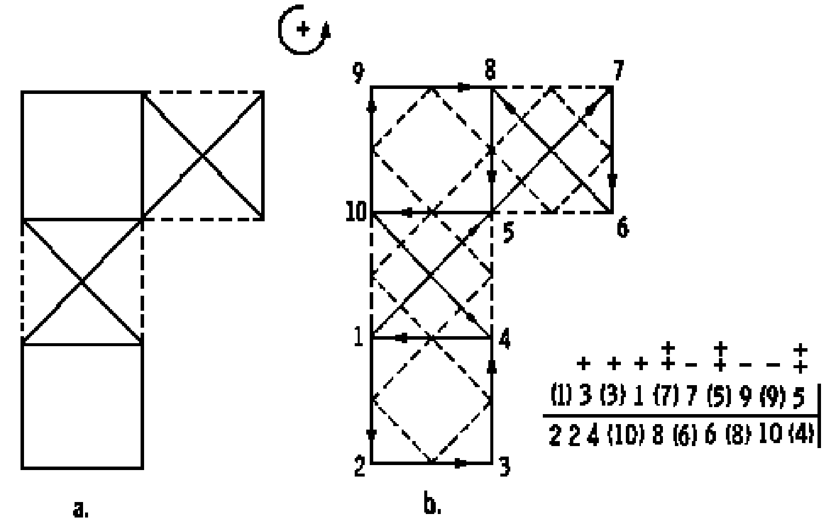



Figure 9.10

sides, each one of the $M$ leaves must take a turn at being the only leaf in a pat. Since the hinge difference across the entire degree $M$ pat is equal to the sum of the hinge differences of each leaf in the pat, this sum must equal the sum of the hinge differences of the operations encountered in the $0-$cut cycle.

Now we have a valuable tool with which we can determine the nature of any tubulation we wish to. In a given map such as the one in figure 9.11, we may orient all the $1-$flexes and assign a positive direction. All $1-$flexes on the $0-$cut cycle should be assigned + or – values according to whether their vectors go with or against the positive direction. Each $0-$cut greater-than-1-flex may be assigned values by finding another path consisting of <u>1–flexes</u> by which the gap across the tubulation may be bridged. In figure 9.11a, side 2 may be reached from side 3 either by the tubulation 3–2 or by the sequence of 1–flexes 3, 1, 2. If this 1–flex path goes with the vectors drawn on the map, the hinge difference for each 1–flex is $+1$; if it goes against the vector, the hinge difference is –1. The hinge differences for all the 1–flexes are then added up and the sum is equal to the hinge difference across the tubulation. Care must be taken with the direction of traversing the tubulation. In our example, the 3–2 flex is in a negative direction, and since it is equal to the two $+1$–flexes, 3–1 and 1–2, the value of the tubulations is –2. The other tubulations are handled similarly. (see figure

9.11b). Now with all of the values known for the 0−cut cycle, the polygon network can be drawn, and as the network touches each face in turn, the hinge value is recorded as being identified with that particular point.

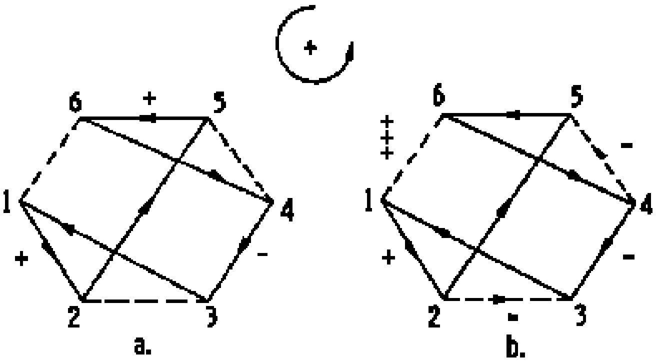



Figure 9.11

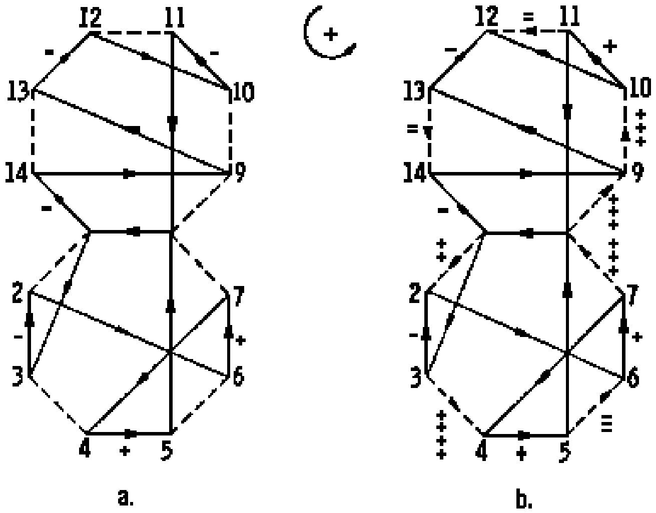



Figure 9.12

Example:

To find the plan of the octaflexagon pictured in figure 9.12, we will arbitrarily orient the 1–flexes by drawing vectors as in the figure, and we will arbitrarily assign counterclockwise a positive sense. Now we must look at each 0–cut 1–flex and discover whether it goes with or against this arbitrary positive sense. When the sign of these 1–flexes is being determined, the 1–flex in question must be referred to the map as a whole; not to the individual cycle of which that 1–flex is a part. The 1–flex values for this map are as follows:

| | | | |
|---|---|---|---|
| 3-2 | –1 | 10-11 | +1 |
| 4-5 | +1 | 13-12 | –1 |
| 6-7 | +1 | 1-14 | –1 |

Now we tackle the 0–cut tubulations. First let us consider the tubulation from 1 to 2. There are two possible 1–flex sequences to travel from 1 to 2: 1, 3, 2 and 1, 8, 5, 4, 7, 6, 2. In the first sequence, the flexing proceeds in the direction of the preassigned vectors, so all 1–flexes are positive. There are two such 1–flexes, 1–3 and 3–2, so since the hinge difference across the tubulation must equal the sum of the hinge differences of the 1–flexes encountered, the flex 1–2 is a 2–flex. Furthermore, since the vector 1–2 is in a counterclockwise direction, the tubulation is a +2–flex when traversing the 0−cut cycle in the assigned positive (counterclockwise) sense. In the second sequence $(1, 8, 5, 4, 7, 6, 2)$, the direction of flexing is against the vectors, so the flexes are −1 flexes. By this scheme, the flex 1–2 is a −6−flex. This, however, is identical with the results obtained from the first sequence, for a $-(n)$−flex is identical with a $+(G-n)$ flex.

All the rest of the 0–cut tubulations may be evaluated in the above manner. Following is a chart of the results obtained:

| 0–cut flex | Shortcut sequence | Hinge value | Sense of Tubulation | 0–cut value |
|---|---|---|---|---|
| 3–4 | 3,2,6,7,4 | +4 | + | +4 |
| 5–6 | 5,4,7,6 | –3 | + | –3 |
| 7–8 | 7,4,5,8 | +3 | + | +3 |
| 8–9 | 8,1,14,9 | +3 | + | +3 |
| 10–9 | 10,12,13,9 | –3 | – | –3 |
| 12–11 | 12,10,11 | +2 | – | –2 |
| 14–13 | 9,13 | +2 | – | –2 |

The first column specifies the tubulation under consideration and the direction the flex is to proceed (thus 3-4 signifies the flex from 3 to 4). The second column designates the sequence of shortcut flexes used to avoid the tubulation. The third column tells the sum of the hinge values of the shortcut flexes. The sum is positive if the flexing proceeds with the preassigned vectors and negative if against them. The fourth column tells whether the direction in which the 0–cut flex was effected was with (+) or against (–) the assigned positive sense. The last column gives the final hinge value for the 0–cut flex if flexing were proceeding in a consistently positive direction along the 0–cut cycle. Now, with the value of each 0–cut flex known, the polygon network may be drawn and followed. As each face is visited in turn, its constant order number and hinge value may be noted down. The plan may be constructed with this information:

$$\begin{array}{cccccccccccccc|}
\begin{smallmatrix}+\\+\end{smallmatrix} & - & \begin{smallmatrix}+\\+\\+\\+\end{smallmatrix} & + & \begin{smallmatrix}-\\-\\-\end{smallmatrix} & + & \begin{smallmatrix}+\\+\\+\end{smallmatrix} & - & \begin{smallmatrix}-\\-\end{smallmatrix} & - & \begin{smallmatrix}-\\-\end{smallmatrix} & + & \begin{smallmatrix}+\\+\\+\end{smallmatrix} & \begin{smallmatrix}+\\+\\+\end{smallmatrix} \\
(1) & 3 & (3) & 5 & (5) & 7 & (7) & 1 & (13) & 13 & (11) & 11 & (9) & 7 \\
\hline
2 & (2) & 4 & (4) & 6 & (6) & 8 & (14) & 14 & (12) & (10) & 10 & (8) & 8
\end{array}$$

Each operation shown by the map may now be given flex values by this same method. To find the value of a tubulation from any side $A$ to any side $B$ in a given cycle all we have to do is add up the value of the flexes along a 0–cut cycle from $A$ to $B$. Thus the value of the tubulation from 1 to 4 in figure 9.11b is $(+) + (\substack{+\\+}) + (-) = -2$. To do this systematically for every path on the map we write all the 0–cut values for each side in order. Then we systematically iterate all possible paths on the map. In the case of the flexagon in figure 9.11 these possibilities are 1-2, 1-3, 1-4, 1-5, 1-6, 2-3, 2-4, 2-5, 2-6, 3-4, 3-5, 3-6, 4-5, 4-6 and 5-6. The value of the path 1-2 is found by looking at the hinge difference between 1 and 2; that of path 1-3 by adding the hinge difference of 1-2 to the hinge difference of 2-3, that of 1-4 by adding 1-2, 2-3, and 3-4, etc. (see figure 9.13). In the manner above described it is possible to build and predict the properties of any proper or improper flexagon we have so far considered.

| 1-2 | 2-3 | 3-4 | 4-5 | 5-6 |
|---|---|---|---|---|
| + | = | - | = | + |

| | | | | | |
|---|---|---|---|---|---|
| 1-2 | + | 2-3 | = = | 3-5 | $\begin{smallmatrix}-\\-\\-\end{smallmatrix}$ or $\begin{smallmatrix}+\\+\\+\end{smallmatrix}$ |
| 1-3 | - | 2-4 | $\begin{smallmatrix}-\\-\\-\end{smallmatrix}$ or $\begin{smallmatrix}+\\+\\+\end{smallmatrix}$ | 3-6 | = |
| 1-4 | = | 2-5 | $\begin{smallmatrix}-\\-\\-\\-\\-\end{smallmatrix}$ or + | 4-5 | = |
| 1-5 | or | 2-6 | $\begin{smallmatrix}-\\-\\-\\-\end{smallmatrix}$ or $\begin{smallmatrix}+\\+\\+\end{smallmatrix}$ | 4-6 | - |
| 1-6 | or | 3-4 | - - | 5-6 | + |

Figure 9.13

Now, at last, it is possible to see why there are two possible ways of attaching two given improper cycles by any two given 0–cut tubulations of the same order. Let us go back to the flexagons in figure 9.6b. The right hand cycle may be determined by arbitrarily assigning a + direction and 1–flex vector (see figure 9.14a). The tubulation in common with the two cycles is then determined as $^{+}_{+}$ when going from 3 to 5. This tubulation then determines the two 1–flexes of the second cycle which also bridge the gap between 3 and 5. If in traveling across from 3 to 5 the tubulation is $^{+}_{+}$, then in traveling from 3 to 4 to 5 the sum of the hinge values for those flexes must be $+2$, or each single flex must be $+1$. These flexes may now be given vectors (see figure 9.14b). When the other two sides are added to complete the new cycle, the two 1–flexes which bridge the tubulation have a choice as to which position they take (see figure 9.14 c & d). In both cases, the 1–flex which remains a 0–cut goes in a positive direction, thus satisfying all requirements. In figuring out the plans to improper flexagons with cycles attached to the tubulations, one cycle at a time must be figured out so that the value of the tubulation in common with two cycles is known first and can be used to determine the other cycle. It should be emphasized, however, that two cycles can only be attached by a flex whose absolute value is the same for both cycles. Thus a $\pm 2-$flex in one cycle of a hexaflexagon cannot be used to attach it to a $\pm 3-$flex in another (see figure 9.15a).

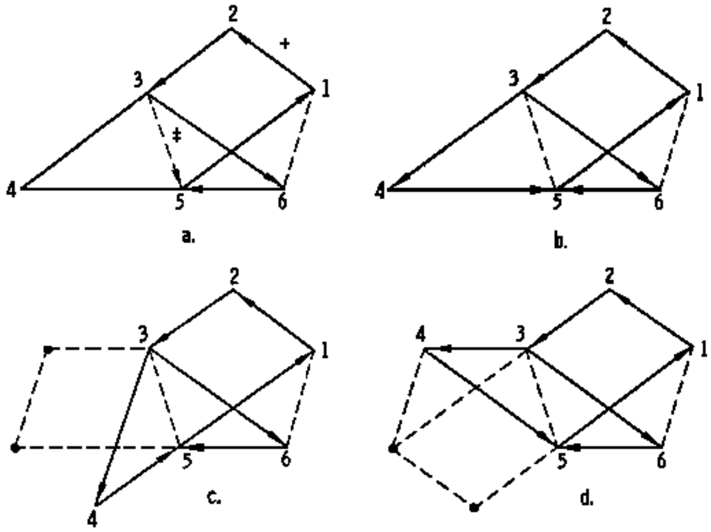



Figure 9.14

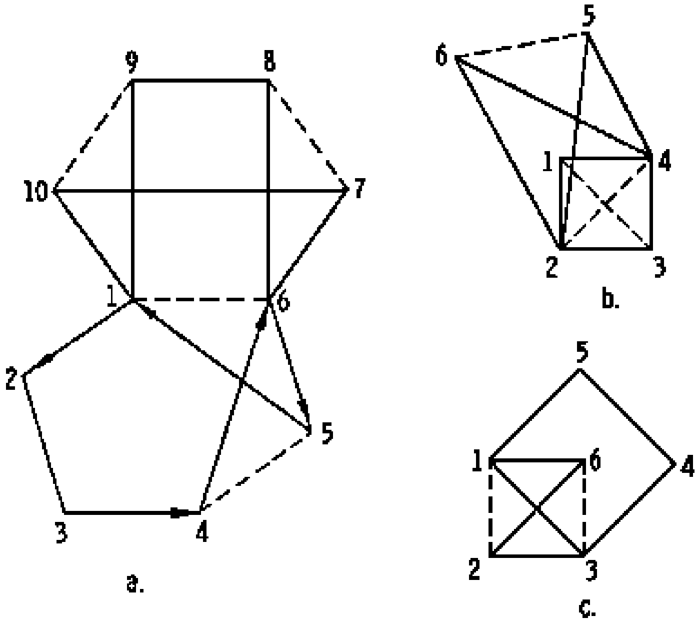



Figure 9.15

Example:

To find the plan of the hexaflexagon in figure 9.16, we assign a positive direction, and arbitrarily orient the 1,2,3,6,5,4, cycle, as in figure 9.16a. The values for the $\pm1-$flexes in this cycle may then be determined, and from them, the two 0–cut tubulations. The value of the flex from 6 to 1 may be determined as $\begin{smallmatrix}+\\+\\+\end{smallmatrix}$ from the 1–flex cycle 6,5,4,1. Now, in the second cycle, the 1–flex cycle 6,8,9,1 also spans the tubulation. The sum of these three $\pm1-$flexes must equal the tubulation value $+3$. However, either three $+1$–flexes or three $-1-$flexes will work since for hexaflexagons a $+3$–flex is the same as a $-3-$flex. So, we have a choice for orienting the second cycle. Let us choose the orientation in figure 9.16b. We can now specify values for all the 0–cut flexes and in particular, we find that 9 to 14 is a $+2$–flex. Since 9–12–14 is the only set of two $\pm1-$flexes of the third cycle bridging this tubulation, they must be $+1$–flexes, and again we can determine all the 0–cut flexes. The number and sign sequences may now be written down:

$$
\begin{array}{cccccccccccccc|c}
 & & + & & & & + & & & + & & & & & \\
 & & + & & & & + & + & + & + & - & & + & & \\
+ & + & + & - & - & - & + & + & + & + & - & + & + & - & + \\
(1) & 3 & (3) & 5 & (5) & 1 & (9) & 11 & (11) & 13 & (13) & 9 & (7) & 7 & (1) \\
\hline
2 & (2) & 4 & (4) & 6 & (14) & 10 & (10) & 12 & (12) & 14 & (8) & 8 & (6) & 2
\end{array}
$$

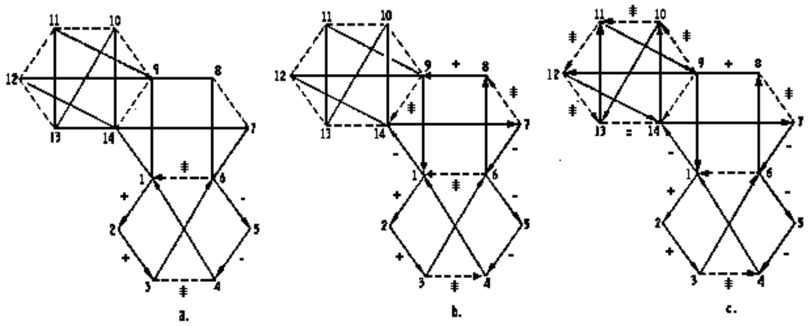



Figure 9.16

# Chapter 10

# Class Distinctions

In all the flexagons with which we have dealt thus far, the notion of "class", relating to the structural components of a flexagon, has been both clear and useful. We have been able, using the three general concepts of class, cycle, and order, to give a general idea of the shape, operation, and size, respectively, of any given flexagon. Several relationships have arisen among these three quantities: In complete flexagons, the cycle is found to be equal to the class, and the order is then of the form $n(G-2)+2$, where $n$ is an integer; the cycle is always less than the class and less than or equal to the order; and so forth.

However, in this section and those following, we will begin to come upon flexagons in regard to which the concept of "class" becomes vague and the rules regarding it no longer hold. These flexagons will reveal that the class, as a description of a flexagon, actually has far less significance than the cycle. As the distinctions between classes is lessened more and more, we will find that the very concept of "class" will change. Before proceeding with more radical changes in the flexagon structure, we can treat a number of variations that grow directly out of flexagons we have already discussed. Each of these deserves individual observation.

### A. Irregular Leaves

Nowhere in the discussion of the flexagon of cycle greater than 3 has the angle between the various sides played a crucial position. It is true that we can create a relationship between the angles of the leaves and the angles of the map polygons, but this relationship is soon seen to be purely superficial. What is important in the map is not the shapes of the polygons involved, but the number of sides they have, i.e., as predicted, the cycle. It would seem therefore that, as in the triflexagons, we can change the angles and lengths of the sides of the leaf polygons, yet maintain the same mode of operation as in corresponding flexagons with regular leaves. Of course, more units may be required to allow flexing at all the faces, thus possibly decreasing the tendency of the flexagon to hold together at certain other faces (the same was true in the triflexagons). However, we will assume that an unlimited number of hands are available to operate the flexagon once it has been constructed, and go right ahead without considering practical difficulties. As in the triflexagons, we make the irregular polygons coincide in the assembled flexagon, and use at least $180^{\text{O}}a$ units, where $a$ is the smallest angle of the leaf.

What happens if not enough units are employed? Those faces at which the angles pointed toward the center do not make up $360^{\text{O}}$ will not permit further flexing. The effect on the map is as if the face were broken in half (See fig. 10.1). At which faces will this occur? As in triflexagons, any two map paths, lying in adjacent cycles the midpoint of whose common side is in line with the midpoints of the two paths, represent faces having like angles at the center of the flexagon. This fact can help us in the manufacture of flexagons with irregular leaves. If we can indicate faces on the map at which like angles of the flexagon will be pointed toward its center, then each map path may be marked with the name of some such angle corresponding to an angle of the irregular leaf. Each face of the map must then be marked so that the angles of the hinge network polygons are identified with angles of the leaves, the angles occurring in the same order in the two places. One convenient way of doing this is by making the map so that the hinge network polygons will be similar to the leaf polygons. In building the plan, one leaf vertex between the incoming hinge of each leaf polygon and the next hinge (where we are traveling about the leaf in the direction, + or −, indicated by the sign corresponding to that leaf) must have an angle equal to the leaf angle designated for that leaf by the appropriate face in the map. Differently labeled angles in the plan should never have a common vertex.

For example, to build the improper flexagon shown in fig. 10.2 with the irregular polygon $ABCD$ so that the angle $A$ would be at the center of the flexagon at face 1, we would proceed as shown.

The proper tetraflexagon of order 4 has two interesting plans made from irregular leaves. The first is made from a hexagonal array of rhombuses and the second from a straight strip of isosceles trapezoids (see fig. 10.3). As is seen, the results of alternating leaf shapes are even more interesting in higher cycle flexagons than in triflexagons. They are so varied that other examples will be mentioned as needed, rather than all presented here.

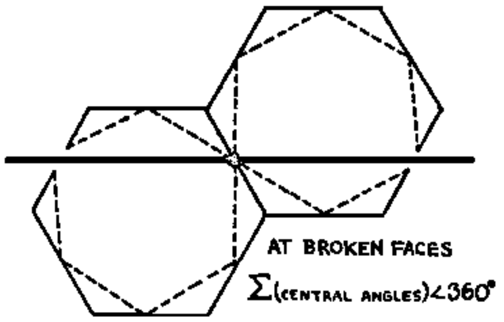



Figure 10.1

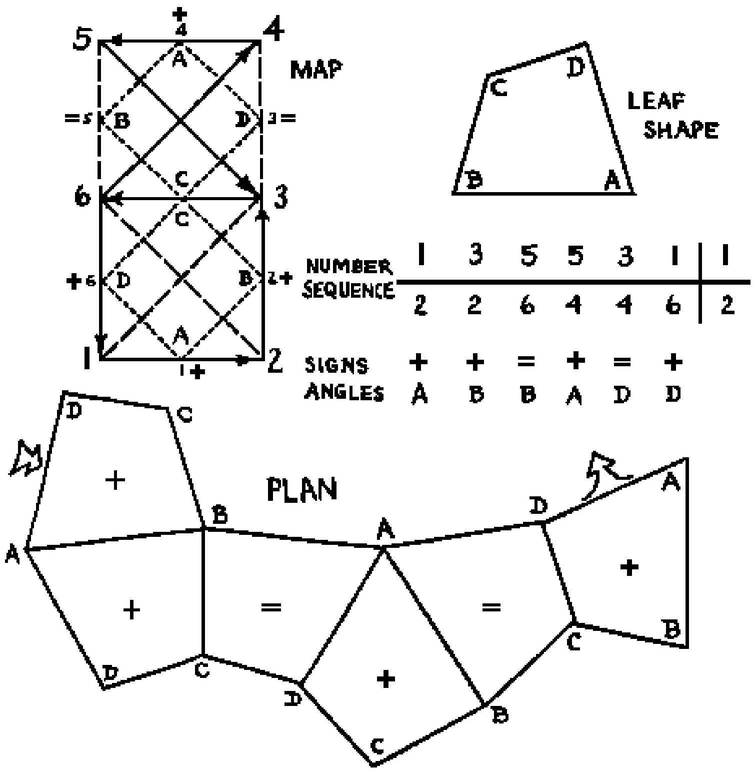



Figure 10.2

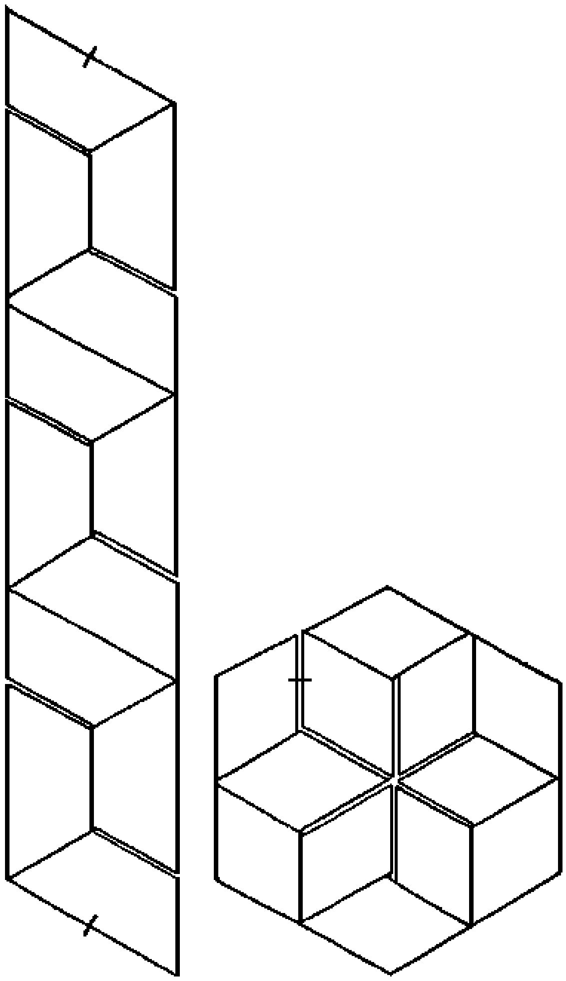


Figure 10.3

**B. Coincidence**

Heretofore it has been assumed that leaves must be congruent, with alternate leaves mirror images of the others. In breaking this convention, we can at the same time begin to show the weakening of class distinction. Consider, for example, the regular(_leafed) proper tetraflexagon of order four.

In this flexagon, two adjacent edges of every leaf are occupied by hinges; the others have no function other than maintaining order in the pat. Suppose, then that we chop off the unused corner of each leaf (see fig. 10.4). It is interesting to note that this leaves the plan in the shape of a straight strip of paper, folded off in right isosceles triangles. The flexagon still holds together, it is still of order 4 and cycle 4 (4 hinge positions), but it is now of class 3. Yet, when the flexagon is folded up, it does not at all assume the characteristic shape of a class 3 flexagon: three colors are exposed on each side of each face, and some of the leaves do not meet at the center (fig. 10.5). Is it right, then, for us to be calling this a class 3 flexagon? We will have to reexamine the definition of "class". The "class" is the number of sides of the component leaf polygons. This flexagon indicates that something more is needed. "Class" must also indicate the relative orientation of the leaves. For convenience then, we state that the "class" of a flexagon must give an indication, when called for, of this orientation.

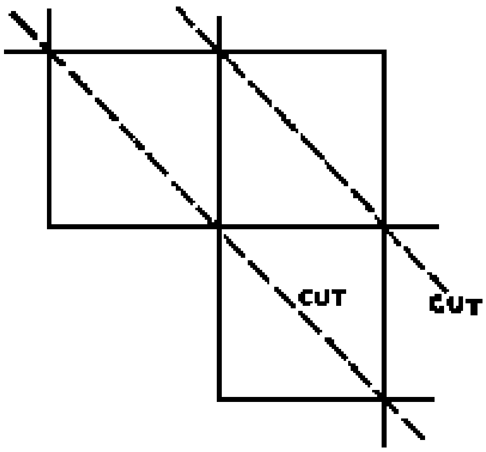



Figure 10.4

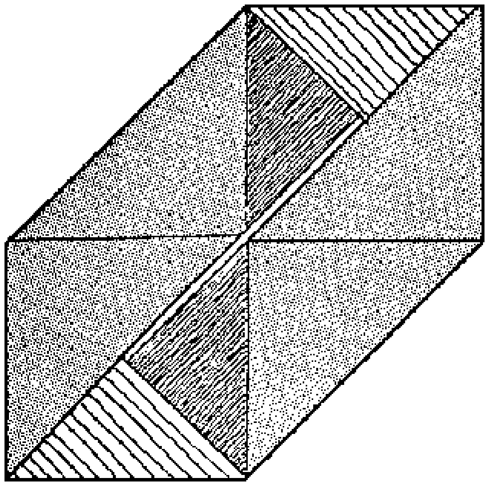


Figure 10.5

It seems clear that we could, in a way analogous to that just shown, chop off any combination of unhinged edges in any flexagon ($K > 3$). Hence, the class of a complete or incomplete flexagon of cycle $K$ cannot be established without further information. Also, just as edges can be removed, they can be added, or the leaves may be perforated or curved leaves may be used. Anything is allowed, so long as on the one hand it does not decrease the size of the leaf to the point where it allows the flexagon to collapse or, on the other hand, it does not increase the leaf in size sufficiently to interfere with the hinging of other leaves. In performing these other alterations, the "class" motion is completely demolished, so that "class" now becomes a description of the general appearance of each leaf. Thus the class comes to be identified more and more with the plan, except that it does not include the sign or number sequences. In fact, the plan is the sum of these three: class, number sequence, and sign sequences.

**C. Faces**

Another descriptive quantity closely associated with the class is the degree of each face. This is defined as the number of hinge positions between the two hinges to each pat, measured about the center of the flexagon, plus one. So long as every hinge position in each pat remains occupied, the face degree is easily determined. However, once we are allowed to mutilate the leaves indiscriminately, by making them from irregular noncoincident leaves, strange things begin to happen. First, if all the hinge positions are occupied, making the leaves irregular will have no effect on the face degree. However, at tubulations on the outer edge of the flexagon's map, for example, as seen in incomplete flexagons, there will always be at least one pat in which not all the hinge positions are taken. Then, if non-coincident leaves are allowed, the number of hinge positions between the incoming and outgoing hinges may be altered by simply chopping some off or adding in some new ones, as in fig. 10.6.

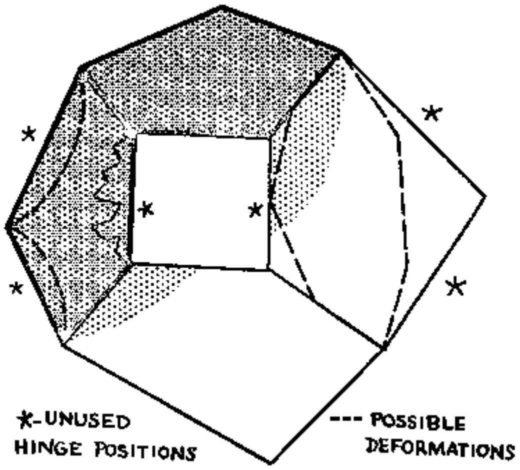



Figure 10.6

As we shall see in the section on heterocyclic flexagons, these additional (or fewer) hinge positions may actually be used, along with more non-coincident leaves, to change the cycle of the flexagon. What has this done to the face degree? It has not changed the angle between the hinges to the incoming and outgoing pats, but the degree of only one pat in each unit has been altered. In fact, it may not be clear just how many hinge positions there are between incoming and outgoing hinges. In view of this fact, it would seem sensible to redefine the face degree as the angle formed by the perpendiculars to the incoming and outgoing hinges, as shown in fig. 10.7, since this value remains constant when the number of hinge positions does not. Yet we shall see that here is the basis for yet another later section, for in compound faces the face degree actually does assume 2 values for each unit. Thus the face degree, like the class, becomes purely descriptive in nature. It, together with the number of units, tells how the pats will meet at the flexagon's center (if at all). Henceforth, when the face degree is not referred to in degrees, it must be understood that it refers to a flexagon of some given cycle with a given shape of leaf (usually a regular polygon of $G$ edges). This definition is also superior to the older one in that it clears up the difficulties that would have arisen in determining the difference between, say a 2−face in a hexaflexagon and a 1−face in a triflexagon whose center had been cut out to facilitate flexing [1].

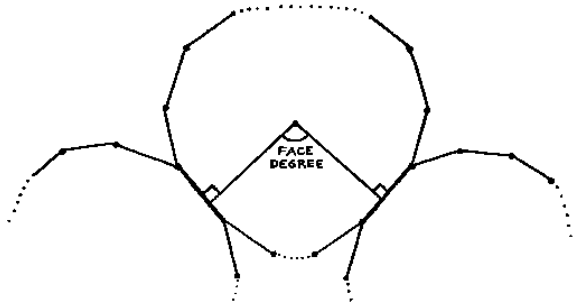



Figure 10.7

**D. More Faces**

Yet another problem of face degrees and leaf angles lies before us. To introduce this problem, we return to the "tubulation" of tetraflexagons. Let us study this type of face more carefully. The improper flexagon shown in fig. 10.8 will suffice for demonstration purposes. Suppose we leave face (1,6) and open up so that sides 1 and 4 show [2].

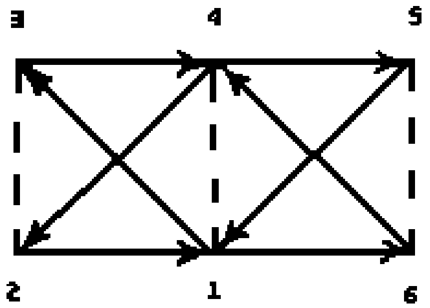



Figure 10.8

[1]The notation "$d-$face" is used to indicate a face of degree $d$.
[2]"$(m, n)$" refers to the face showing side $m$ on top and side $n$ below.

Side 4 will be on the inside, side 1 on the outside. We find that we can change faces a number of ways; folding the tube flat the way it was opened out gives back (1,6) if we reopen from the bottom, (1,5) if we reopen from the top. Folding the tube flat the other possible way gives the two corresponding faces (1,3) and (1,2). If we turn the tube over, so that the other end is upward, we get the faces (6,1), (5,1), (3,1) and (2,1). These are all the faces possible.

In order to deal effectively with such flexagons as this, we must correlate the operations just described with those already employed with $1-$faces. In order to obtain a system as simple and universal as possible, we maintain our former policy of minimizing class differences. Thus, since flexing has heretofore replaced one side (the lower side) by a new side, while moving the other (upper) side, we define this as flexing, for all kinds of faces. This, then, establishes which side of the flexagon is up; the sides of each face can now always be ordered. Turning over will be the operation that changes the order of the sides in the face. This ordering of the sides is, of course, distinct from the ordering of the pats. Just as turning over reverses the order of the sides, so we define rotation as reversal of the pat order. As we would naturally expect, rotation of the $2-$face meant squashing it flat another way. As for turning over, it was done in the $2-$face by rotating it $180^{\mathrm{O}}$ end over end. Again as expected, this merely reversed the order of all the succeeding pats. However, flexing does not receive so simple a diagnosis as this operation.

Since we flexed from face (1,6) to a new face with sides 4 and 6, side 6 must, in the tubulation, be the side defined as the "lower" side. This makes sense if we think of the tubulation as a limiting case of face degrees less than $180^{\mathrm{O}}$, as in Section V. To flex again, we will be required to fold together the surfaces marked 6. However, we cannot do this without moving 6 to the inside of the tube. Previously, we have been able to "push through" the flexagon, this operation being that which moves the flexagon from the position in which the upper side can be folded together to that in which the lower side can be folded together. This operation is inherently dependent upon the class and face degrees of the particular flexagon, therefore having no relationship to the other operations, except in that it is necessary to "push through" a flexagon to flex it. Thus, if a flexagon cannot be "pushed through" at a given face, it will not be able to leave that face except through using the side by which it arrived at it. The effect of this on the map is as if the face line were broken. A face can be pushed through only if the sum of the angles between the hinges, about the center of the flexagon, is at least $360^{\mathrm{O}}$. Thus we can break map lines by building flexagons some or all of whose faces do not satisfy this requirement. In cup flexagons, for example, every map path is broken. Other interesting flexagons of this type may be made from $45^{\circ} - 45^{\circ} - 90^{\circ}$ or $30^{\circ} - 60^{\circ} - 90^{\circ}$ triangles, with two units (See fig. 10.9). In faces of degree $180^{\circ}$, such as the tubulation, the hinges are parallel, and no number of units will suffice to allow pushing through. Hence in this case the map path is always broken. If we should extend the "pushing through" operation to include the non-trivial case of actually cutting open the tubulation and winding it up the other way, flexing would occur unimpeded. Thus we choose to say that "pushing through" is impeded, rather than flexing itself. This lets flexing remain independent of the sum of the angles about the flexagon's center.

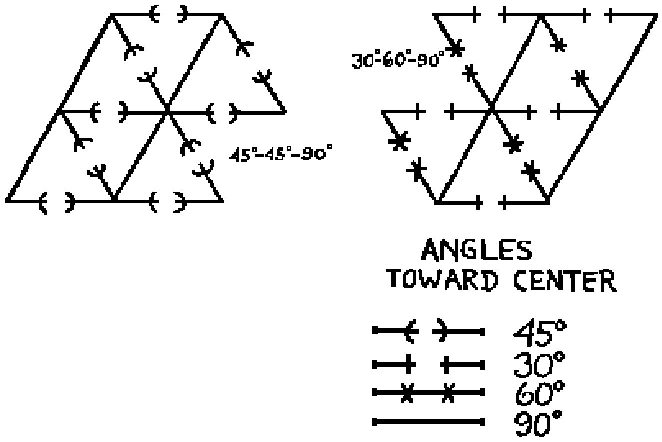



Figure 10.9

Well, if we cannot flex away from face (4,1) without turning over or cutting the flexagon, how exactly did we attain the faces that we did? In terms of the operations already known, the transition from (4,1 ) to (1,6) was clearly equivalent to turning over, flexing and turning over again. Not counting the rotation this was the same operation that brought us to (1,3). It is called a back-flex, since it is the reverse of flexing. A back-flex differs from a flex only in the side (lower or upper) that is folded together, since turning the flexagon over makes the back-flex a flex and the flex a back-flex. The operation bringing us to faces (1,5) and (1,2) must have been a back-flex, since it folded together side 4.

It may have been noticed that something is strange about faces (1,5) and (1,2). Although side 1 is technically the "upper" side (the side to be folded together by the next back-flex), it appears on the underside of the flexagon. Thus a subsequent back-flex will look exactly like a normal flex. Such a face, in which the "lower" side is on top, is described by affixing a negative sign to its face degree. Doing this is justifiable in several ways.

First, it has been seen that the hinge position used in flexing determines the face degree of the next face. The hinge position, like the face degree, can be measured in degrees, and is, in fact, the same as the face degree of the next face. Since any face in which the new side is to appear on the bottom of the flexagon opens out from the bottom, and therefore has its new hinge toward the top of the folded-together flexagon, the new hinge will be in a position between $180^{\rm O}$ and $360^{\rm O}$. For face degrees, however, we will use only those angles between $-180^{\rm O}$ and $180^{\rm O}$, so that the upside-down faces have negative face degrees. We can see that it is not sufficient to give only the face degree of a given face, since there are now four face degrees per face. For a face comprising the two sides $A$ and $B$, either side may be on top, and either side may be the "upper" side. Thus the form faces are $(A, B)^n$, $(A; B)^{-n}$, $(B, A)^n$, $(B, A)^{-n}$. [3]

There, will be four faces, related like these, whenever there is at least one. Given one of these faces, we will be able to reach other faces, made up of other sides, using flexes and rotations. Using these operations only, we can reach two and only two faces in each set of four, by following routes similar to that shown in fig. 10.10a. As we see from the figure the two related faces look exactly alike, but different sides are folded together in flexing. Doing this is equivalent to changing the flexing direction without turning the flexagon over. Thus the total number of faces breaks down into two equal families, the member of one of which can only be reached from members of the other by means of turning over. To distinguish among the various faces in the map, we mark each path with a face degree, and also indicate the "upper" side, by drawing an arrow from the "lower" to the "upper" side. Thus the faces made up of the two sides, $A$ and $B$, are represented as in Figure 10.10b. Turning the flexagon over, which changes face $(A, B)^n$ into face $(B, A)^n$ , is seen to reverse the direction of the arrow. The two faces that can be reached from one another by changing the flexing direction as in figure 10.10a, and which look alike, have both arrows and signs reversed. In actually using the map, we choose one in each set of four faces and mark it upon the map. The others must be figured out as needed. It helps to remember (1) that flexing backwards means that the flexing direction has been reversed, so that one travels against all the arrows and the signs of all the faces are wrong, and (2) that turning the flexagon over makes all the arrows point the wrong way or (if the flexing direction is reversed also) makes all the face degrees have the wrong sign. Thus, since in reaching faces (1,5) and (1,2) we traveled against a $1-$face arrow, the result is a $(-1)-$face with 1 rather than 5 as the upper side, although 5 is on top; the map shown in figure 10.8 could also be drawn as in figure

[3] $(A, B)^n$ is the face $(A, B)$ having face degree $n$.

10.11a.

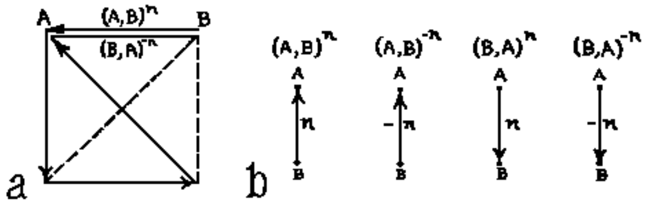



Figure 10.10

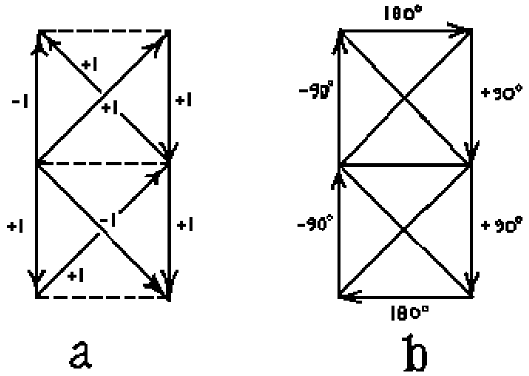



Figure 10.11

This raises another question, which results in fairly fruitful study. Just which of the four possible faces in each set are we to mark upon the map? In all cases, we will label a given map with the members of either one of the two families of faces (but not both), thus establishing a distinction between the flexagon in a given position and the same flexagon turned over. Which family we choose is irrelevant. Between the two possible faces remaining at each position, there is no such distinction, so that they may be mixed in any way. So far, we have consistently chosen the face in each pair with the positive face degree. There is, however, another method worth considering. This method deals only with the paths on the outer edge of the map; other faces may be marked with either possible face. Considering only the outer faces, then, we find that they form a simple closed curve, enclosing the rest of the map. The arrows are placed on this curve so that they always point the same direction, clockwise or counterclockwise about the map. They are then labeled with the appropriate signs (See fig. 10.11b).

The interesting thing about this method of labeling is that it gives the sign sequence signs corresponding to each outer face. These are equal to the face degrees shown. We can now, for the first time, make use of the generality with which flexing has been treated, in proving this fact. Flexing along an $0-$cut cycle without rotations shows the sub-pats corresponding to each face, with each one alone making up a pat at its face. Also, since the single-sub-pat pat alternates between the two pats, as well as inverting itself, during each flex, the sign or sum of the signs of the leaves in each single-sub-pat pat is always equal to the face degree (or the negative of the face degree; this point is irrelevant). In alternate cycles, we will travel the opposite way about the cycle. However, in changing cycles, we rotate after flexing so that the single-subpat pat occurs in the same position as it last did. This makes the face degree equal to the negative of the sum of the signs of the leaves in the single-subpat pat, rather than equal to it. Thus, if we draw the vectors the wrong way about alternate cycles, as we do when we draw them head-to-tail all about the edge of the map, the face degree will always equal the sum of the signs in the single subpat. These are the signs themselves, in the edge of the map. Q.E.D.

Since this is true, we can now easily find sign sequences for any flexagon, and, moreover, by drawing vectors all in the same direction about the edge of a map whose face degrees are unknown, and then labeling the faces with corresponding signs from a given sign sequence, we can determine the type of faces present in an unexplored map.

### E. Negative Angles

Another justification for the use of negative face degrees is the use of negative leaf angles. We have already built triflexagons from leaves having negative angles, in section V. Negative angles were the result of continuously deforming the leaves from ordinary positive angles, through $0^{\rm O}$ angles, to negative angles. To again distinguish such flexagons from ordinary ones, we assume that the face degree, which equals $180^{\rm O} - a$ $(0^{\rm O} \leq a \leq 360^{\rm O})$, where $a$ is the angle between the hinges of the leaf, is continuous for all values of $a$. This gives the accepted positive and negative values for the face degree.

In section V, we treated flexagons of cycle 3, such as that shown in figure 10.12, for various possible face degrees. Since the sum of the face degrees, taken with the arrows all pointing in the same direction, must be zero, we let two of the faces be of degree $+n$, making the other of degree $-2n$. Thus, for $n = 120^{\rm O}$, as when equilateral triangles are used, $-2n = n = 120^{\rm O}$, and all the faces are alike. For higher classes, $n = 360^{\rm O}/K$; $-2n = 360^{\rm O}(K-2)/K$; and $-2n$ is either $\pm 180^{\rm O}$ or negative.

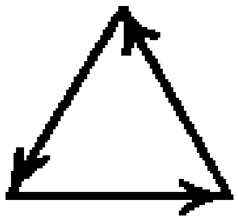


Figure 10.12

It is interesting that, in deforming the leaves to give them negative angles, not only is the upper side placed underneath the flexagon, but its center moves to its outer edge. We defined the face degree as the angle between the perpendiculars on the hinges entering and leaving each pat. This definition allows enough ambiguity to explain the presence of two face degrees at each position, for we did not specify the direction of measurement of the angle. Consistent with the observation that in a negative face the "central angles" of the flexagon point away from the center, is the fact that if the angle between the perpendiculars is measured not so that it includes the center, but across the outer edge of the flexagon, where the negative leaf angles "point" (See fig. 10.13), it turns out to be negative. Also, when a flexagon is turned over, the signs of the two angles are automatically reversed (see fig. 10.13). When the sign of a face degree is not specifically given, it is to be assumed that we are referring to the face with a positive face degree.

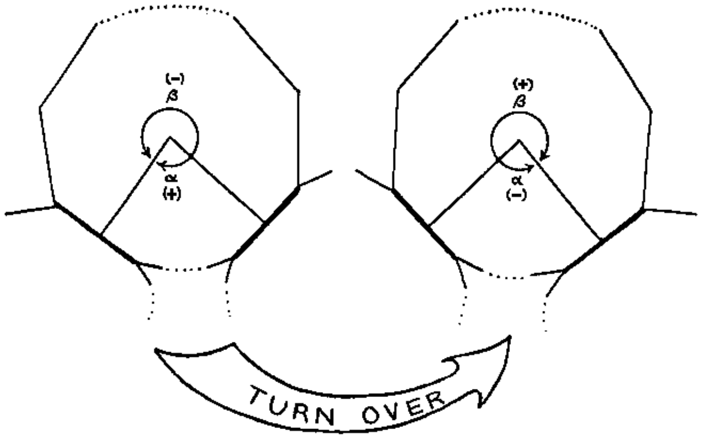



Figure 10.13

When we come to making negative-angled flexagons of cycles other than 3, we find ourselves putting hinges across places that were considered in section V to lie "at infinity", or were not (for some reason or other) allowed to have hinges. Thus, the flexagon in fig. 10.14 has hinges at "infinity" during both tubulations. Fortunately, the dubious concepts of "angles at infinity", and so forth, are not critical to the theory, and everything said so far applies equally well to flexagons of all cycles. The only difference is that an increase in cycle prohibits a decrease below $K$ of hinge positions. Since we have mentioned the tubulation in tetraflexagons, it is worth pointing out that its face degree, $180^{\circ}$, makes it meaningless to place an arrow along it, since all of its faces have equal face degrees.

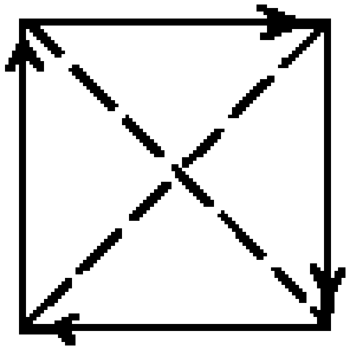


Figure 10.14

In conclusion, it might be said that when one actually comes upon a face with a negative face degree, probably the simplest, least confusing procedure is to turn it over and forget all about the whole thing.

# Chapter 11

# Zero Angles

Amid all our concern for face degrees equal to and greater than $180^{\rm O}$, the reader may have noticed the conspicuous absence of $0^{\rm O}$–faces, corresponding to $180^{\rm O}$ leaf angles, which would complete the range of possible values.

Two equivalent types of $0^{\rm O}$–faces can be built both of which involve placing the hinges in line with one another. The first method superimposes the hinges, as in fig. 11.1a. The second puts them side by side (fig. 11.1b). The only advantage of one over the other is that the second method allows us to reverse the position of the $0^{\rm O}$ pat (or leaf) in respect to the two leaves it joins (See fig. 11.1b. Forcing is still necessary).

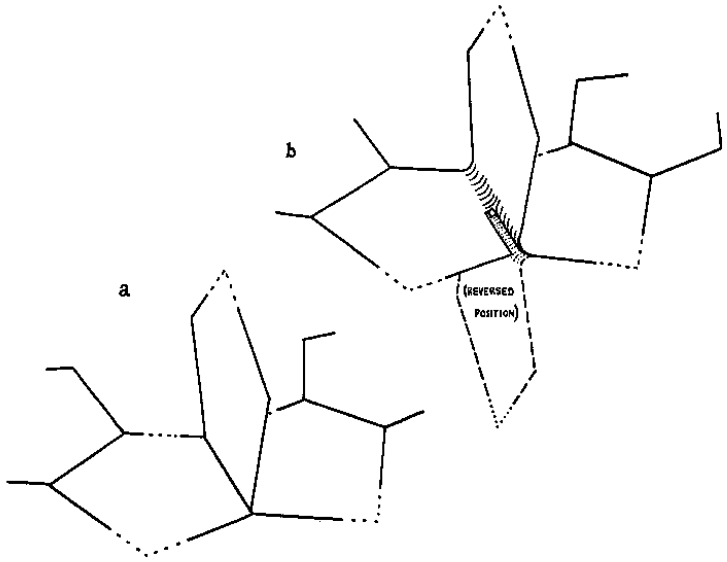



Figure 11.1

Clearly, a $0^{\text{O}}$–face will be made up of a set of flaps, joined along a line at the center of the flexagon. Since this is like the folded-together position in non $0^{\text{O}}$–face flexagons, we are tempted to open it out. Nothing prevents doing this, since the flaps of the $0^{\text{O}}$–face cannot lock sides out of sight. Thus most $0^{\text{O}}$–faces are not only $0^{\text{O}}$–faces, but exhibit a third side as well, which may have any face degree; if $0^{\text{O}}$, we can open out again, until one face degree $\neq 0^{\text{O}}$ is obtained. In a sense, then, $0^{\text{O}}$–faces can be completely ignored in working the flexagon. On the other hand, since they are incapable of locking shut, $0^{\text{O}}$–faces tend to lessen the flexagon's stability greatly. For this reason most of the research done heretofore seems to have avoided them. However, Dr. F. G. Mannsell and Miss Joan Crampin [1], present a notable exception; the irony of the situation being that their interest was apparently due to a misunderstanding of the significance of the regular triflexagon of order 6, which was presented in an article by Miss Margaret Joseph[1]. The order 6 flexagon they discuss will, however, serve ideally for introducing $0^{\text{O}}$–faced flexagons. It is built from a straight strip of 18 equilateral triangles, folded up in the constant order 1 2 3 4 5 6. Thus its map is that shown in figure 11.2a. It is, of course, a hexa-flexagon. We were told that these could not be made of triangles without hinges overlapping: this is precisely what happens in this flexagon. A simple calculation shows which sides will appear together as $0^{\text{O}}$–faces: those connected by dotted lines in the figure. When we have built this flexagon, we are surprised to see that, except for numbering, it is identical with the triflexagon of order 3, made with 6 units. This, too, is explained by the map. Suppose we erase the numbers off the leaves. Then sides making up $0^{\text{O}}$ faces will be indistinguishable, and, if they are drawn as one point, we get the map in figure 11.2b, from the map in fig. 11.2a. This second map is much more easily read than the first, but is difficult to use in making the flexagon. It is therefore recommended that each form be used for the better-suited purpose, making or operating, only.

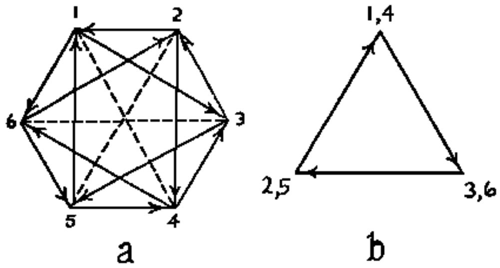



Figure 11.2

[1] See bibliography

Miss Crampin points out that any combination of the sides at one vertex of a map such as that in fig. 11.2b and the sides at an adjoining vertex can occur. Not only is this true, but the sides at one vertex can be mixed, so that a number appear at one time. This is a further reason to consider $0^{\text{O}}$ face sides as identical. An interesting figure that demonstrates how the sides at each vertex can mix is shown in fig. 11.3. It should be marked according to the number sequence $\frac{1\,3\,3\,5\,5\,1}{2\,2\,4\,4\,6\,6}$. The flap on this object travels around the object's center. The $0^{\text{O}}$ faced flexagon may be made to do the same thing, with its three flaps.

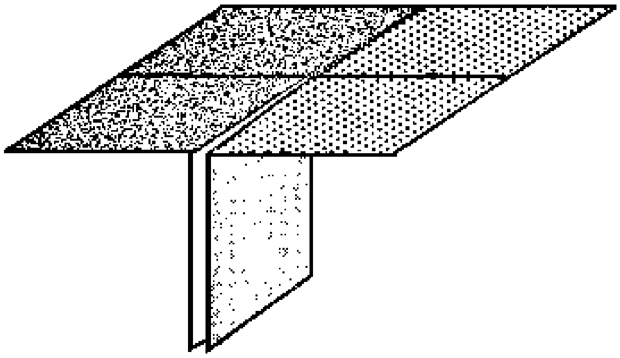


Figure 11.3

Aside from the points already mentioned, $0^{\text{O}}$ faced flexagons have little to distinguish themselves from other flexagons. Both types (either may be used in most cases) are made using only slight alterations of the usual construction method. The map requires no changes at all. To see how any $0^{\text{O}}$ faced flexagon can actually be put together we consider the flexagon in figure 11.4. The first thing that strikes one is the comparative simplicity of the second map. The first map, however, shows each possible pairing of the sides into faces separately. Since this is true, we also notice in the first map that paths leading to opposite ends of $0^{\text{O}}$ face from the same side are of equal degree. This is characteristic of $0^{\text{O}}$ faced flexagons, and, if no $0^{\text{O}}$ faces are present in the 0–cut cycle, so that their signs are readily seen, the presence of $+n$ and $-n$ signs in the sign sequence, for one cycle, may help indicate their presence. Another sure indication of the presence of $0^{\text{O}}$ faces in a flexagon is that its class may be less than its cycle. This forces some of the hinges to overlap, so that $0^{\text{O}}$ faces result. The $n$ used in the maps given indicates that any angle will be adequate in place of $n$. Let us choose $120^{\text{O}}$, so that the map becomes like fig. 10.12.

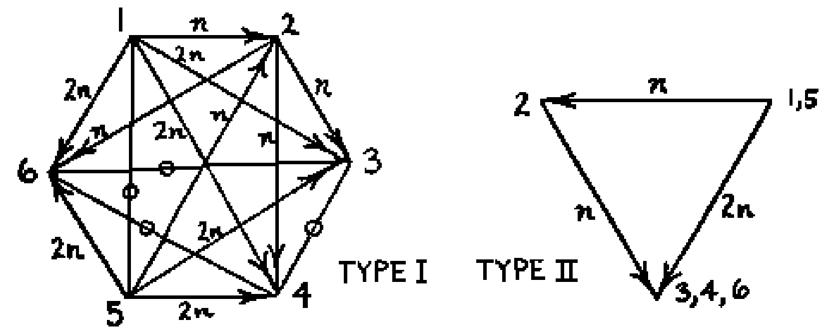



Figure 11.4

By the method described in the last section (sect X, part D), we find, for the sign sequence, $++0^{\circ}+-+$, where $+=120^{\circ}$. The number sequence is of course $\frac{1\,3\,3\,5\,5\,1}{2\,2\,4\,4\,6\,6}$, with three units. The only irregularity comes in the handling of the $0^{\circ}$ leaf. The leaf following this leaf can be placed either above it or below it; unless we either use the double hinge position type of zero flex or superimpose the new hinge in the correct place, the flexagon will not fit together. To determine the correct position; count either up or down away from the number on top of the $0^{\circ}$ leaf, just so long as the number on the bottom of the leaf is not encountered. When a number occurring in either the last leaf before the $0^{\circ}$ leaf or the next leaf is encountered, stop. If the number encountered was on the leaf before the $0^{\circ}$ leaf, the new leaf is placed underneath. If the number was on the new leaf, put the new leaf above. The proof of this little theorem is not difficult, but it is awkward, so that it is omitted here.

When it has been placed correctly, the new leaf must be folded over (or under) the leaf preceding the $0^{\circ}$ leaf, before other leaves are added on. This puts the right surface on the top side of the leaf and orients it correctly for use with the sign sequence. The remainder of the assembly is routine. When we are finished, the face (1–5, 3–4–6) should look like fig. 11.5.

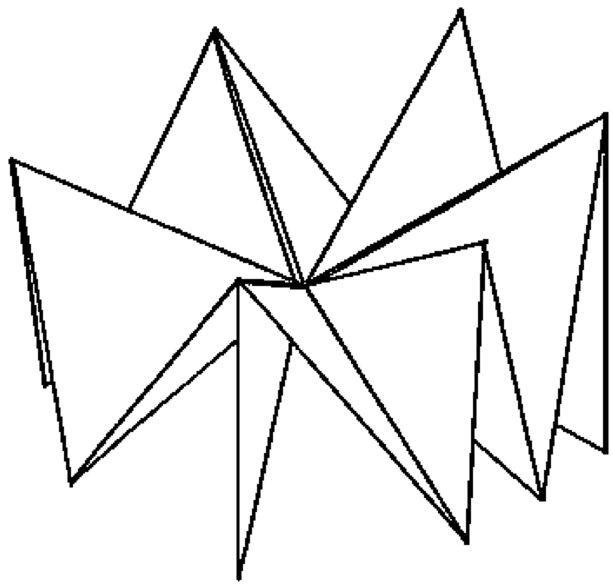


Figure 11.5

The main consequence of the inclusion of $0^O$ faced flexagons in our discussion is the enormous increase in the number of flexagons we can build. For example, there are now 13 possible 4–flexagons or order 4, made from squares. All of these have been made; few prove interesting to any great extent, since most of the type II maps (See fig. 11.4b) are trivial. When we get around to building $0^O$ faced flexagons of more than one cycle, though, a new difficulty arises: flexagons in which any two cycles have a $0^O$ face in common require two hinge positions at the $0^O$ face. Even when these are given, as when we use the 2–hinge position type of $0^O$ face, the change of cycle cannot occur. To see why all of this is so, and to find a remedy for the situation, we must consider the past structure of an intercycle face. There are two pats, one for each cycle. Each of these is made up of a pile of subpats (and/or single leaves) arranged in the pat structure $ABCD \ldots N$, where each letter represents a subpat, and $A$ is connected to $B$; $B$ to $C$, etc. When these are joined, they form the constant order $ABCD \ldots NN'M'L' \ldots B'A'$, where $A$ is connected to $N'$ and $N$ to $A'$. However, if we try to use only one inter-pat hinging position during the $0^O$ face, the hinge $A - N'$ blocks off $N$ from $A'$, and the two hinges must intersect. (See fig. 11.6, upper half). The figure shows, even if the hinges are allowed to intersect, by using a double hinge position type of $0^O$ face, one pat $(ABC\ldots N)$ will be locked shut. This is what prevents further flexing in one of the two cycles. Yet we can see that if the intersection could be moved, so that it would occur in front of the subpat $A'B'C'\ldots N'$ as shown in fig. 11.6, rather than in front of $ABC\ldots N$ then flexing could occur. The only way to move the intersection is to rotate both pats about themselves through a half twist, in opposite directions, so that the crossover is unwound off of the second pat onto the first. Doing this would add no twists to the strip, but it does require flexible leaves. Specifically, it requires that the leaves be creased along the perpendicular drawn to the intersecting hinges at their midpoints. The ends of the hinges toward the top of the flexagon during the $0^{\text{O}}$ face may then be folded down next to the bottom ends, after which the bottom ends are to be folded up to replace the top ones. This operation produces the desired result. Fig. 11.7 gives the picture for the 4–flexagons; other cases are analogous.

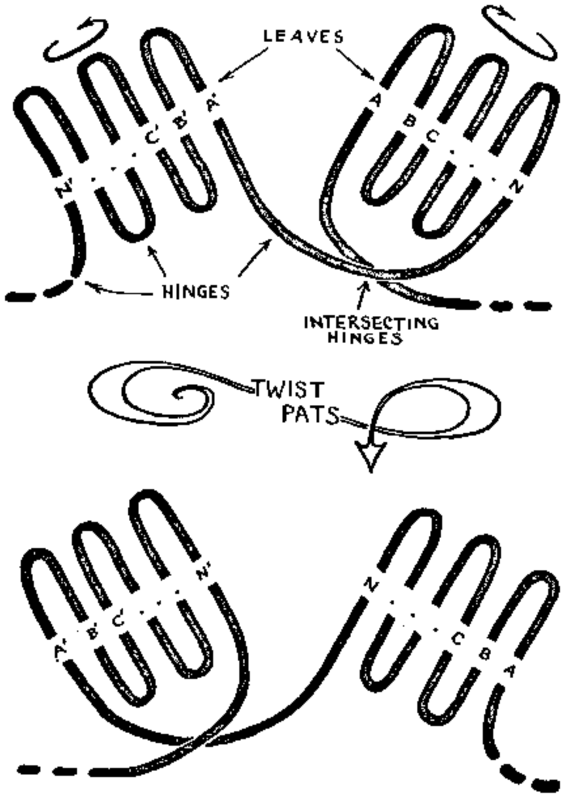



Figure 11.6

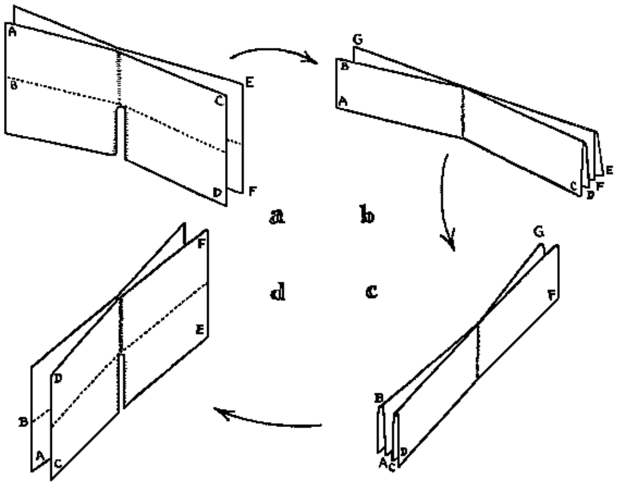


Figure 11.7

Even this does not clear up all of the difficulty. In the flexagon shown in fig. 11.8, for example, each of the three hinges $a$, $b$, and $c$ will intersect the others. Suppose that we cut away the bottom 2/3 of hinge a to make room for hinges $b$ and $c$. Then we must cut away the top (1/3) of the hinges $b$ and $c$. However, to make room for hinge $b$, we must the cut away the bottom 1/3 of hinge $c$, which completely severs it. Hence it is impossible to have a $2n$ cycle of $0^{\text{O}}$ faces, each of which joins to another cycle. This difficulty will disrupt the counting of the number of possible $0^{\text{O}}$ faced flexagons quite a bit.

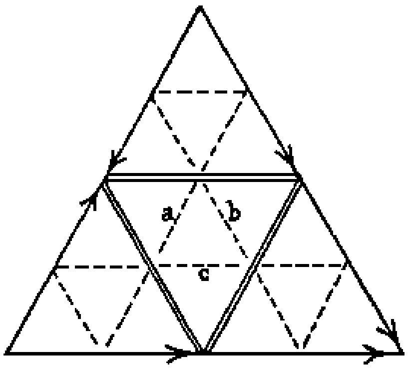


Figure 11.8

Since each pat of a $0^{\text{O}}$ face has a sign sequence sum of zero, the flexagon could be slit all the way along each $0^{\text{O}}$ face hinge and put back together into two smaller flexagons. The reverse is also possible. Thus the two flexagons in fig. 11.9a, when each is cut while folded together at side 1, can combine to form the $0^{\text{O}}$ faced flexagon in fig. 11.9b.

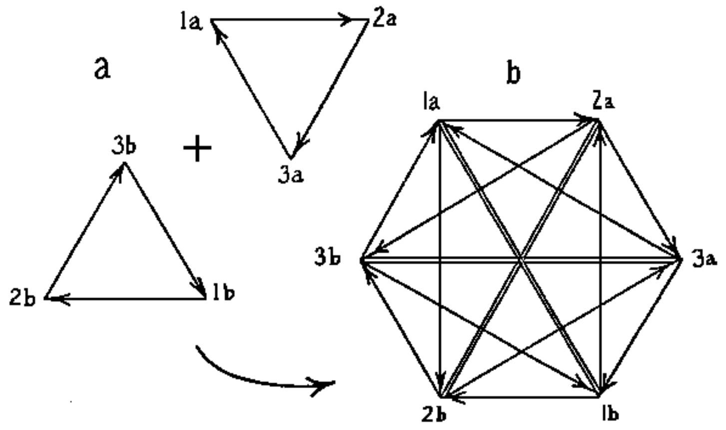



Figure 11.9

$0^{\text{O}}$ faced flexagons go through other interesting contortions. As was mentioned, they are quite unstable, so that a number of types of distortions are possible. The most interesting of these occurs in any flexagon in which a cutoff $0^{\text{O}}$ face and a $90^{\text{O}}$ face have a side in common. Call the $0^{\text{O}}$ face $(a, b)$ and the $90^{\text{O}}$ face $(a, c)$. Then consider another cutoff face, $(d, c)$, which crosses $(a, b)$ (See fig. 11.10). During this face, the leaves or subpats corresponding to $(a, c)$ and $(d, a)$ will make up one pat, those corresponding to $(b, c)$ and $(d, b)$ forming the other. The subpats $(b, c)$ and $(a, c)$ will both be either on top or on bottom. The three hinges on these two subpats will make two right angles, so that they will be able to open out away from the other two subpats. The simplest example is the map in fig. 11.10, made from single leaves rather than subpats and with $n$ a multiple of $90^{\text{O}}$. The position in which the two leaves $(b, c)$ and $(a, c)$ open out is shown in fig. 11.11 ($n = 180^{\text{O}}$). Since, as was mentioned before, flexagons with extra units may be considered identical with high-cycle flexagons of the same class with fewer units, this same distortion can be produced in some unsuspected places. The proper tetraflexagon of order 4, for example, can be made with 3 units. It may then be considered the same as the mixed flexagon, half of whose map is that shown in fig. 11.12.

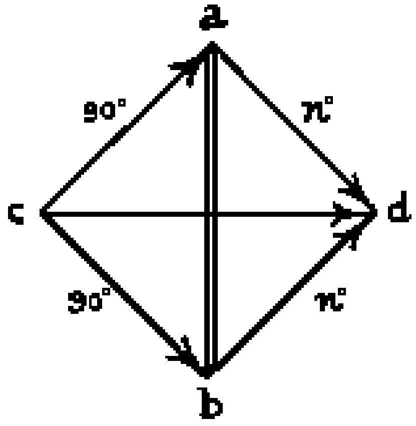



Figure 11.10

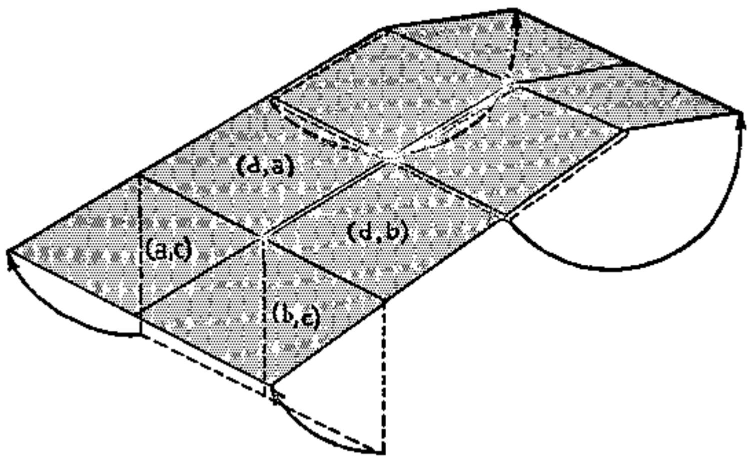



Figure 11.11

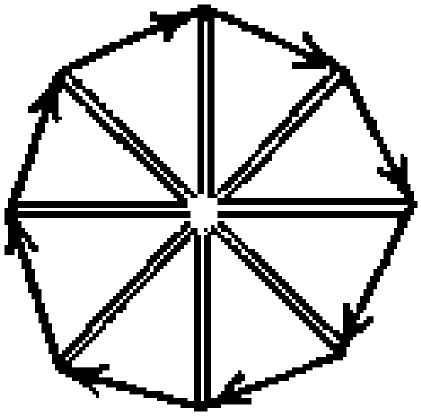


Figure 11.12

This map satisfies the conditions to the distortion. The reader may recognize that this distortion is the same as flexing half of the order 4 flexagon frontwards, the other half backwards. By adding in still more units, the flexagon may be given more flexes backwards and frontwards at the same time.

$0^{\mathrm{O}}$ faces may open out at other times, also in a most unexpected manner. The real reason for this unprecedented phenomenon is to be found in the flexagon pictured in fig. 11.11. When it is opened out, it is merely a sheet of paper with a hole in it; it is not twisted at all. The fewer the twists in a plan, the better the chance of its falling apart, With an almost arbitrary sequence of face degrees for each cycle, there is a good chance of getting very few twists. Here is a clue, at least, to instability.

A final ambiguity brought about by $0^{\mathrm{O}}$ faces may be seen in the flexagon shown in fig. 11.13. When we draw the second type of map for this flexagon, as in fig. 11.14, we find that the cycles added on to that containing the $0^{\mathrm{O}}$ faces become indistinguishable. In fact, if the flexagon is treated as an ordinary regular triflexagon with too many units, the problem sides do, indeed become a $0^{\mathrm{O}}$ face. The production of this $0^{\mathrm{O}}$ face is parallel to the flexing both backwards and frontwards at the same time in the 3 unit proper tetraflexagon or order 4.

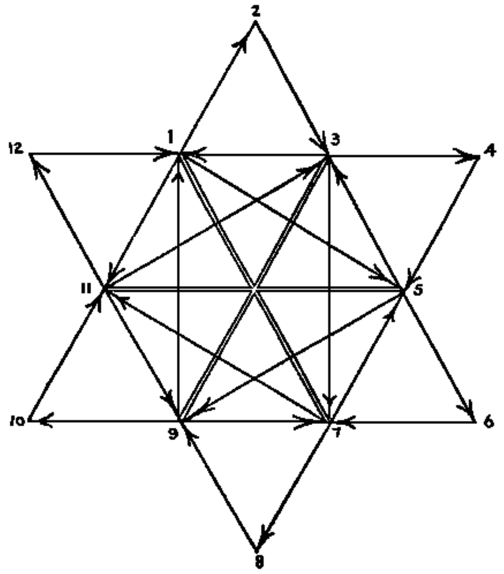



Figure 11.13

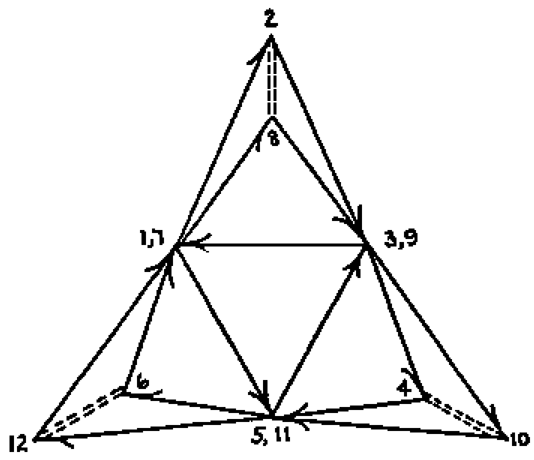



Figure 11.14

The flexagon with $0^{\text{O}}$ faces of the double-hinge-position type may also be considered as flexagons made of altered $2n$–gons, with alternate $2n$–gon vertices pushed in to form an $n$–gonal shape. Using this observation may make the planning and manufacturing of this type of flexagon slightly simpler, since they can then be treated exactly like other flexagons made of altered leaves. Certain of the faces (to be determined in the usual way) will, of course, be $0^{\text{O}}$ faces. In the same way as $n$ vertices of these leaves were forced flush with the others, so any kind of leaf can be altered to give an assortment of $0^{\text{O}}$ face positions; by pushing in one or more vertices.

As we have seen, the $0^{\text{O}}$ face fits neatly in with the rest of our flexagon theory. Just as $0^{\text{O}}$ faces can be used in conjunction with the alteration of leaf shapes, so they can be applied to the other flexagon "dimensions". We will not, therefore, specifically mention $0^{\text{O}}$ faces, to be distinguished from other faces, in the remainder of our discussion. It is to be understood that they may be used in any possible situation as desired.

By pushing leaf vertices still further in toward the center of the leaf, we can, for the first time, produce leaves with tangible negative angles, which appear as concavities (See fig. 11.15). Previously, negative angles have been produced only by continuously altering the leaf angles through $0^{\text{O}}$. Now they are produced by passing through $0^{\text{O}}$ faces, which have leaf angles of $180^{\text{O}}$. The only notable feature of these flexagons is that they are exceedingly difficult to handle. Since the hinges, when extended, cut the leaf, the leaf must be creased along the extension for the flexagon to open out at all the faces. Also, due to the wide variation in angles necessary to obtain negative angles, there tend to be a large number of units, always a sign of instability.

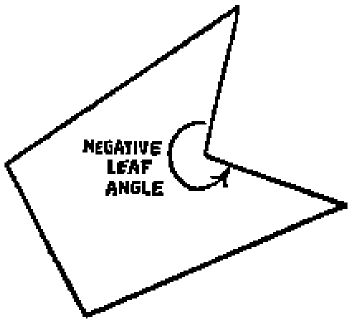



Figure 11.15

In the sign sequence we give the angles between pairs of consecutive leaf hinges. It can now be pointed out, however, that this information does not give the order of the hinge positions about the edge of the leaf. This order was determined so long as the leaves were assumed to be convex polygons. Now, though, a considerable variety of leaves may be used, all having the same sign sequences and maps in completely different-looking flexagons. For example, the leaves shown in fig. 11.16 all have same hinge positions, but in different order. The job of assigning an order to the hinge positions must be given to that old standby, the class, since the order of the hinges helps to determine the shapes of the leaves.

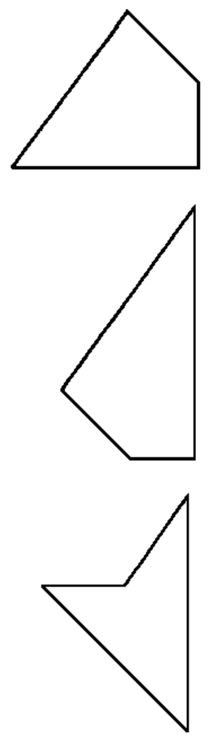


Figure 11.16

# Chapter 12

# The Heterocyclic Flexagons

Heterocyclic flexagons are defined as those made up of two or more unequal cycles. Thus we have been dealing with heterocyclic flexagons for a long while, since all incomplete flexagons are heterocyclic. In fact, the only distinction between incomplete flexagons and other heterocyclic flexagons is the class, which as we have just seen, is a weak one indeed. However, the heterocyclic flexagons do present a number of interesting problems, e.g., their construction, for which reason they are examined separately.

In introducing the heteroflexagons, we will momentarily not require class uniformity within the flexagon and assume that, rather than working with polygons all of the class of the highest cycle, we are working with regular polygons of the classes equal to the particular cycles in which they are employed. To strengthen the new approach we will start from the beginning and attack the problem with the technical knowledge that we now have available.

We have seen how the addition of a second cycle to a one-cycle $G$-flexagon can be accomplished in steps by the addition of one new side at a time. It seems clear, then, that the number of sides to be added in to complete the cycle depends only on the shape of the polygons used. It is found, naturally enough, that by changing the type of polygon used, when building the second cycle, we can make this cycle take on any desired size. For example, by using pentagons, in the proper places, we can add a pentagonal cycle to a tetraflexagon. A flexagon can then be built of any combination of cycles, using various plan polygons. Such flexagons are called heterocyclic flexagons.

The justification for suddenly changing the type of leaf may be more easily seen in another example. Consider a proper flexagon of cycle $G$ in which one cycle is complete, and a second cycle is one side short of completion. This means that one more leaf is to be slit, at the empty hinge position. Suppose, however, that we use only half of this hinge position, saving the rest by adding in a new edge to our $G$–gons at this point. The flexagon will clearly operate as effectively as if the leaves had remained unchanged, for we have done no more than change the angle between hinges and cut off a corner. However, we do now have another hinging position, so that we have changed the value of $G$. Yet we need not disturb the completed cycle at all, since it occupies a distinct and relatively independent position in the pat structure. We thus obtain a flexagon with cycles of both $G$ and $G + 1$. This method can clearly be extended to produce any combination of cycles.

Heterocyclic flexagons will, of course, have maps made up of mixed polygons. This does not, however, impair the power of the map of predicting pat structure and plan shape. The only alteration is that we must carefully record the signs in terms not only of multiples of a basic pat rotation unit but in terns of $G$ itself, since $G$ assumes varied values. That is, aside from recording the usual "+1" or "−2", we must indicate the type of polygon to which this rotation value corresponds. All of this information is given in the map: the sign in the usual manner and the polygon by the polygon in which each hinge network vertex lies (see fig. 12.1). It is interesting to note that there can be no confusion at the junction of two cycles, since no sign or number is associated with such a position in the map. Furthermore, we can have a heterocyclic flexagon in which one cycle is completely surrounded by different cycles, so that there are no leaves in the plan having its shape. Such a cycle is the triangle in the map shown in fig. 12.1. Clearly, the plan for this map is made up entirely of squares and pentagons.

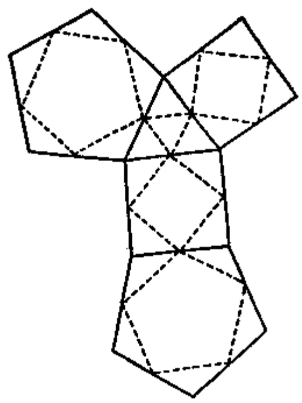


Figure 12.1

So far, the heterocyclic flexagon seems to be quite adequately treated with no major extension of flexagon theory. However, there are two aspects of these flexagons that have been momentarily overlooked. Each of these is so important to later concepts that it requires special treatment.

### A. The Leaf-Shapes

When we actually try to assemble a heteroflexagon, we find, naturally enough, that we are superimposing various different types of polygons. The problem inherent in this situation is seen when we try to fold together a strip of polygons such as that shown in fig. 12.2, folding it into a multicyclic pat structure like 213. There are a number of ways in which this problem has been attacked.

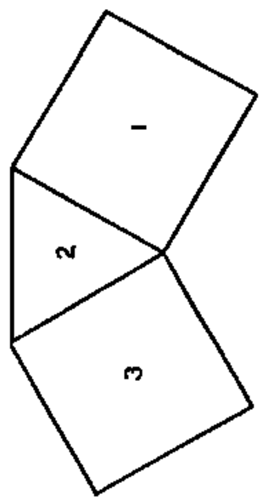



Figure 12.2

First, the shapes of the leaves could be changed. Thus, using the problem just cited, we could alter the leaves to one of the shapes shown in fig. 12.3. As we can see, the alteration must allow leaf 1 to fit between leaves 2 and 3. That is the only shape requirement. They do not need to coincide or even be the same size. In fact, if we added in two straight lines rather than one between $A$ and $B$ in fig. 12.3, the plan would be made up of squares, and yet it would seem essentially unchanged. Similarly, it could be any curve. Then the "class" becomes unimportant at this point, since the number of sides of the polygons used need have no effect on the way the flexagon works.

The second way that we could have solved the problem is necessary

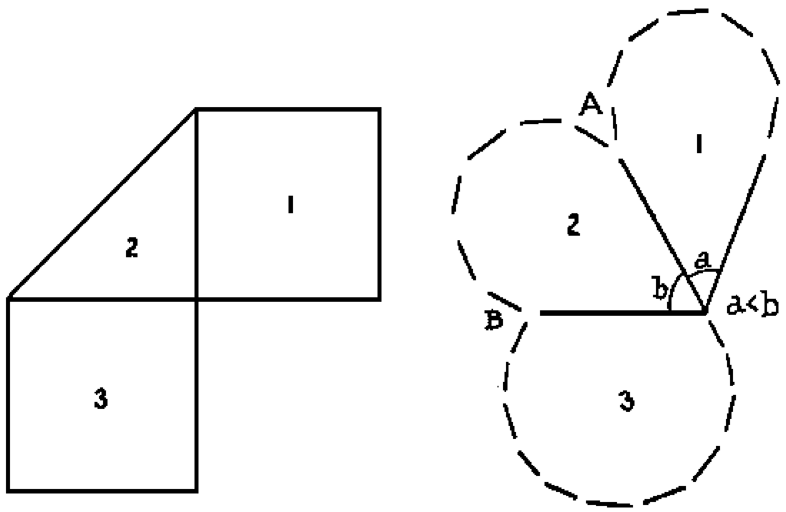



Figure 12.3

only if we still wish to use regular leaf polygons. The basis for this method is observation that, in leaves the number of whose sides are not relatively prime, such as the triangle and hexagon shown in fig. 12.4, we can trim the corners of the triangular leaves so that they look like hexagons yet are hinged like triangles. In this condition the leaves will fold together without difficulty. For the general case, then, we find the Least Common Multiple of the numbers corresponding to the types of polygons in the plan and make all the leaves in the shape of these $LCM$–gons. We thus systematically chop off all the corners that might get in the way. This method is quite interesting from the point of view of class, since the alteration makes the flexagon totally incomplete if two of the component cycles are relatively prime but with different degrees of incompleteness in various cycles. The class of this flexagon is now simply the $LCM$. Therefore, the face degrees, since they are still the same in terns of central angles, cause there to be more hinge positions between incoming and outgoing hinges at each face: if the number of these hinges had been $m-1$ in the original polygons, say $n$–gons, then it will be $m \times p - 1$ in the $n \times p = LCM$–gons.

A combination of these first two methods gives what we know as incomplete flexagons. We make all the leaves in the shape of the polygon with the greatest number of edges and then use only an appropriate number of

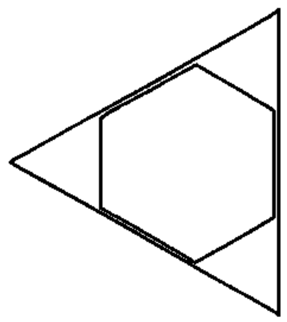


Figure 12.4

the sides for hinges, in cycles requiring fewer edges for the polygons. This is equivalent to changing the angles of one of the leaves and then cutting off corners (see fig. 12.5). Those made as in fig. 12.5b will not, of course, be able to make use of all the possible flexes. In this method, the class of all the "incomplete" cycles is changed.

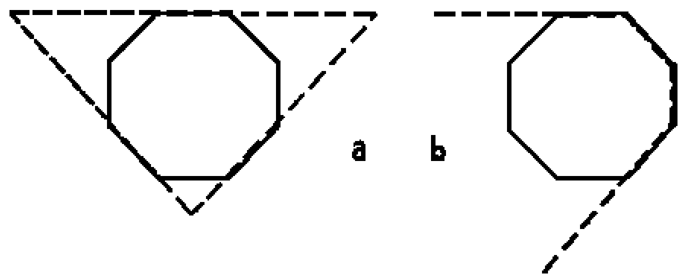



Figure 12.5

The final method for dealing with conflicting hinges is the use of circular leaves. As we saw in the first method the only important factor in the shape of the leaves is the angle between hinges coming into and leaving each pat. Aside from this, the only requirement is that the leaves overlap each other, more or less, so that the flexagon is locked together and will not readily collapse. These requirements are both fully satisfied if we allow the leaves to assume the shape of overlying circles, hinged along tangents (fig. 12.6). The length of the hinges depends only on how crowded together the hinges are. We have thus, so to speak, eliminated all corners by using a polygon with an infinite number of sides. The angles between hinges may be computed from the map by imagining that the circular leaf is the circle inscribed in a polygonal leaf.

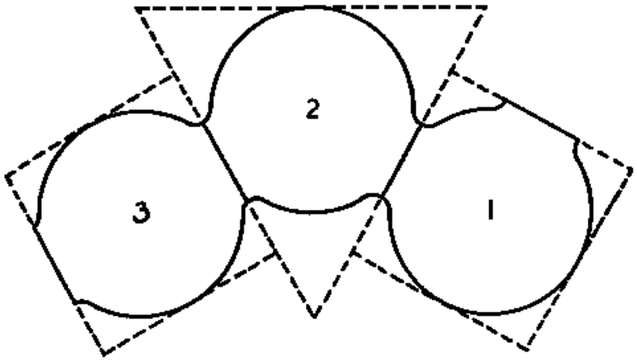



Figure 12.6

The use of circular leaves suggests several possible changes in notation. First, it would be very convenient to give the sign sequence in terms of degrees of the central angle between hinges in each leaf, thus removing reference to the kind of polygon used in the leaves of the flexagon, which is in this case quite ambiguous anyway. Also, we might suggest this ambiguity of class by using the map notation developed independently by R. F. Wheeler [1], who uses circles to represent the map cycles. The cycle circles are tangent to one another at various points, each of which represents a face. The arc between two faces represents a side. The arrows indicating direction of travel from face to face without turning the flexagon over now are the same as the direction of rotation of a gear assembly having the same appearance. Most interesting of all, the map and hinge network, when both built in this way from circles, are indistinguishable. Hence the building operation is considerably simplified; more so if the central angles between face points in the same cycle are used to indicate the class of the leaves used in that cycle.

### B. The Sign Sequence Sums

Now that "class" has dissolved out from under us, the sign sequence can no longer be a simple string of signed numbers; we must now indicate the central angle between the hinges of each leaf. How this is done is irrelevant; the point is that this is the new meaning we must assign each of the terms

[1] See bibliography

in the sequence. The real problem, however, arises when we try to use the sign sequence.

Until now, we have adhered to a basic rule of flexagon building which states that the sum, $D$, of the terms in the sign sequence must be evenly divisible by the class; or if we choose to speak in terms of symbols, that $D \equiv 0 \pmod{K} \equiv 0^{\circ} \pmod{360^{\circ}}$. The physical interpretation of this rule is that, when a unit is folded up with one leaf upon the other, the hinges entering and leaving this unit pat will coincide.

It is not difficult to see why this rule has always held, heretofore. All previous flexagons were produced by the slitting process, or, in the sign sequence, by replacing a given sign with a group of $K-1$ opposite signs (for $K > 3$, incomplete cycles may be viewed in this case as deletions of complete cycles. Since deletion clearly does not change the sign sequence sum, incomplete cycles pose no difficulty.) Hence $D$ remained constant for a given $G = K$. Considering only the one-cycle flexagon of cycle $G$, therefore, we see that its sign sequence is composed of $G = K$ like signs, so that

$$\begin{aligned} D &= Kx \text{ (unit sign) } = Kx\frac{360^{\circ}}{k} \\ &\equiv 0 \pmod{K} \equiv 0^{\circ} (\bmod 360^{\circ}). \end{aligned}$$

However, in the heterocyclic flexagons made from regular polygonal leaves, slitting did not replace a sign by $K-1$ opposite signs; slitting replaced one sign in the old cycle, where $G = m$, say, by $n-1$ opposite signs, where $n = G_{\text{new cycle}} \neq m$. Hence, the sign sequence sums for this slitting are unequal:

$$\begin{aligned} D_{m \text{ cycle}} &= 1\,x \text{ (unit m-sign)} & D_{n \text{ cycle}} &= -(n-1) \text{ (unit n-sign)} \\ &= \tfrac{2\pi}{m} & &= -(n-1)\tfrac{2\pi}{n} \\ &\equiv \tfrac{2\pi}{m} (\bmod 2\pi) & &\equiv \tfrac{2\pi}{n} (\bmod 2\pi) \end{aligned}$$

At this point we will either force the difference $\frac{1}{m} - \frac{1}{n} = D \pmod{360^{\circ}}$ to become congruent to zero, by adding in other cycles or by using one of the techniques for destroying class distinctions given previously, or we will leave $D \neq 0 \bmod 360^{\circ}$.

In the latter case, when the flexagon unit is folded together into a unit pat, as it is, for example, in flexing, the incoming and outgoing hinges will not coincide, and the ability to flex will depend only on the magnitude of $D$ and the elasticity of the materials. Of course, if we wish to abandon class altogether, the sign sequence may be juggled around at will, and we need only make the angles have any values that will satisfy the rule. If we wish to use regular polygons as leaves, $D$, the sum of all discrepancies such as $(\frac{1}{m} - \frac{1}{n})$, must be computed. In doing this, the terms representing a given cycle must always be given the same sign in the difference (whenever $\frac{1}{n}$ occurs it will always be preceded by a minus sign; $\frac{1}{m}$, never). When $D \equiv 0^{\circ} \pmod{2\pi}$, the flexagon will operate without difficulty (except, of course, the difficulty encountered in overlapping assorted polygons). It is interesting to note that, as predicted, $D$ for the flexagon shown in fig. 12.7 is uninfluenced by the presence of the triangular cycle, but is also uninfluenced by the pentagonal cycles. The computation for this flexagon goes:

$$\frac{D}{2\pi} \equiv \sum(\text{differences}) \pmod 1$$

$$\begin{aligned}
\sum(\text{differences}) &= (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{4}) \\
&= 3\,(\frac{1}{3} - \frac{1}{4}) \\
&\equiv \frac{1}{4} \pmod 1 \\
D &\equiv \frac{\pi}{2} \pmod{2\pi}
\end{aligned}$$

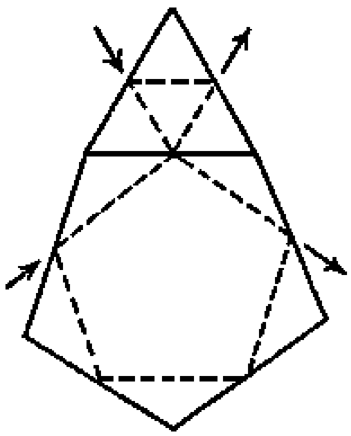


Figure 12.7

Using this method, the feasibility of a given flexagon may be guessed almost at once. Thus $D$, for the flexagon shown in fig. 12.8, is clearly quite near to zero, since the difference between one-fourth, and one-fifth is almost the same as that between one-third and one-fourth. Or, actually doing the computation, $\sum(\text{differences}) = (\frac{1}{5} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{4}) = \frac{1}{30}$, and $D = 12^{\mathrm{O}}(\mathrm{mod}\ 360^{\mathrm{O}})$, which is close enough to allow flexing with reasonably elastic materials.

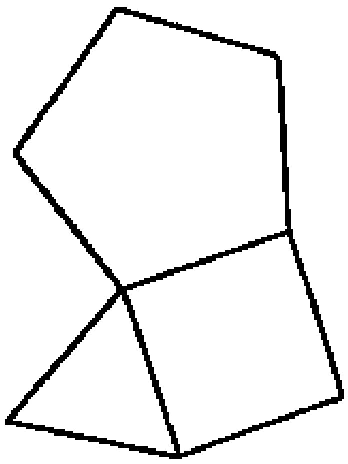


Figure 12.8

A graphic demonstration of these angle differences is given when the hinge network is constructed. Where two polygons of different types meet, the network lines must be bent (See fig. 12.7).

# Chapter 13

# Compound Faces

Just as we had met heterocyclic flexagons before they were discussed as such, so among the heterocyclic flexagons will find flexagons with compound faces.

A compound face is one in which the two pats are of different face degrees. Now, the face degree of a pat is the sum of the signs for the leaves in that pat, except that alternate pats will have the sign of the face degree reversed. The reason for this latter fact is that the sum of the entire sign sequence is the angle between the perpendiculars on the hinges entering and leaving the folded-together unit (see fig. 13.1). When one pat is unfolded from the unit, as shown, it is turned upside down, so that its face degree becomes negative in respect to that of the other pat. If the face degrees of the two pats $i$ and $j$ are equal, we obtain the followings: [1]

$f_i \equiv D_i \pmod{360^{\circ}} = f_j \equiv -D_j \pmod{360^{\circ}}$ ; $f_i - f_j = D \equiv 0^{\circ} \pmod{360^{\circ}}$

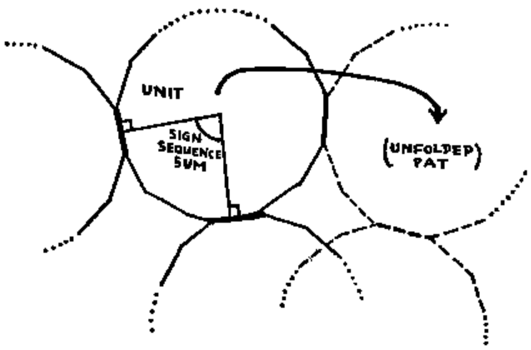



Figure 13.1

[1] $f_n$ is the face degree and $D_n$ the sign sequence sum for the pat $n$; $D$ is the flexagon's total sign sequence sum.

Here, then, is merely a restatement of the rule that the sum of the sign sequence must be congruent to zero, mod $360^{\circ}$; either this, or its faces will be compound. Thus the magnitude of $D$ is a measure of how compound a given face is; $D$ is the difference in face degree between the two pats making up any given face of the flexagon.

In our discussion of heterocyclic flexagons, it was said in effect that compound faces would be allowed only so long as $D \equiv 0^{\circ} (\mathrm{mod}\ 360^{\circ})$. The reason for this was said to be that when $D \not\equiv 0^{\circ}$, binding results during the flexing operation. This is still true. Most compound flexagons (those whose faces are compound) can be operated only with difficulty, if at all. In fact, they must be made of rather pliable material, as with polycyclic flexagons of cycle greater than 4, since rigid leaves would make flexing totally impossible. The reason for this annoying complication is rather easier to observe than to explain. If the hinges entering and leaving the folded-together unit do not coincide, a certain amount of twist (varying with $D$) will be imparted to the flexagon when each unit is folded completely together. When we attempt to fold all the units together simultaneously, this twist builds up and unwinds part of the flexagon before we are finished. If, then, we try to flex one unit at a time, we encounter trouble with the twist given the flexagon by flexing, itself. However, this twist problem is of less magnitude than the preceding one, so that it is practical, in most cases, to strain the flexagon to this extent and flex each unit separately. Either this, or we can eliminate the twist problem entirely by not folding the unit completely together during flexing. In this way, the flexagon will flex all at once, but the pats will be found to interfere with one another, thus binding the flexing operation. In most cases the best procedure is to keep in mind what one is attempting to do, and then simply feel around for the easiest method of attaining the desired structure.

Why do we trouble ourselves with compound flexagons at all, if they do not flex in a simple, respectable manner? It is because there are so many compound flexagons. There are far more compound flexagons than noncompound, of course, but there are also far more compound flexagons than there are heterocyclic flexagons. To find the compound flexagons made up of one type of cycle only, we return to our proof that $D \equiv 0^{\circ} (\mathrm{mod}\ 360^{\circ})$ in flexagons other than heterocyclic flexagons. This proof depends upon the sign sequence substitution of groups of signs having equivalent sums, mod $360^{\circ}$. The fact is that the substitution process, basic as it is, depends only upon the changes that occur in the number sequence. These alter the flexagon's structure; sign sequence changes alter only the positions of the hinges and the shapes of the leaves. These latter have been shown to have little relevance to the former; the two can actually proceed independently.

Therefore, why can we not use arbitrary signs in the sign sequence corresponding to a given number sequence, so long as hinges are not allowed to intersect? The reasons that we have created rules to prevent doing this are twofold: first, zero-faces might result. This would happen whenever the leaf polygons would be made to coincide and complete cycles would be used. Second, $D$ might not be congruent to zero. Both of these complications have finally been accepted as members of the flexagon family, so that the objections are overruled. With only one necessary sign sequence restriction remaining, the shape of the flexagon can take on unlimited variety.

The main difference between compound and non-compound flexagons, as far as general appearance is concerned, is that compound flexagons do not lie about the center of the flexagon in an even ring of pats. Alternate pats lie closer to the center than the others. The reason for this will be seen by the reader as we progress. Compound flexagons usually require more units than other flexagons. To determine how many units will be used in a specific case, we must find the angle between the hinges entering and leaving each unit. If the face degrees of the two pats are $a$ and $b$, then this angle, $c$, is less than or equal to $360^{\circ} - a - b$. It may be less than this value if the pats are folded partly together (see fig. 13.2). We may note that if the polygon $ABDCE$ in the fig. 13.2 is concave at $D$ or $E$, a correction of $-180^{\circ}$ for each concave angle must be added to the expression above.

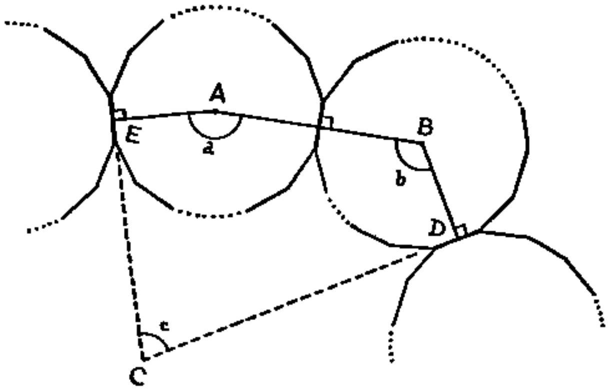



Figure 13.2

The number of units used must then be at least $360^{\circ} - a - b$ and at least 2. Since this number does not contain the constant $D \equiv (b - a)(\text{mod } 360^{\circ})$, the number of units depends upon the face chosen for the determination. We therefore choose the smallest face degree possible in the flexagon as $a$. Notice that $b$ is congruent to the sum of the signs of the leaves in that pat, while $a$ is congruent to minus the sum of the signs of the leaves in its pat.

In order to give an indication of the variety of compound flexagons, we will arbitrarily limit ourselves in this discussion to flexagons made up of coinciding regular leaves. These flexagons are perhaps the most picturesque of the compound flexagons. To begin, we examine the possible faces for $K = 3$. Other than $D \equiv 0^{\circ}$, we have here only one possibility: $D \equiv 120^{\circ}$. Then the face degrees will be either $120^{\circ}$— $0^{\circ}$ or $240^{\circ}$— $120^{\circ}$. (Faces one of which is the negative of the other, such as $240^{\circ}$— $0^{\circ}$ and $120^{\circ}$— $0^{\circ}$ , are not counted separately here. For this reason, we limit ourselves to just one of the two possible values thus presented for $D$; $D = -120^{\circ}$ is not considered.) For the $120^{\circ}$— $0^{\circ}$ face, we need $\frac{360^{\circ}}{180^{\circ} - a - b} = 6$ units (see fig. 13.3).

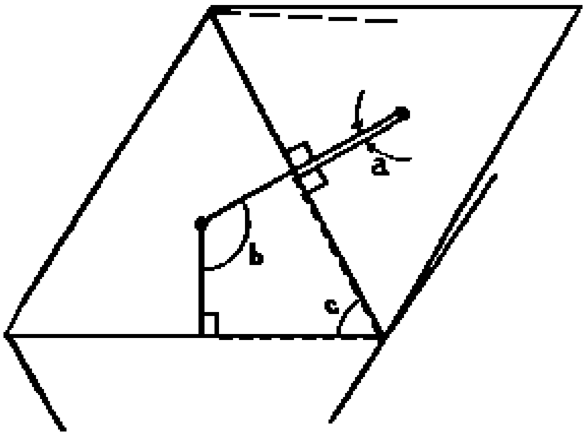



Figure 13.3

These lie in a hexagon, with flaps between the units (see fig. 13.4). Clearly, whenever one pat has face degree $0^{\circ}$, the flexagon will require twice as many units as the corresponding non-compound flexagon with face degree $\neq 0^{\circ}$, yet look the same. The $240^{\circ}$— $120^{\circ}$ face (fig. 13.5) apparently needs an infinite number of units, since its hinges are parallel.

When this is the case, we merely give the flexagon two units and do not permit it to lie flat. This face then would look like an octahedron with two opposite faces removed.

When $K = 4$, we have the possibilities $D = 90^{\circ}$, $D = 180^{\circ}$ and thereby the faces $90^{\circ}$— $0^{\circ}$, $180^{\circ}$— $90^{\circ}$, $180^{\circ}$— $0^{\circ}$ and $270^{\circ}$— $90^{\circ}$. From parallel cases already considered we can eliminate $90^{\circ}$— $0^{\circ}$, $180^{\circ}$— $0^{\circ}$ and $270^{\circ}$— $90^{\circ}$. As for $180^{\circ}$— $90^{\circ}$, we need $\frac{360^{\circ}}{360^{\circ} - 270^{\circ}} = 4$ units, which will be arranged in the hollow square shown in fig. 13.6.

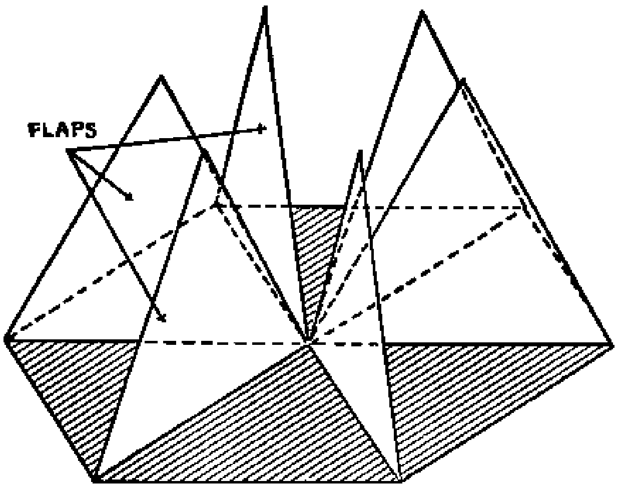



Figure 13.4

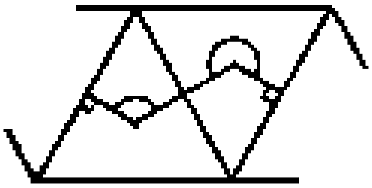


Figure 13.5

The flexagon $\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ \hline 2 & 2 & 1 & 1 \\ +90^{\circ} & +90^{\circ} & +90^{\circ} & \end{array}$, $G = 3$, has three faces of this interesting type, and is the simplest interesting compound flexagon, in its construction. For $K = 5$, all cases but $144^{\circ}$— $72^{\circ}$ and $216^{\circ}$— $72^{\circ}$ cannot lie flat: $\frac{360^{\circ}}{360^{\circ}-216^{\circ}} = \frac{5}{2}$ units. But the face $216^{\circ}$— $72^{\circ}$ will lie flat, with five units, in the handsome star shape of fig. 13.7. As before, we can make a flexagon of this type with all "+" signs, and therefore with all like faces, having $G = 3$. In fact, for $K$ equal to any integer $n > 3$ the flexagon with the sign sequence $+\frac{360^{\circ}}{n}, +\frac{360^{\circ}}{n}, +\frac{360^{\circ}}{n}$ will have three like faces the degree of which will be $(n-2)\frac{360^{\circ}}{n} - \frac{360^{\circ}}{n}$, and all of which will lie flat. If $n = 6$, the flexagon is shown in fig. 13.8. The other noteworthy faces for $K = 6$ are $120^{\circ}$— $60^{\circ}$, $180^{\circ}$— $60^{\circ}$ and $180^{\circ}$— $120^{\circ}$. These have the forms shown in fig. 13.9.

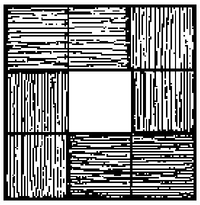


Figure 13.6

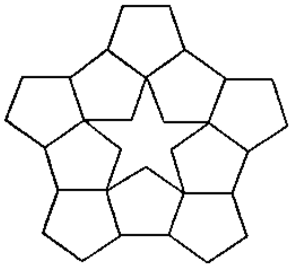


Figure 13.7

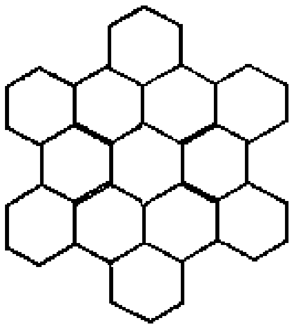


Figure 13.8

The face degree $180^{\circ}$— $60^{\circ}$ is that of all the faces of the flexagon with the sign sequence $+60^{\circ}$, $+60^{\circ}$, $+60^{\circ}$, $+60^{\circ}$, except the cuts across the diagonals of the square map. These faces are of degree $240^{\circ}$— $120^{\circ}$, so that they look like octahedrons with 2 opposite faces removed and the corners clipped (fig. 13.10). The compound flexagons already given indicate what will be found when we increase $K$ further.

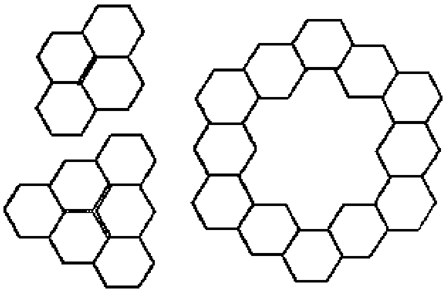


Figure 13.9

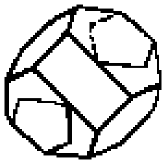


Figure 13.10

One very important question that remains unanswered is how we can tell the face degree of a compound flexagon without actually experimenting. This problem is easily dispelled when we remember that $D = b - a$ is a constant, so that a computation of each value of $a$ or of $b$ would give each value of the other. If we are given the sign sequence and its map, we can compute $D$ immediately. To distinguish between the two different face degree values, we can place them on opposite sides of path in the map. To determine the face degrees, we begin by placing the sign sequence terms on the outside of the map, next to their corresponding faces. The remainder of the face degree numbers are computed by the rule that the face degree term on the outside of a path is equal to the sum (mod $360^{\circ}$) of the sign sequence terms lying on that side of the path, taken in the direction indicated by the path vectors. Hence, the term across a path from a given sign sequence term will be $(s - D) \bmod 360^{\circ}$ where $s$ is the term. This rule is merely an adaptation of the old method of computing face degrees, pointing out that the two possible ways to calculate each face correspond to the two possible values for each unit. It follows inmediately from the fact that those leaves making up faces lying on one side of a given face line in the map make up one pat of this face; those lying on the other side make up the other pat. In computing face degrees, careful track must be kept of the directions of the face vectors. Around the outside edge of the map, these are understood to travel in one direction about the map. Why have we said that the term across a path from a sign is $(s - D) \bmod 360^{\circ}$, when it is to be equal to the sum of the signs other than $s$? It is because of the direction of the face vectors. Another way of viewing this is that the difference of the two face degrees of each face must be $D$: $s - (s - D) = D$.

Three examples are shown in fig. 13.11: one for a flexagon with regular coinciding leaves worked in degrees, one for a flexagon with regular coinciding leaves worked on in terms of a definite $K = 3$, and one for a heterocyclic flexagon made from regular leaves.

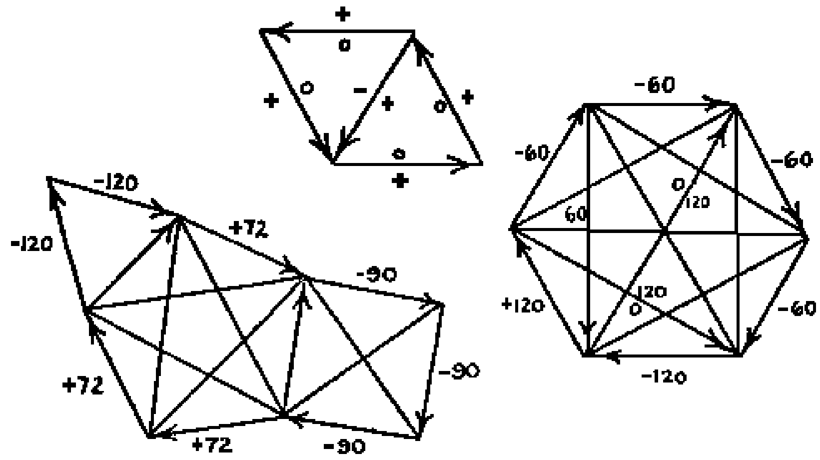



Figure 13.11

This method at last gives us a strong tool for working effectively with heterocyclics in which $D \not\equiv 0^{\circ}$. We can now accurately predict such otherwise confusing data as which faces will be nearly or exactly zero faces and which faces will not lie flat. We can also compute how many units will be required to make any given face lie flat.

# Chapter 14

# Playing Cards

The past few sections have been devoted mainly to the generalization of the previously rather strict notion of class. In this section we find the culmination of this tendency, in the complete abandoning of class as a relevant flexagon characteristic.

We have seen that face degree is determined by class, and that is all. Thus, where the hinges between two leaves are placed in relation to the other hinges to those leaves makes no difference whatsoever in the shape of the map. Nor, for that matter, does the shape or size of the leaves involved. Any variation in this direction has been shown acceptable; any shape of plan may be assembled into a flexagon with any given kind of map, so long as the hinging – or, what is equivalent, the numbering of the leaves is done correctly. Let us, then, make the shapes and sizes of the leaves completely arbitrary, irrelevant. This is difficult to do in practice, so we make use of the fact that hinging and numbering are equivalent by using one in place of the other. Using numbering only gives a "plan" made up of separate leaves, to be arranged in specified order, and to be rearranged only by the allowed operations: flexing, rotating, etc. The hinging is left to the imagination of the operator. What is the class of this flexagon? As first proposed, the class makes no difference at all (within limits: if the "class" is "lead plates 10 ft. square", practical difficulties may arise in performing the various operations). For the sake of convenience, we may use a set of rectangular cards, numbered on both sides. Also we may limit ourselves to one unit per flexagon, since the flexagon no longer has angles meeting at a center. If desired, a holder (which fixes the constant cyclic order) may be made for the cards, so that they are not rearranged by accident.

Now that we have a general idea of what the "cards" look like, we must learn to operate with them. This will be slightly more difficult than it was in flexagons where there were hinges to serve as guides, but it will also demonstrate far more clearly the nature of each operation. First we notice that all the cards actually do is give us a rearrangeable ordered set of ordered number pairs. Thus we can easily represent all the essential details of a card flexagon by a sequence of numbers. This may bring to mind the flexagon representation of C. O. Oakley and R. J. Wisner (see section 4) and the temptation to call the number sequence the flexagon, the object itself a "flexagon model". As far as the present authors are concerned, the two are equivalent.

As for the numbers actually used on the cards, it must be recalled that there are several systems for labelling flexagons, all of then acceptable: the number sequence, constant order, and pat structure systems.

The first of these, the number sequence system, results in a set of cards labeled with the numbers $(1,2),(2,3),(3,4),\ldots(N-1,N),(N,1)$ and arranged so that like numbers are together, in the order shown above, but with the cycle broken in two spots to form a pair of pats. (All the card sequences such as the one above, must be considered cyclic: the first term is understood to follow the last.) The trouble with this system is that any two flexagons of order $N$ are indistinguishable unless further information is given. If we are given the map, we can follow it in operating the flexagon, but we would prefer to be independent of the map.

The constant order system incorporates all the information given by the map into the cards. To construct a constant order card flexagon, the outer faces in the map are numbered in the order in which they are approached by the traversal of the hinge network (large numbers in fig. 14.1). This establishes a correspondence between this system of numbering and the number sequence numbering system. To change system, we can now easily substitute the numbers of the constant order into the positions of the terms of the corresponding number sequence cards. The card flexagon for fig. 14.1 would be $(1,8),(8,5),(5,4),(4,3),(3,2),(2,6),(6,7),(7,9),(9,1)$, again broken into two pats. As can be seen, all the process really amounts to is the copying down of the constant order numbers from about the edge of the map. Since in both the constant order and the number sequence systems adjacent leaves (number pairs) have like numbers facing one another, we can eliminate one number in each pair and let a leaf be represented by only one number, with the understanding that the surfaces of two adjacent leaves that face together are actually to be colored alike (they make up the same side). Then the flexagon of fig. 14.1 becomes, in constant order cards, $1,8,5,4,3,2,6,7,9$ and customary flexagons, with hinges, may be so labeled by simply numbering the leaves in the plan from 1 to $N$ in the order in which they are attached to one another.

The pat structure system, unlike the other methods, views the flexagon

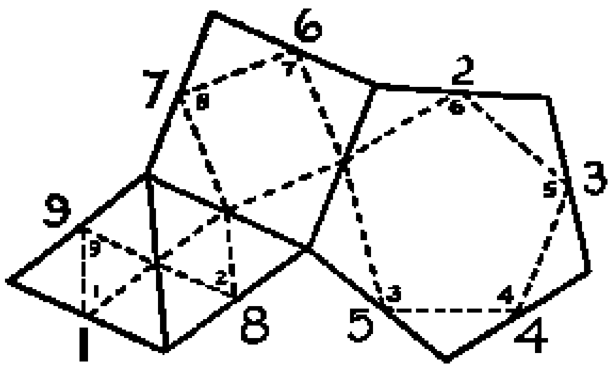



Figure 14.1

not from the point of view of a fixedly numbered unit of leaves, but as an ordered pair of structures of leaves, We have seen (section IV) the relationship between the pat structure system and the constant order system in triflexagons; it is analogous in the general case.

Our next problem is to interpret the flexagon operations in terms of operations upon the sequences of numbers that we have obtained. The simplest of these is rotation, which reverses the order of the two pats, without altering them structurally. Turning over the flexagon inverts the structure of each pat, without reversing their order. These operations can be clearly interpreted in all the systems. All remaining operations involve alteration of the pat structure. Of these other operations, which include distorting, flexing, and any other operations that one should choose to allow, only flexing will be considered here.

In describing a flex, the first thing to notice is that it acts upon three sets of leaves, each of which may or may not change pats, may be inverted, etc., but is not broken up. One of these sets is an entire pat; the other two make up the second pat, and are separated by a thumbhole. To recognize a thumbhole, we notice that it must be any spot at which the leaves above, connected to another pat along one edge of the given pat, are connected to the leaves below, which connect to another pat along a different edge of the given pat, by a single hinge. For this to be the case, the leaves in the pat must be divided by the thumbhole into two sets of leaves, each of which is made up of leaves lying together in the plan. Here we find our relationship to the constant order system, for this means that one must be able to arrange all the numbers between the thumbholes in each pat consecutively. There will be just $G-2$ thumbholes per cycle, not including the two thumbholes separating the two pats in each unit. The fact that pats must be separated by thumbholes lets us know which groupings of leaves are possible as faces.

Now, flexing is the operation which folds together two pats, thus forming a $(G-1)$st thumbhole, and then removes a different thumbhole, so that the face is changed. The critical position is that at which there are $G-1$ thumbholes, in the folded-together unit. We must first know which of the sides is to be folded together; where the extra thumbhole is to be formed. The formation of this thumbhole will eliminate any thumbholes that might be present from another cycle. Then we pick any one of the $G-1$ thumbholes remaining of the thumbholes previously present, in the folded together unit, and using this, lay the flexagon flat again. The reason why the thumbholes must now be thought of in terms of the folded-together unit, or in terms of a single cycle, is that in flexagons of $G>3$ there will be a mixture of left -and right- flexes, so that thumbholes of other cycles cannot be separated by assigning them all to, say, the left-hand pat, as we did in triflexagons. However, there is one difficulty in finding which of the thumbholes remain in the folded-together unit. Since the flexagon plan is cyclic in structure, we have no way of knowing the order of the two pats, unless we invent some way of distinguishing the two different hinges joining the two pats. If we cannot distinguish these hinges, we will be unable to tell which of the spaces between two pats has become a thumbhole when the flexagon has been folded together, and, if the ambiguity persists, the folded together structure will be divisible into thumbholes at any point, due to the flexagons cyclic construction. To prevent this we say that the hinge between the highest-numbered leaf in the left-hand pat, as seen written out in numbers, and the lowest-numbered leaf in the right-hand pat is the hinge that will be folded together to make the extra thumbhole. That is, the constant order numbers of the right-hand leaves follow those of the left-hand leaves. In the number sequence system there is no possible way to tell the thumbholes without a map, anyway. In the constant order system, the remaining break in the constant order will sort out the desired thumbholes. To keep in mind that it is a break in the cyclic constant order, the two leaves connected by the unfolded-together hinge can be encircled: the lowest term of the left-hand pat, and the highest term of the right-hand pat. In the pat structure system, we need merely keep in mind that, in folding together, the right-hand pat must receive higher numbers than the left-hand pat in the pat structure of the folded-together unit. Then the pat structure need not be considered cyclic. Although the method used in the pat structure system may seem simpler, it does require renumbering of leaves, and therefore, while well-suited to work with sequences of numbers, is not well suited to card flexagons.

Supposing that we are able to eliminate the thumbholes of any other cycle, we can set up the actual mechanism of flexing in all three systems. Suppose that a comma represents a thumbhole, a semicolon the space between two pats. To represent some specific permutation $P_i$ of the set of $y$ consecutive integers $x, x+1, x+2, \ldots, x+y$, we will use the notation $P_i\{y_x\} \cdot P_i^*\{y_x\}$ will indicate the permutation $P_i\{y_x\}$ with the terms taken in reverse order. Then, for the three systems, we have the following, where $k+m+n=N$, the cycle is arbitrary, and all addition is mod $N$.

### A. Number Sequence System.

In this system, as in the constant order system, we understand that the numbers on the leaves will be taken down starting at the "upper" side of one pat, passing through to the "lower" side of that pat, and then from "lower" side to "upper" side through the other pat. It will always be the "lower" side of the flexagon that will be closed together in flexing. We can describe the operations rotation (R), turning over (T), and flexing (F) as follows:

$$
\begin{array}{lllr}
 & a\pm 1 \quad a\pm 2 \ldots a\pm k \;;\; a\pm k\pm 1 \ldots a\pm k\pm m \;,\; a\pm k\pm m\pm 1 \ldots a(\pm N) & & (I) \\
\rightarrow & a(\pm N) \quad a(\pm N)\mp 1 \ldots a\pm k\pm 1 \;;\; a\pm k \quad a\pm k\mp 1 \ldots a\pm 1 & & (R) \\
\rightarrow & a\pm k\pm 1 \quad a\pm k\pm 2 \ldots a(\pm N) \;;\; a\pm 1 \quad a\pm 2 \ldots a\pm k & & (T) \\
\rightarrow & a\pm k\pm m \quad a\pm k\pm m\mp 1 \ldots a\pm k\pm 1 \;,\; a\pm k \ldots a\pm 1 \;;\; a \; a-1 \ldots a\pm k\pm m\pm 1 & & (F)
\end{array}
$$

The letter "I" indicates the initial position.

### B. Constant Order System.

The notation here is as in part A. The flexing operation is shown in two steps to point out that the thumbholes to be considered available for flexing must remain as thumbholes even after the folding together. The folding together may itself create a thumbhole, but such a thumbhole cannot bo used, since it was not a thumbhole in the un–folded together pats.

$$
\begin{array}{llllllr}
 & P_h\{k_c\} & ; & P_i\{m_{c+k}\} & , & P_j\{n_{c+k+m}\} & (I) \\
\rightarrow & P_{ij}^*\{(m+n)_{c+k}\} & ; & P_h^*\{k_c\} & & & (R) \\
\rightarrow & P_{ij}\{(m+n)_{c+k}\} & ; & P_h\{k_c\} & & & (T)
\end{array}
$$

$$
\left.
\begin{array}{lllllll}
 & (1) & P_h\{k_c\} & , & P_i\{m_{c+k}\} & , & P_j\{n_{c+k+m}\} \\
\rightarrow & (2) & P_i^*\{m_{c+k}\} & , & P_h^*\{k_c\} & ; & P_j^*\{n_{c+k+m}\}
\end{array}
\right\} (F)
$$

**C. Pat Structure System.**

Here the pats, numbers are taken down from the "upper" side of the pat to the "lower" side in both cases; in flexing, the "lower" side is folded together. Again, both steps are shown in flexing.

$$\begin{array}{llllllll}
 & P_h\{k_0\}\ ;\ P_i\{n_m\} & , & P_j\{m_0\} & & & & (I) \\
\rightarrow & P_{ij}\{(m+n)_0\} & ; & P_h\{k_0\} & & & & (R) \\
\rightarrow & P_{ij}^*\{(m+n)_0\} & ; & P_h^*\{k_0\} & & & & (T) \\
 & (1)\quad P_h\{k_0\} & , & P_j^*\{m_k\} & , & P_i^*\{n_{k+m}\} & \left.\right\} & (F) \\
\rightarrow & (2)\quad P_j\{m_k\} & , & P_h^*\{k_0\} & ; & P_i^*\{n_0\} & &
\end{array}$$

In each of the above cases, the leaves were numbered in a specific direction along the plan. This direction is that in which the pats are arranged in the sequences of leaves shown. If the direction is changed, the signs associated with terms in part $A$ must be changed and changes must be made in the subscripts of part $C$. Since changing the direction of labeling the strip is equivalent to changing which pat is broken up in flexing, we will give the flexing operation for a "left-handed" flex, which will then suffice for both of these contingencies: A left-handed flex = right-handed flex in a plan numbered backwards.

**A. Number Sequence System.**

$$\begin{array}{lllll}
a\pm 1 & a\pm 2 \ldots a\pm k & ,\ a\pm k\pm 1 \ldots a\pm k\pm m & ;\ a\pm k\pm m\pm 1 \ldots a & (I) \\
\rightarrow\ a\pm k & a\pm k\mp 1 \ldots a\pm 1 & ;\ a \ldots a\mp k\pm m\pm 1 & ,\ a\pm k\pm m \ldots a\pm k\pm 1 & (F)
\end{array}$$

**B. Constant Order System.**

$$\begin{array}{llllllll}
 & P_h\{k_c\} & , & P_i\{m_{c+k}\} & ; & P_j\{n_{c+k+m}\} & & (I) \\
\rightarrow & P_h^*\{k_c\} & ; & P_j^*\{n_{c+k+m}\} & , & P_i^*\{m_{c+k}\} & & (F)
\end{array}$$

**C. Pat Structure System.**

$$\begin{array}{llllllll}
 & P_h\{k_0\} & , & P_j\{m_k\} & ; & P_i\{n_0\} & & (I) \\
\rightarrow & P_h^*\{k_0\} & , & P_j\{m_0\} & , & P_i^*\{n_m\} & & (F)
\end{array}$$

To use these directions for right-handed flexes in backwards numbered flexagons, let the pats referred to be interchanged, or interchange the pats shown without changing them.

The operation given above as "flexing" (F) cannot be repeated upon a flexagon without the use of a new thumbhole, from a second cycle. Thus it should be distinguished from a flex followed by a rotation, which is the operation that is used traveling about a single cycle. This relationship may lead us to look for other properties of the operations that have been established. For this purpose, let us establish the following notation: $XY$ signifies that an operation $X$ is to be performed, to be followed by an operation $Y$. Flex-rotation is $FR$. If $I$ is the identity, then $RR = R^2 = I = T^2$. This "multiplication" is not necessarily commutative: $FR \neq RF$, but $TR = RT$. We can see the symmetry of the flexing operation if we observe that $(FRT)^2 = I$. If we flex-rotate, the resulting structure can invariably be made to flex again, since it can again be plugged into the flexing expression. Thus it can be proved that if a flexagon can once be flexed it can be flexed in at least 3 distinct situations, in such a way that $(FR)^3 = I$. We could define the cycle, $G$, of a given sequence of flexes, by $(FR)^G = I$, where it is understood that the group of leaves represented by $P\{n_x\}$, any $x$, in the instructions for flexing, must contain no thumbholes.

The operations which we have described provide means for a fairly complete discussion of flexagons using paper and pencil only. We can build flexagons up, using sequence of numbers: a flexagon is an ordered pair of pats. A pat is a set of smaller pats, each taken in inverse order, as demonstrated by the diagrams in figures 14.2 and 14.3. We can take any flexagon we build and, by plugging it into one of the operational descriptions we have given, put it through its paces. These arrangements of numbers and the various operations that can be performed upon them may even prove interesting for their own sake.

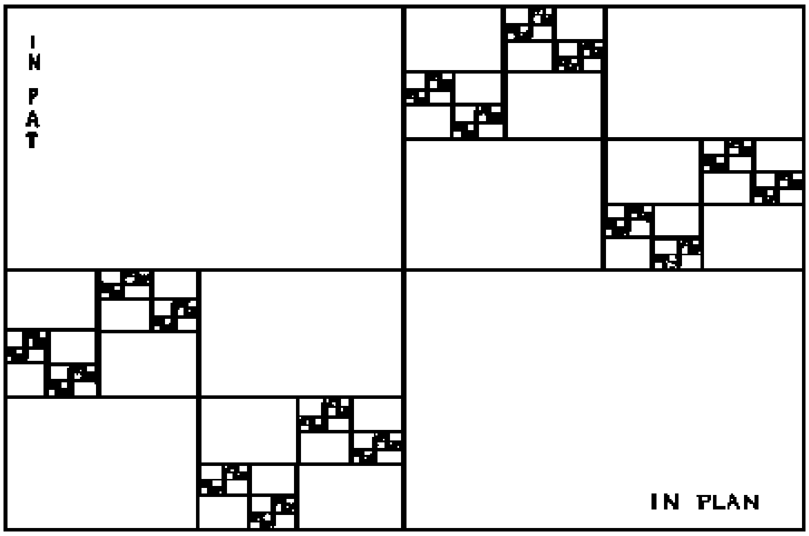



Figure 14.2

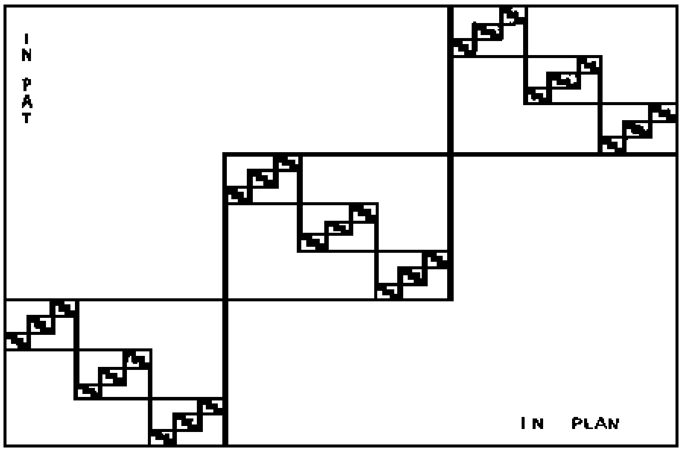



Figure 14.3

The operations of flexing and rotating may be used to create yet another representation of the flexagon. To reach a given face of a flexagon from some starting point, there is a unique shortest sequence of 0–cut flexes and rotations. No matter what path is used, the faces taken by this shortest route must be included in the route. Hence, using the identities $R^2 = I$ and $(FR)^G = I$, any path between the two faces may be reduced to the unique shortest path. The process is shown in fig. 14.4.

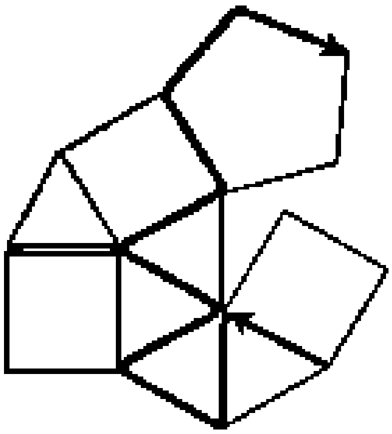


Figure 14.4

This is because at a given face a rotation will either lead away from the desired face or toward it. If it leads away, backtracking will be necessary. Using this fact, we could let a sequence of flexes and rotations represent each face. In fact, this principle is the basis for the map, as we have seen in the discussion of the covering space (see section 3). Notice, however, that in the pat structure system, faces are not necessarily distinct. Symmetrical spots on the map will be indistinguishable, so that the flex-rotation method of face representation will fail to create the equivalence classes that actually exist in this system.

Now that we can deal effectively with flexagons made up of sequences of numbers, let us return to the cards themselves. Flexing a card flexagon, it will be found, is a real job. Finding which thumbholes are available is the chief problem; the actual flexing is really not too difficult. In fact, all it requires is the putting together of the two pats and their disunion at the chosen thumbhole. Actually, this is all the information that is imparted by the number manipulating operations in the three systems we have discussed. To assist in finding the thumbholes, we may resort to using a map (as is necessary in the number sequence system). On the other hand, a slightly more subtle method is the coloring of the numbered cards according to a system which will embody the map. This is the same as using both number sequence and constant order, and permits us to shuffle the cards and then replace them in order very easily.

Using both number sequence and constant order may lead to the discovery that certain flexagons are dual with other flexagons, or with themselves; that is, the constant order of one is identical with the number sequence of the other, and vice versa. Of course, the sequences may differ (each term of one from the corresponding term of the other) by a constant, but this means that they are identical, and may be detected by comparing the differences of adjacent terms in each of the two sequences, as in section 4.

## DUALS

More generally, we might ask if, given any constant order, and thus the flexagon specified thereby, might there not be another, flexagon having this constant order as its number sequence? This does, in fact, turn out to be the case, and, moreover, the constant order of the second flexagon is found to be the number sequence of the first. That is, the two flexagons are duals. And, furthermore, the proof of this fact is constructive in nature; we describe a process for obtaining the Tuckerman tree of the dual directly from the original tree. Essentially, all the chain structures in the tree must be converted to fan structures, and vice versa. More precisely, alternate tree vertices must have the branches inverted in order, so that the righthand most branch becomes the lefthand most, etc. (See fig. 14.5). That this construction actually provides the dual is shown by induction from the segment of tree shown in figure 14.6, which has the dual shown there. All trees are made up of segments of this form, essentially, and dualizing then merely reverses the orientation of one vertex with respect to the next. The generalization of this process to flexagons of any cycle is clear.

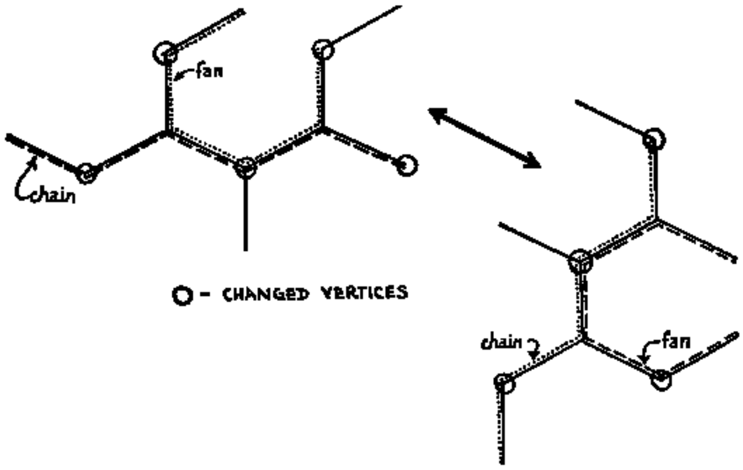



Figure 14.5

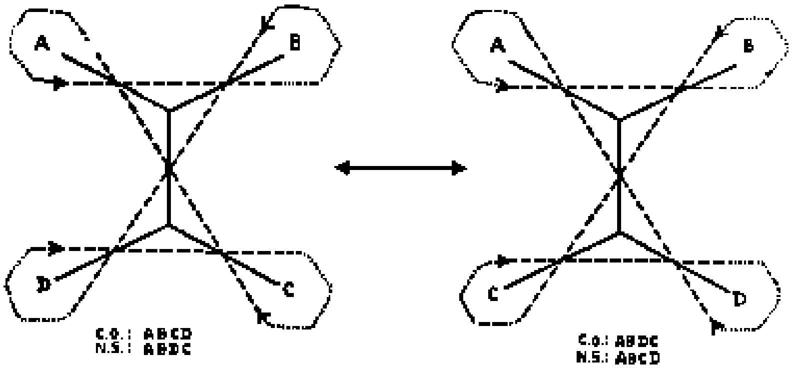



Figure 14.6

Since every flexagon has a dual of the same order, but there are an odd number of flexagons of some orders, it follows that some flexagons are self-dual. Thus, among tri-flexagons of order 6, the fan and chain are dual, and the regular flexagon is self-dual (See fig. 14.7) [1].

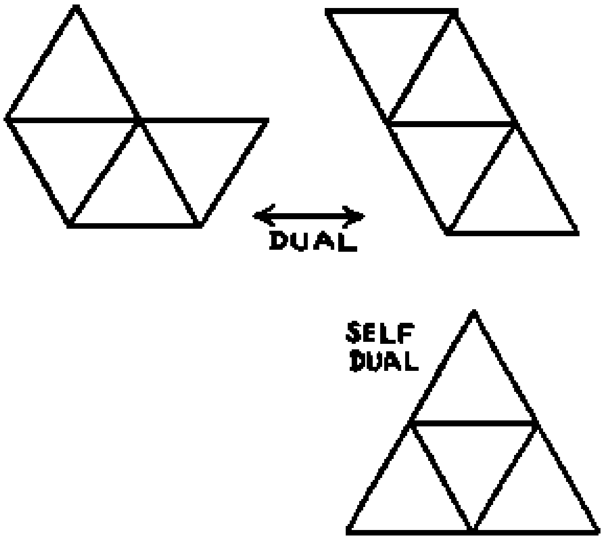



Figure 14.7

If, rather than use both the constant order and the number sequence in labeling cards, we restrict ourselves to the constant order, we will easily be able to construct any card flexagon of order $N$ from a set of cards numbered from 1 to $N$. This will include flexagons with all possible combinations of all kinds of cycles. Which of the possible permutations of the $N$ cards are we able to use? Or can we shuffle the cards, and have any random constant order make a flexagon? How many possible flexagons of order $N$ can we make? These questions lead us on, away from the card flexagons and the problems of class, and open up a whole new field of inquiry.

What requirements must we satisfy to build a flexagon? This is the question that lies at the bottom of all the problems now before us.

[1] In fact, all regular flexagons are self-dual.

# Chapter 15

# Requirements

In order to build any flexagon, it is necessary to satisfy enough requirements to specify it completely; these include such classifications as the number of leaves, their shape, arrangement in the plan, and the order in which they are folded together. So far it has been made possible to specify a given flexagon precisely under each of these categories; we now hope to establish more general requirements, to specify general classes of flexagons more broadly.

What sign sequences, in general, will generate flexagons? Two limitations might occur to us. First, we want the sign sequence to be reducible to the sequence of a known flexagon. For then this known flexagon could be extended to give the desired plan. The second rule is that the sum of the signs be congruent to zero, mod $360^{\mathrm{O}}$. However, if we allow flexagons to have compound faces, this second requirement disappears, and if zero degree faces are allowed, the first rule will not necessarily take effect. As for the first rule, it may be simplified considerably by noticing that the only irreducible sign sequences are of the form $n, -n, n, -n, \ldots$, where $n$ is an angle and, in reducing, adjacent terms are replaced by their sum (mod $360^{\mathrm{O}}$), provided this sum is non-zero. In fact, these are the only sequences that cannot be reduced to sequences of the form $n$ or $n$, $-n$. Then these flexagons (except $n$, $-n$ flexagons) are those that must have zero-degree faces.

Considering only flexagons without zero-degree or compound faces, and of class 3 or 4, we can further specify requirements on the sign sequence. Considering the class 3 flexagons first, notice that all possible equilateral plans may be arranged on a lattice of equilateral triangles (for nonequilateral leaves, the sign sequence works if and only if the corresponding sign sequence for equilateral triangles works). This lattice may be thought of as three interlaced hexagonal lattices (see fig. 15.1).

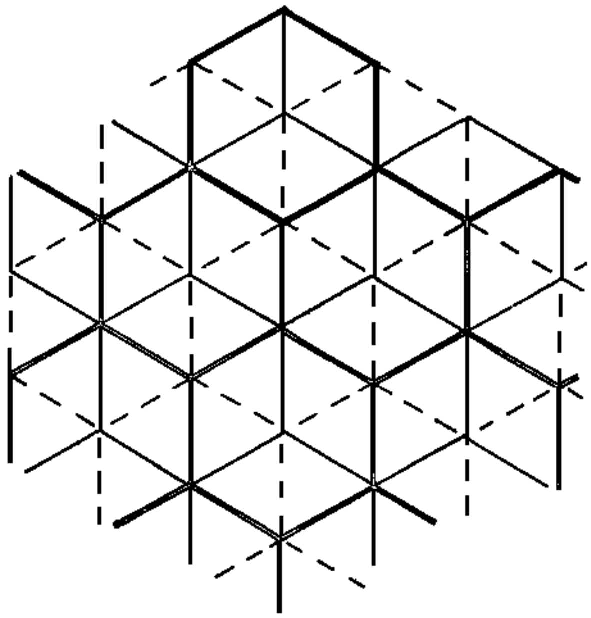


Figure 15.1

Then the sum of the sign sequence terms will be congruent to zero (mod $360^{\text{O}}$) in those plans in which the entering and leaving hinges lie on edges of the same hexagonal lattice. The proof of this depends on the fact that passing through any single hexagon does not change the sign sequence sum. As we see in fig. 15.2, no matter where we leave the hexagon, the sum of the signs within the flexagon is zero; passing about the center of the flexagon once or more does not change the sums. A similar demonstration will give the corresponding result for tetraflexagons: when the plan is placed on a square lattice, the incoming and outgoing hinges must be parallel and they must fall on lines separated by an even number of squares: $0, 2, 4, \ldots$ (see fig. 15.3, in which a plan beginning on a dark line must end on a dark line).

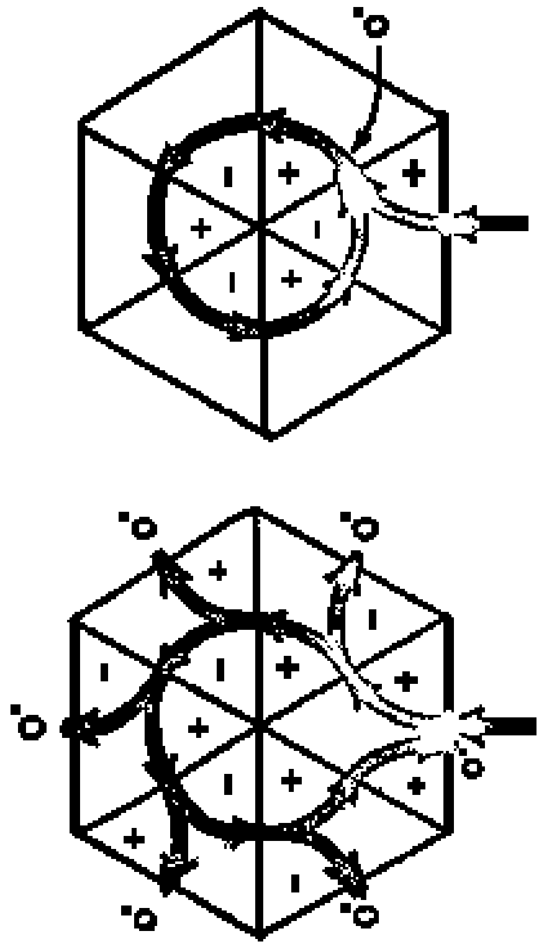


Figure 15.2

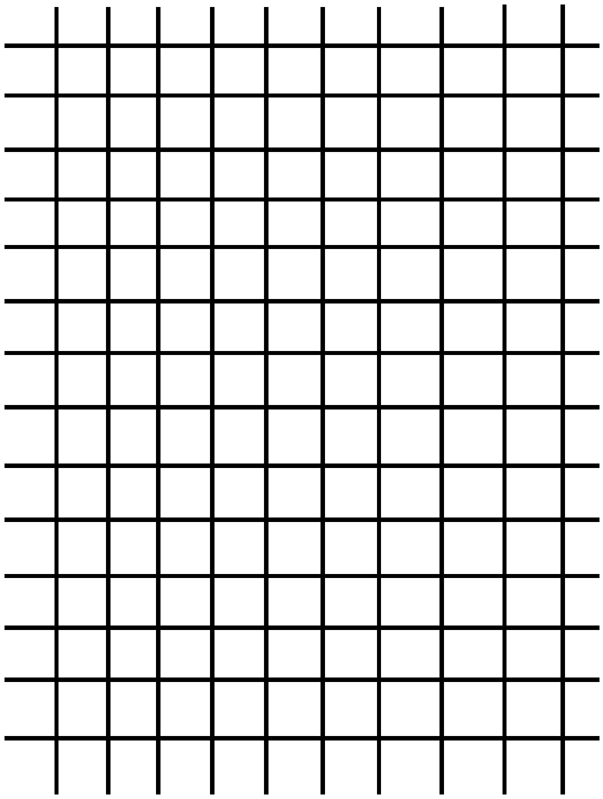


Figure 15.3

These examples lead us to ask not what sign sequences, but what plans, or actual arrays of $n-$gons, can be folded into a flexagon. In this case, there are $n-1$ possible positions for the hinges on either end. Since there are only $n$ different possible values for the sign sequence sum, it will follow that there are at least $n-2$ ways of hinging the ends together that allow the sign sequence sum to be zero. Thus, for example, any linear array of triangles except a $+-+-\ldots$ spiral can make at least one flexagon.

Returning to the problem of $+n$, $-n$, $+n$, $-n$,... sign sequences, it may be pointed out just how the zero degree faces arise. At some point in the map, two consecutive faces of one cycle will lie on the edge of the map. Then they will correspond to the signs $+n$, $-n$, and the $1-$cut across them will have to be a zero-degree face (see fig. 15.4). The leaves which corresponded to these two signs would have to be folded together, whereupon the two hinges going to and leaving them would coincide. The only case where this problem does not arise is in the sequence $+n$, $-n$, the trivial flexagon.

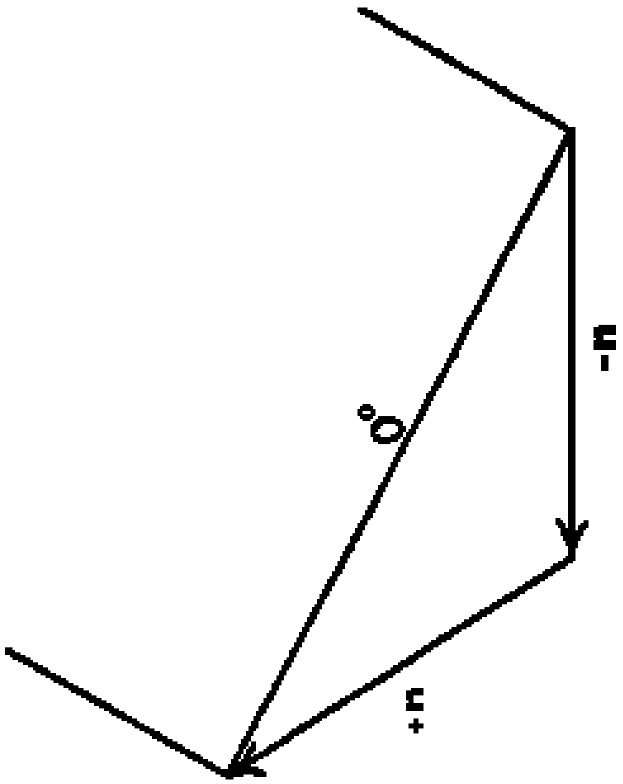



Figure 15.4

Certain restrictions must be placed on the constant orders that stand for flexagons. Notice, for example, that the permutation 14253 is useless, for it leads to the polygon network and map shown in figure 15.5. Flexing is impossible in this flexagon, because the Tuckerman tree is a closed loop. Impossible constant orders can be described most easily by their polygon networks, in which at least two polygons will be joined along an edge, rather than a vertex only. This in turn happens precisely when a pair of network lines is joined by more than two disjoint routes in the network.

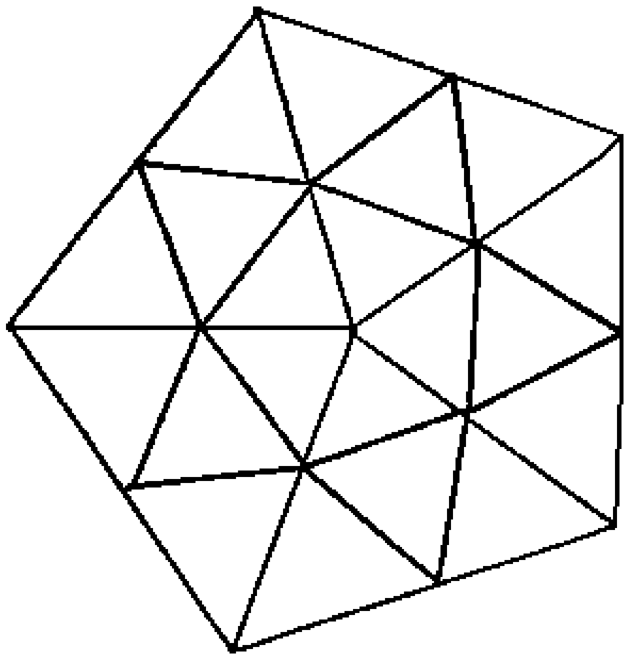


Figure 15.5

# Chapter 16

# The Number of Flexagons

It might be said that the most interesting mathematical problems concerning flexagons arise when one begins to count them. Certainly Oakley and Wisner's treatment of the triflexagons shows both that these combinatorial problems are non-trivial and that their solutions may throw light on the flexagon problem in general.

However, aside from the results which Oakley and Wisner have obtained, it appears that little is known in this direction, and the present authors cannot at this time produce many answers. Still, it seems worthwhile to at least formulate some of the remaining problems, in the hope that this will encourage solutions.

By analogy with the problem of counting triflexagons of a given order, we might ask how many G-flexagons there are of order $N$. This question still allows ambiguity: are we to allow incomplete cycles or not?. If we do, we are essentially counting the total number of different flexagons of cycle less than or equal to $G$ or of mixed cycles less than or equal to $G$. This seems superficially related to the problem of the total number of flexagons, with mixed or non-mixed cycles, with order $N$; i.e., the number of dissections of a polygon into polygons by non-crossing segments joining the vertices. Then again, we might not include incomplete cycles at all, thus seemingly simplifying the question considerably.

In any of these cases, we have not considered face degrees at all. We might ask, for instance, how many different 1-flex cycles one could have about a single cycle of a G-flexagon, as shown in figure 16.1 for $G < 7$. Or, if partially tubulating flexagons are not considered, how many different possible arrangements in all are there per G-cycle?. Then again, we may decide not to allow zero-faces in each of these cases. All of these problems are extended by asking how many different G-flexagons can be made with K-gons, for $K > G$.

There is always the problems of counting the flexagons whose maps possess some special property, such as rotational or mirror-image symmetry or being made up entirely of some pat structural unit, as with the regular triflexagons. The number of self-dual flexagons is not known even for $G = 3$, and no fruitful work at all has been done on the number of flexagons derivable from a given sign sequence, nor on the number of distinct sign sequences of a given order.

There are one or two easier questions whose answers can be supplied readily, to end on a promising note. The total number of faces in the G-flexagon is twice the number of paths. If cut-offs are not included in the count, there will be $2\frac{NG-N-G}{G-2}$ faces; if cut-offs are included, this becomes (G+1)(N-2)+2. These figures hold only when applied to complete flexagons.

In order to find the number of map paths touching a given side, without drawing the map, proceed as for triflexagons, but reducing sequences of $G - 1$ consecutive numbers instead of mere pairs of consecutive numbers. In this way a minimum number $M$ of different leaves are eventually left. The number of paths touching the given side is then $\frac{M-2}{G-2} + 1$; note that this is just $M-1$ for $G-3$ and $1/2M$ for $G-4$. If cut-offs are to be counted among the map paths touching the face, then the number of such paths is simply $M - 1$.

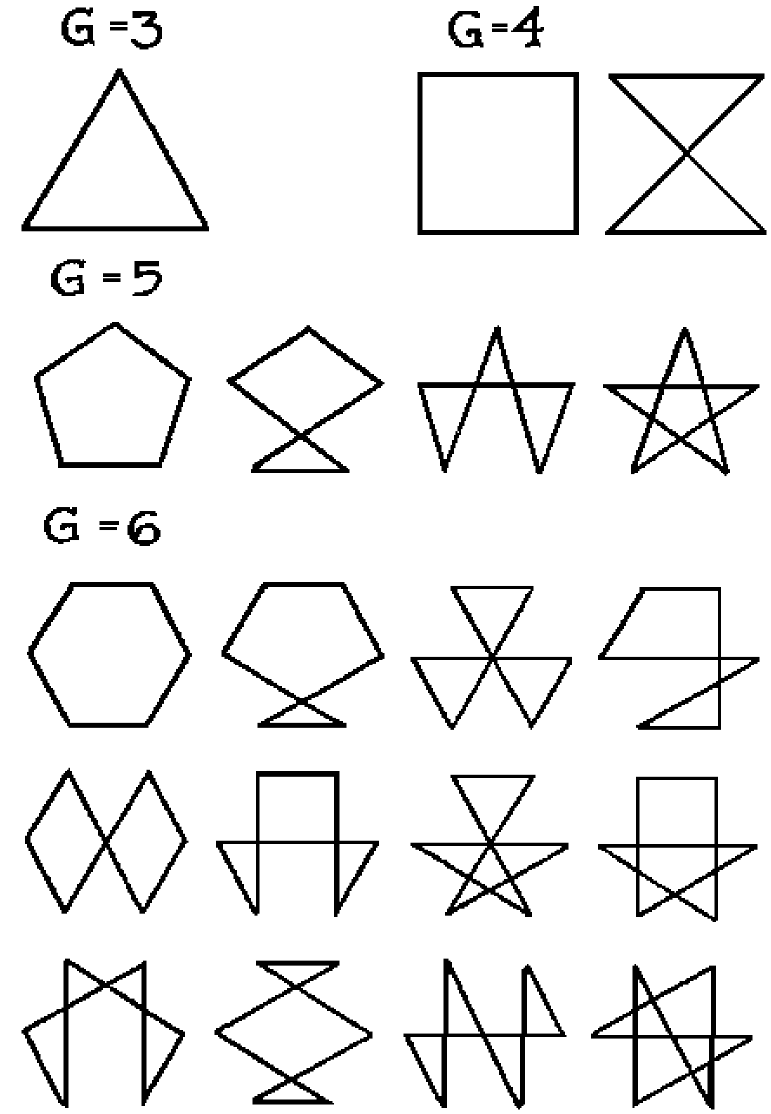



Figure 16.1

# Chapter 17

# The Distant Relations

Thus far we have encountered quite a few different sorts of flexagons, most of which have represented generalizations of the first simple ones. This process of generalizing, or breaking rules intentionally, may lead in still new directions. There are also objects that, though they are not deserving of the name flexagon, still bear a more or less remote resemblance to the flexagons. All of these objects are worth looking into for their own sake. Various such entities will be discussed; it will be clear from each what its relationship to the family of true flexagons is.

## A. Hinged Tetrahedra

One direction to extend the flexagons would seem to be to higher dimensions. Though this has not yet been done, the thought led to some study of hinged tetrahedra.

Two tetrahedra may be joined by a long double hinge so as to roll around one another. The two hinges are arranged so that while one winds off the first tetrahedron onto the second, the other hinge winds from second to first tetrahedron. This type of hinging, similar to flexagon hinging, is exactly like that on swinging doors. The hinge is ordinarily long enough to wrap around all four faces of the tetrahedron, and has one end fastened to each tetrahedron (see fig. 17.1). Several different arrangements of hinges are possible, exhibiting the faces in different orders, but that shown seems most stable. Its hinge is a $+ + ++$ strip of equilateral triangles, split down the middle.

A little more interesting but a little bit less related to the flexagons are smokerings, which are rings of polyhedra hinged together end-to-end, each

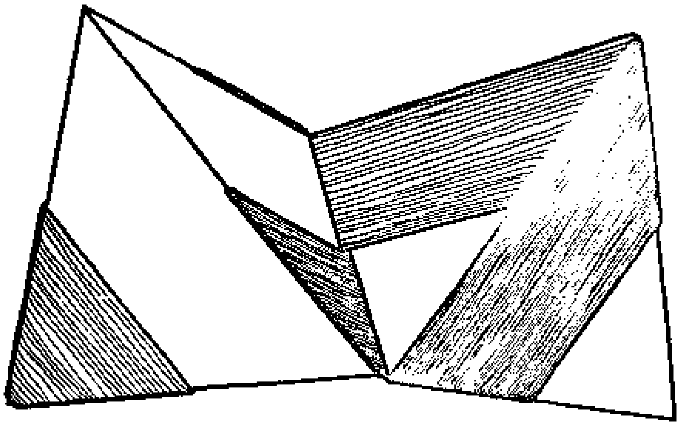


Figure 17.1

being hinged to the next along one edge. Although many polyhedra may be made to do this, tetrahedra are the only regular polyhedra which will form a ring when joined by diametrically opposite edges to the adjoining tetrahedrons.

The common smokering is a loop of eight regular tetrahedra joined end to end (see fig. 17.2). The number of tetrahedra is generally even, since the opposite edges are perpendicular, and the hinges at the ends of a chain of an odd number of tetrahedra are also perpendicular. Eight is the smallest number of tetrahedra that will both form a ring and exhibit the interesting property of smoke-rings called twirling. In twirling, all of the tetrahedra are simultaneously rotated about an axis passing through the midpoints of the hinged edges. The overall effect of the process is that the smokering as a whole turns inside out. Although there is nothing theoretically complicated about twirling, the effect is fascinating, especially if the smokering is brightly colored.

Six tetrahedra may form a ring, but the ring is too tight to twirl. If more than eight tetrahedra are used, the loop becomes looser and looser, until 22 tetrahedra can be tied into a knot. If the tetrahedra are stretched slightly, a ring of an odd number may be formed. If moderately irregular tetrahedra are allowed, a smokering may be made of five tetrahedrons. This

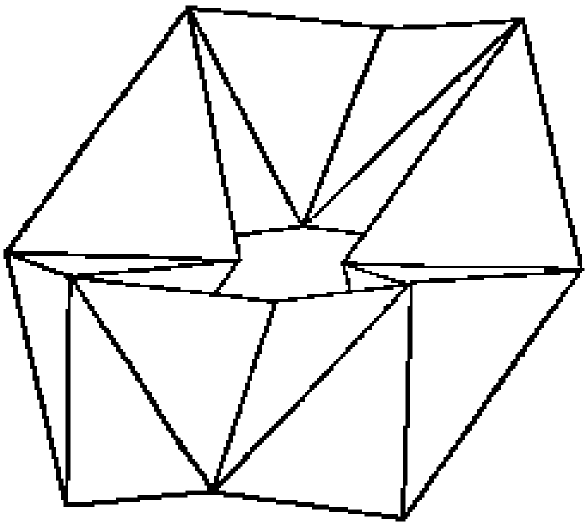


Figure 17.2

ring will twirl one piece at a time, with only two tetrahedra moving at once (see fig. 17.3). Four tetrahedra may form a non-rigid ring, but they will not twirl completely. A smokering of three tetrahedra is rigid.

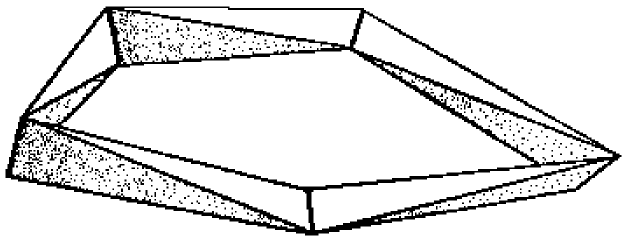


Figure 17.3

A cube may be dissected into three sections, one of which is a smokering of six irregular tetrahedra. The two remaining pieces are identical (see fig. 17.4). The smokering has two different types of positions, shown in figure 17.5, due to the dissimilarity of the component triangular faces.

Suppose we mutilate a smokering by removing two corresponding sides of each hollow tetrahedron, but so that the ring will still hold together.

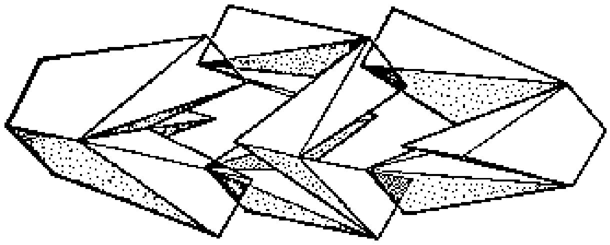


Figure 17.4

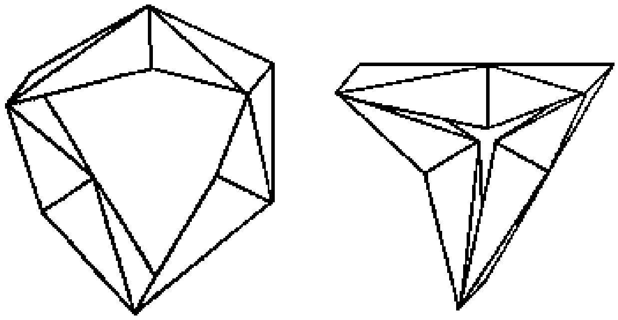


Figure 17.5

Then each tetrahedron introduces one quarter-twist into the resulting strip (of paper). There will be one half-twist for every four triangles in the strip. Thus a smokering might conceivably be described as two intertwined flexagons, each of which has the wrong number of twists to actually be a flexagon. One of the very few cases in which a real flexagon will produce a real smokering is that of a mixed flexagon of six units, four of these of order three and the others of order two — a fairly sad specimen of a flexagon.

## B. The Mushrooms and Bregdoids

Mushrooms are related to smokerings only in that they both can be said to twirl. That is, mushrooms are ring-shaped, and their peculiarity is that they rotate about a looped axis passing through the body of the mushroom. Beyond this, however, the similarity is slight.

Mushrooms are made from two or more strips of polygonal paper leaves. The strips are intertwined so that no flaps result and so that the result is a pile of leaves which holds rigidly together. Mushrooms are not as satisfying as flexagons, in a way, in that their properties rest on the fact that paper is not, after all, rigid, and that if long enough strips are interwound, the mushroom can be bent into a ring and the ends may be joined.

The mushroom that may be made from equilateral triangles is wound up quite tightly, as shown in fig. 17.6. Two strips are used, both with the sign sequence $++++\ldots$. The finished mushroom, which may require several hundred leaves, appears in fig. 17.7.

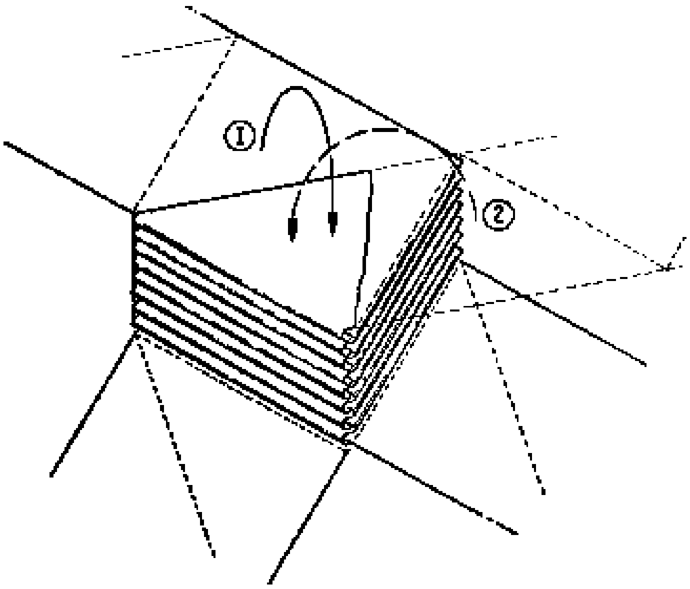



Figure 17.6

Two straight strips of squares, if folded together like soda straws, form a much looser mushroom, with four faces. Two or three $+++\ldots$ square strips may be wound (braided) together to give other four-faced mushrooms. Similarly, for the higher polygons many different strips produce corresponding mushrooms, with little significant structural variation.

Returning to the four-faced mushroom made up of two straight strips of squares, we find that it is not nearly as rigid as was the mushroom made of triangles. The problem presents itself of finding the shortest possible

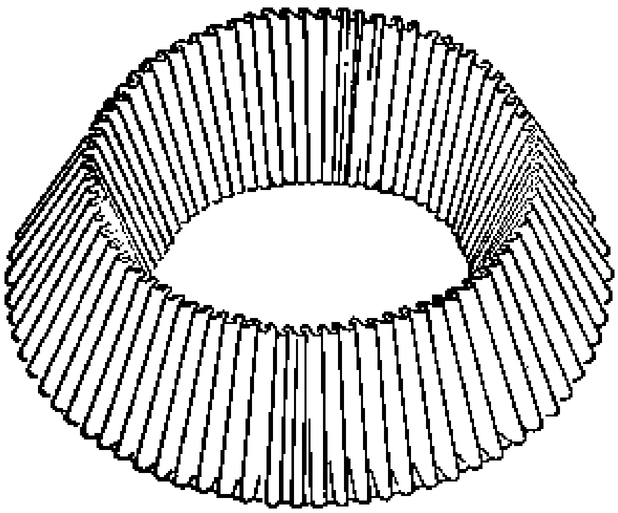


Figure 17.7 Mushroom of Three Faces

strips of squares for building this mushroom. Surprisingly enough, it turns out that each strip need be only eight squares long. However, such a small mushroom cannot be obtained without several concessions; each square must be creased along both diagonals, there is no longer any hole in the center, the mushroom lies flat, and twirling no longer occurs as a simple continuous operation. In regard to the latter requirement, we recall the equilateral triflexagon of order 3, which resulted from the imposition of similar minimality requirements on a 3-twist Moebius band. It was found necessary to divide the strip up into triangles, the central hole disappeared, the flexagon lay flat, and there was no longer the simplicity and continuity of the process in the Moebius band corresponding to flexing (i.e., rotating the band along its length, about the central hole, while keeping the position of the twists fixed). And just as the flexagon became a new object, the squashed-flat mushrooms are no longer called mushrooms, but <u>bregdoids</u>.

The bregdoid we have just described is shown in fig. 17.8. As in flexagons, we have two surfaces showing, each of which is to be called a <u>side</u>. However, a third set of leaves may also be found, between the two sides, in the <u>pockets</u> (see fig. 17.8). Since the mushrooms formed from squares have four faces, it should hardly seem unnatural for this bregdoid to have four sides. In fact, four positions or <u>faces</u> are possible, depending

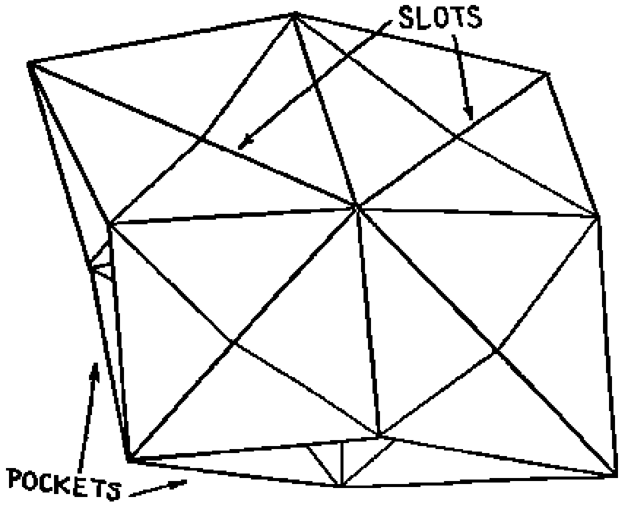



Figure 17.8

on how the ends of the strips are joined together. Each of these faces shows two sides, but the sides come in pairs, so that each of two faces will show the same sides. To clarify, if the sides are numbered 1, 2, 3, and 4, two faces will show sides 1 and 3, say, and the other two faces will then exhibit sides 2 and 4. The difference between the two faces showing each pair of sides is the side that may be found in the pockets.

The problem now is to find out how to twirl the bregdoid. From observing twirling in smaller and smaller mushrooms, it becomes clear approximately how it will work for bregdoids. Each face exhibits three sides; the fourth side is hidden, and corresponds to the one side that is usually hidden in a mushroom. Thus twirling commences with hiding a new side; as in flexagons, the lower side is chosen. This is done by widening the pockets downward. From this position we know that we can open out the bottom end of the bregdoid so that it will lie flat; it then stands to reason that if the slots in the upper side (see fig. 17.9) were converted into pockets, and the pockets into slots, that the top end of the bregdoid would open out flat, completing the twirl. This is done either by the crude method of simply pulling apart the slots, forcing them to open into pockets, or by folding the leaves together above the slots first, then opening out the new pockets, which does not force the leaves, and is therefore a little easier on

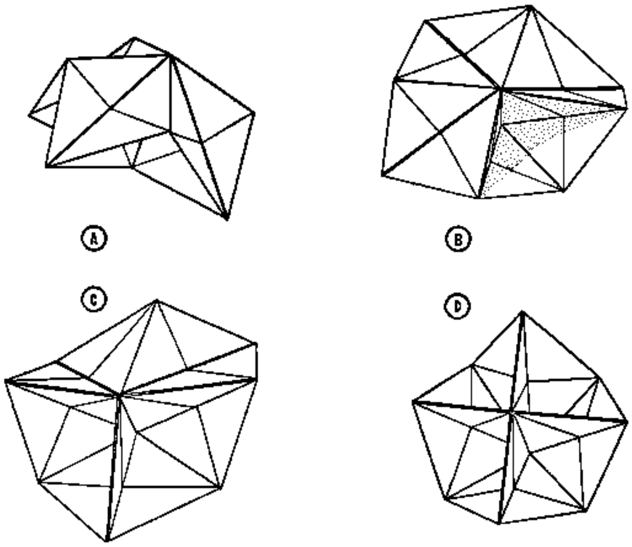



Figure 17.9

the bregdoid.

Once we can twirl, it is easy enough to paint all the sides and find the map. As with flexagons, the map for bregdoids made of squares is itself a square. The faces having common sides lie diagonally across from one another.

If we tally up the area of the inside of the pockets, we see that it is actually twice the area of one side. Moreover, if the sides are painted, half of each pocket remains white. In fact, just half of the area of the bregdoid ever turns up as a side, unlike the situation for flexagons. When the bregdoid is unwound, it turns out that, in each component square leaf, there are four right triangular sections; one is painted on both sides, the two joining to the adjoining leaves are painted on one side only, and the remaining triangle is blank. There are four different ways of rewinding the two strips so that triangles folded together are colored alike, and the results are quite different. One method, of course, gives back our colored bregdoid. Two others, however, mix the colored and white triangles in various ways, and the fourth makes all the sides blank. These four arrangements exhibit a kind of duality; the colored with the white, and the two mixed ones with each other; each arrangement can be converted to the dual arrangement

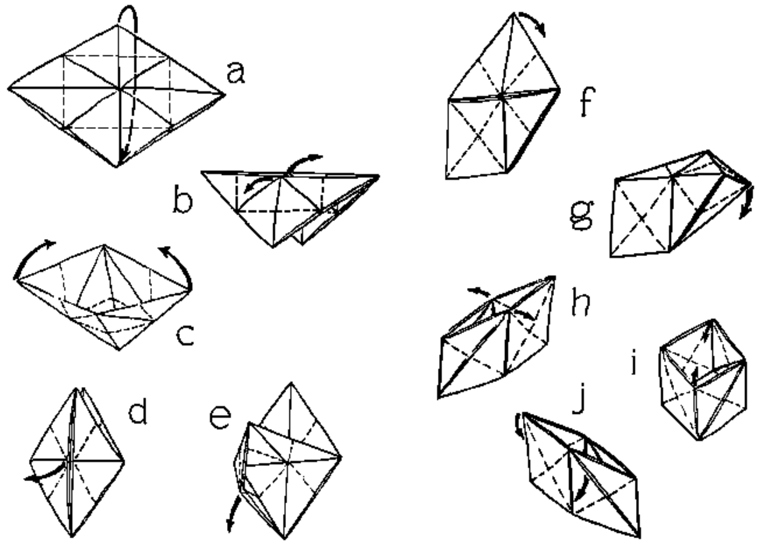



Figure 17.10

by a special process due to Peter Hartline (fig. 17-10). In each pair, the two strips are wound in the same direction, only all the hinges are reversed. Since all four positions have the same leaf structure, the maps are the same, and joined together by the dualization process.

Bregdoids can be derived from other mushrooms, and may be made with more than two strips of paper. A bregdoid has been made from hexagons which does not lie flat; it stands up, with a thickness of about 1/3 its diameter. No bregdoids have been designed having maps of more than one cycle, though. In this respect they are less complex than flexagons. However, there is a case for considering one-cycle flexagons as one-strip bregdoids, made from + + + + ... chains of polygons. From this point of view, flexing becomes a sort of specialized twirl.

Bregdoids, like flexagons, have a basic unit which, in the case of the square bregdoid studied above, was repeated four times. Needless to say, any number of repetitions will suffice, if it is greater than 4, and 3 may even be enough for polygons of more sides. Also, bregdoids have been made from irregular polygons; these may require many units, and do not in general lie flat at any time.

## C. The Fleptagon

The fleptagons have not been studied very thoroughly, since they are in general both complex and physically unstable. One of the simplest fleptagons is materially identical to a four-unit equilateral triflexagon of order 3. To produce the fleptagon, one unit is partially flexed, as shown in fig. 17.11. The resulting object looks like a flexagon with an extra flap, but it is structurally quite different from the flexagon, and it operates quite differently as well. As the figure suggests, the seven pats are structurally independent, in that there is no symmetry such as that which in the flexagon gives rise to the unit. A fleptagon is operated by folding it over double along any one of the central diagonals marked A, B, and C in figure 17.11. It may then open out, somewhat like a two-unit flexagon, to produce a new fleptagon structure.

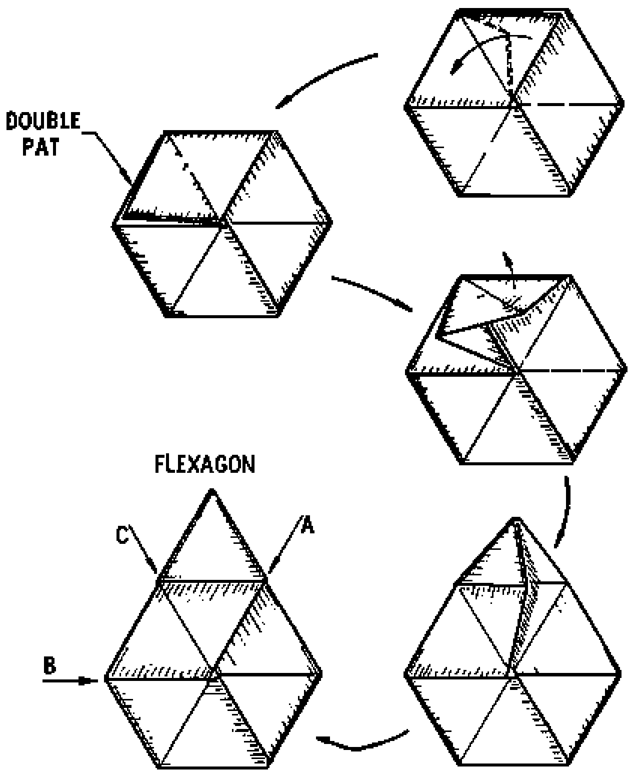



Figure 17.11

The biggest difference between flexagon and fleptagon is that the latter, in "flepping", may not completely change all of the leaf surfaces showing. That is, the "side" has lost any meaning or integrity, when thought of as independent of the faces. Thus a fleptagon map shows only the relationships among the faces, or possible states, of the fleptagon. The faces, since they may have common leaf surfaces, cannot be specified in the usual way. Also, there may be many more faces possible in a fleptagon than in a flexagon of corresponding size, since more permutations of the leaves may be allowed. Then again, there may be far fewer faces, depending on the structure of the fleptagon. The number of faces is not uniquely determined by the order of the fleptagon, which is defined as the number of leaves in the plan.

Fleptagon plans are as irregular as the rest of the subject; there need be no repeated unit, as with flexagons. It is not known either how to construct fleptagons, what fleptagons are possible, how many there are, or even along what diagonals they must be folded to flep. Part of the reason why so little has been done in this direction is that even small fleptagons, like flexagons with too many units, tend to disintegrate in various ways.

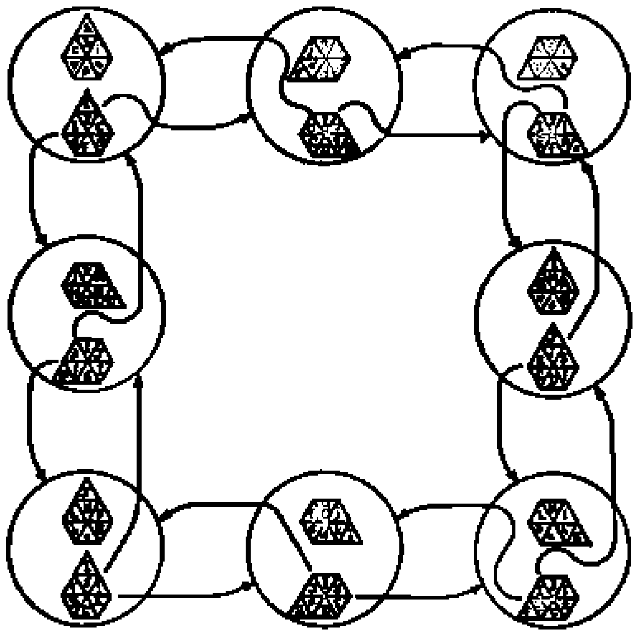


Figure 17.12

In the fleptagon maps given in fig. 17.12 and 17.13, each face has been diagrammed, front and back, to show the structure clearly. Also, the diagonals used in each flep are shown. Each of these fleptagons shows a characteristic of more complex flexagons; one illustrates a branching map, which is characteristic of change of cycle in flexagon maps; the other map shows that fleptagon maps can have a cyclic structure. No multi-cyclic fleptagons have been built, but these two maps suggest their possibility.

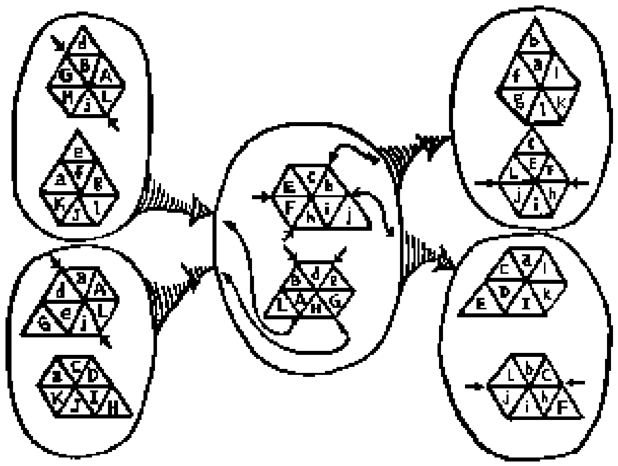


Figure 17.13

It would be nice if analogues of the various face degrees, cycles, and classes of flexagons could be discovered in the fleptagons, since the theory is at present unknown.

# Bibliography


[1] A. S. Conrad, "The Flexagon," RIAS Technical Report 60-24, 1960.

[2] Joan Crampin, "Note 2672," Mathematical Gazette, February, 1957, 55.

[3] Martin Gardner, "Flexagons," Scientific American, December, 1956, 162-166.

[4] Scientific American, "Letters," March, 1957, 8-9.

[5] Martin Gardner, "About tetraflexagons and tetraflexigation," Scientific American, May, 1958, 122-6.

[6] Margaret Joseph, "Hexahexaflexagrams," Mathematics Teacher, Vol. 44 (1951) 247.

[7] F. G. Maunsell, "The Flexagon and the Hexahexaflexagram." (Note 2449), Mathematical Gazette, Vol. 38 (1954) 213-214.

[8] Harold V. McIntosh, "Flexagons," Unpublished paper.

[9] C. 0. Oakley and R. J. Wisner, "Flexagons," American Mathematical Monthly, Vol. 64 (1957) l43-154.

[10] Alan Phillips, "Hexaflexagons," Mimeographed.

[11] W. R. Ransom, "A Six-Sided Hexagon," Vol. 52 (1952) 94.

[12] R. E. Rogers et. al., "Changeable Amusement Devices and the Like," U. S. Patent No. 2,883,195.

[13] N. F. Rutherford, "Toy," U. S. Patent No. 2,245,875.

[14] Mike Schlessinger, "Tuckerman Tree Theory," Unpublished paper.

[15] R. F. Wheeler, "The Flexagon Family," Mathematical Gazette, Vol.42 (Feb., 1958) 1-6.

**SMOKERINGS:**


[16] W. W. Rouse Ball, Mathematical Recreations and Essays, rev. by H. Coxeter Macmillan, 1960.

[17] A. H. Shemet, "Changeable Display Amusement Device," U. S. Patent No. 2,688,820.

[18] B. M. Stalker, "Advertising Medium or Toy," U. S. Patent No. 1,997,022.