A Quick Flexagon Survey

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Abstract

The construction of flexagons is outlined, both as an exercise in paper folding with paste and paint, and as a theoretical construct with maps, diagrams, and some illustrative figures. The script has been revised from a series of e-mail postings, so line drawings have replaced figures which were originally rendered in ASCII.

Contents

1 Materials and Motivations

Supposing that it as alright to play with flexagons just because they are fun to play with, here are some suggestions for getting started.

The historical approach is probably the best way to start, because it was simple and led to a succession of discoveries as more and more experience was acquired. That involved trimming a strip off notebook paper because of the difference between british (probably european A4, maybe something more distinctively british) and american (letter size) notebook paper.

It is easy to take any sheet of paper cnd cut off strips, but getting a roll of adding machine tape will give a strip as long as you want and of a convenient width. This works pretty well for making triangles, either equilateral (sixty degrees) or right isosceles (forty five, forty five, ninety). It is important to back-fold the paper, getting a fan-fold stack of triangles, rather than to fold it over, scroll style. That wad gets thicker and thicker, and it is hard to maintain the outline of the triangles. Also, starting with an approximation to sixty degrees is self-correcting, and after two or three folds, the triangles are pretty regular. With care you can also get good forty-five degree angles.

Later on, other polygons will make their appearance, and for those it is easier to prerare a sheet with a drawing program and print out copies as needed. But that comes later on; don't worry about it at first.

Eventually edges will have to be joined together, which could be done with scotch tape or masking tape. That was a good idea when we made flexagons out of IBM cards, but working with paper it seems better to include an extra polygon at the end of the strip, overlap it with the first polygon and paste them together. Here there is a glue stick made by Henkel called Pritt which we use almost exclusively. Stationery stores or school supply shops in the United States surely have something similar. Obviously dry glue has advantages over wet glue, unless the latter is quick-drying.

You can crease paper strongly and tear it to get lengths you want, but having a nice pair of sharp scissors is quite convenient. Later, when cutting sheets of preformed polygons, they will be essential. By all means get good scissors, you will enjoy their feel as you cut things up. But, school-supply-shop scissors will do if none other are handy.

Those are the essential supplies — paper, scissors, glue. For marking faces, it is more elegant to color them, but mathematically it is better to number or letter them. That can be done with any pen or pencil. Still, decorating them is something that you will eventually want to do, and almost any coloring medium will do. Among school supplies, a box of Crayolas works out well enough, but the Crayolas we get here are too waxy and don't smooth out well enough, and to get a good effect the colored area has to be rubbed with tissue paper or a cloth. There are some better japanese crayons, and we have also used pastel chalk.

Colored pencils work well, but coloring large areas requires quite a few strokes. It helps to choose pencils with thick leads. Still, pencils come in a wide variety of colors, which is often nice. Finally one can think of Magic Markers, which we use quite a bit; again multiple colors are available. Better to use water based ones than spirit based, mainly because of the fumes. But since the coloring is optional, one can wait until one has some flexagons and then experiment with different media.

As for non-essential supplies, it isn't a bad idea to keep some paper clips of various sizes or spring clips, to hold things together while folding them. But those are optional things that may come to mind while you are working along. There is also a question of what to do with the flexagon after you've made it. That is one of the uses to which we put our large collection of cookie boxes. But photo albums with slots of a reasonable size, or just large envelopes, possibly bound together in a loose leaf notebook, serve the purpose. The main thing, is that once you get inundated with flexagons, you need some way to organize all the mess.

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So much for materials. The first exercise is to put together a strip of ten equilateral triangles, fold it around into a hexagon, and secure the ends to get a ring of nine triangles. If the over-and-under sequence is done right all the way around, there will be a hexagon with three single triangles alternating with three double triangles which are hinged together and result from the folds that were made. Nine triangles in total. Six sides visible on top, six on the bottom, and six hidden from view. For a total of eighteen; which is nine triangles with two sides, a top and a bottom.

Two origami terms are convenient; mountain-fold and valley-fold. It will be seen that the hexagon can be folded about some axes. The simplest, but wrong, way to fold is to fold in two, getting three triangular pieces on top of the other three. The correct fold is in three, where alternate edges go up and down (folding in two, some edges don't get folded). Now, looking down upon the figure as it lies in a plane, a mountain fold is to raise an axis up toward the viewer, as though to form a mountain. A valley fold is the opposite, to move it down. Mountains and valleys need to alternate.

Folding two instead of three is the most common mistake which beginners make.

Once the hexagon has been folded up into a three-bladed rosette, it can be opened out again, either by undoing the original fold, or by separating the leaves from the bottom of the symmetry axis. That is what is called flexing, and should be practiced until it can be easily and quickly done. Also notice that there is a natural position for the fingers while this is done, which calls for minimal or no rotation of the flexagon. Finally note that from some positions, a valley fold can be made, and from others a mountain fold. Later on, with more complicated flexagons, both are possible, but for now, just the one.

If you already have done all this, excellent. Otherwise, practice it and wait for further instructions. To anticipate them, try doubling everything first before making the hexagon.

2 Strips of Paper

If one is going to paste and paint flexagons, it is worth adding an old newspaper or piece of carton or whatever to the list of supplies. That keeps from messing up the table top where one is working.

The best way to get started is to begin with something simple. The historical beginning was with a strip of paper folded into equilateral triangles. At first a simple loop made a hexagon which could be folded into a three bladed structure which would close up at one end and open out from the other. While doing so the faces of some triangles which had been hidden from view became visible, while others disappeared. Numbering the faces, or coloring them, revealed a cycle of three, which could be repeated over and over again indefinitely. Coloring with the primary colors fits in well with further experiments.

It is not too hard to make isosceles right triangles instead of equilateral triangles. The same technique of back-folding keeps good angles, and when the strip is finished, it can be twisted in one direction over and over again to make a square. It takes a little more care to join the ends than with the sixty degree triangles, but it can be done. Interestingly, this was a way to fold up a tabloid sized newspaper into a wad which could be thrown up on a porch without stopping when riding by on a bicycle. What the thickness of the wad obscured was the joining and subsequent flexing. Also, some new creases are required in order to fold the square into a hollow cylinder.

The outside of the cylinder can be painted, the figure unfolded back into a square, and then folded again on the other diagonal. This time the painted surface will be inside the cylinder, and the outside can be given another color. Coming back to a square, half will be painted and the other half not. This time fold the diagonal backward instead of forward (or the other way around, depending on the initial choice), and the cylinder will reappear with all the colored faces hidden. So a third color can be used, and then a fourth after returning to the square and changing diagonals. Finally everything has been painted and there is nothing more to do than to watch the cycle of folds.

Martin Gardner [2] described this under the heading of “flexatube” in one of his flexagon columns, emphasizing the way that the folding sequence could be used to turn a paper box (topless and bottomless) inside out without tearing or stretching it. But it is also part of the flexagon family.

There are other artifacts which can be gotten by braiding two or more strips of paper together (using rope, you get Celtic knots and the like). Most likely anyone who has sipped Coca Cola or a milkshake through paper straws has idly braided them back and forth to have something to do after the drink has been finished. Again, the figure is rarely ever closed into a circle, nor studied further, even then. Crossing the strips at sixty degrees gives one result, but crossing them at right angles makes another interesting structure.

It is not hard to take a very long braid and close it, but the shortest which seems to work uses a sequence of eight squares in each strand. Actually it is better to take a strip of nine, and paste the ninth onto the first to get a ring of eight. The first strip can be closed at once; but it is better, in fact essential, to intertwine the ring with the free strip, which goes well enough until the last two (plus tab) squares have to be put in place. Pressing the figure down into a plane, carefully making diagonal creases as needed in the squares (better to crease them all from the beginning, before doing any folding or joining), the end of the strip can be threaded into the folds in the ring, and the ends pasted together.

This is the structure which the flexagon book calls a bregdoid; although it is interesting enough to play with, it is not really part of the lore of flexagons. It also makes a hollow cylinder of squares, and so is related to the flexatube. More complicated arrangements result from adding multiples of four squares to the two starting braids.

Still working with paper strips, the original flexible hexagon (= flexagon) was prepared by folding the paper strip double, before joining the ends. So instead of a string of nine (plus a tenth for joining), a string of 19 (18 plus the joining tab) was first doubled up into 9, then closed into a hexagon. This is a process which is hard to resist, it is just a question of how the string of triangles is twisted. However, this is the point where one has the makings of a puzzle, because it is not so easy to locate and paint all the faces. In fact, Martin Gardner used it as a magic trick by pasting a photo on one of the faces and cutting it up according to the folds. It was then shown to someone with the photograph hidden, with the challenge to find it (“cherchez la femme”) and bring it into view.

Once the strip is folded double, there is no reason not to double it again (36+1) and again (72+1) and again. But at that point or even sooner, the folds of paper have become so thick that the flexagon is unmanageable.

So there are quite a few things which can be done just starting from a strip of paper. One of them is, having built a flexagon, to make a map of the faces, just as one would do when exploring Mammoth Cave; or in real life, finding your way home from the woods. It is also worth opening up a flexagon after it has been colored, to see just how the colors are arranged along the strip, and on which sides. So don't paste it too tightly at first, or use a paper clip instead of glue. Knowing that, you could color the strip beforehand, and avoid a certain amount of messiness. Or have the strip printed up, ready to cut out and fold.

3 Stacks of Polygons

Having folded some paper strips into flexagons, painted their faces, and made a map of their interconnections, one might wonder about the range of possibilities. One of them is to try out different polygons, although small regular polygons such as equilateral triangles or squares are the easiest to work with and a good place to begin. Just folding a long strip invites using equilateral triangles, or with more care and patience, right isosceles triangles.

Since the individual polygons are folded over, preferably as a fanfold, to get the flexagon, it is not hard to see that successive polygons are reflections of one another. One way to build a more general, and possibly irregular, strip is to start with a polygon, preferably made by drawing one on a piece of carton and cutting it out. Then, it can be traced on a sheet of paper to get more copies of exactly the same form.

It is a good idea to number the edges in order, in the same way on both sides of the carton cutout, although turning it over will reverse the clockwise or counterclockwise sequence. Because the eventual string of polygons will be folded along the edges, the polygon ought to be convex. For three or four vertices they can't be made any other way, but from pentagons onward they could have concavities. Actually it *is* possible to make concave quadrilaterals, so we should say there are unusable polygons from three vertices onwards.

The reason for numbering the edges is to foresee that the polygons will be folded in sequence, so that running along a string there will be a last (that is, previous) one and a next one adjacent to whichever one is held at the moment. They don't have to be joined along consecutive edges, so the numbering helps to describe the jumps if there are any.

One of the best ways to begin is to use regular pentagons about two inches in diameter. The size should be something conveniently held in the hand, and for which several can be drawn on a sheet of normal letter size paper, which is the most likely material to have at hand. Of course, sheets of brown wrapping paper or anything else at hand can be used. Newspapers aren't printed on very strong paper, so they wouldn't work as well. Pentagon flexagons won't lie flat when folded, but almost so and strike a balance between simplicity and complication which makes their working much easier to understand.

To build the flexagon, trace out the first polygon near one corner of the paper using the template. Copy the numbers onto the tracing, preferably putting "1" at the bottom left, and tilting so that "2" will make the strip run off at a diagonal.

Now, putting the template back with the same side up, turn it over along edge "2" and trace out a new copy. The numbers which are visible should be transcribed onto the copy, being sure to use the ones which are now visible but which used to be on the bottom of the template. Put the template back, turn it over along edge "3" to get the top side back again. Trace it from the new position, and once again copy the numbers. Because of pen or pencil thickness, the template has to be positioned so that the connecting edge is traced in the same position both times. A little sloppiness won't matter, but reasonable care should be taken so that everything will come together smoothly in the end.

Some practice may be needed to get the figure to fit neatly onto the sheet of paper where it will be drawn. The result should run off in pretty much a straight line, even though it will be a little crinkly. Just how many pentagons should be drawn? To begin with, five or six; six will permit closing a ring of five by pasting the sixth over the first. It will turn out that closing the ring is something you won't want to do, and using a paper clip for the closing instead of pasting will show you why. As the construction goes on, you will want two rings, so it is better to make them both from the beginning from strips of six pentagons, and to join them later.

Once the strip, or frieze, has been drawn, it should be cut out along the edges and the extra paper discarded. The connecting edges should then be folded in both directions, so as to make the paper more flexible, and to locate the places where the frieze is going to be folded. Once that is done, the frieze can be fanfolded to get a paper spiral. Notice that when five pentagons have been so folded, if the first edge was marked "1" then the last edge is also marked "1" and is parallel to that first edge. The sixth pentagon can then be folded back up on top and glued (or clipped).

What has been constructed is a spiral of pentagons, making one turn after each five pentagons because the angle between folds was 360/5 or 72 degrees. That is the exterior angle; the interior angle of the pentagon is 180-72, or 108 degrees. If the frieze had gone on and on, the spiral would have kept on turning. The important thing to notice is that edges become parallel again after five folds, so that different full turns could be separated by folding the place where they are joined back into a plane. Attempting to fold before that won't work because the axes are skewed. That becomes obvious when one turn of the spiral can't be opened out because the axes block each other, whereas two or more turns can be opened out.

Upon opening, there are two spirals lying side by side. If the eleventh pentagon was pasted over the first, a turn consisting of five pentagons spiralling in one direction lies next to another turn of five spiralling back up in the opposite direction. The eleventh can now be pasted to the first without any reservations. At this point, we see that the joint to make the pair is operative, and that some movement is possible since the bottom of the first spiral now connects somewhere else, no longer blocking the other hinges.

Here is where the third dimension comes in handy, because the top pentagon from one stack can now be transferred to the bottom of the other, and the other way around. That is the same as if the stack had been started with the second pentagon rather than the first. Repeating the operation, each of the pentagons will be exposed in turn. Therefore if they had been given different colors, it is possible to fold the stack and run through the succession of colors.

On the other hand, if they hadn't been colored, now is the time to do so, either of which will make the changes which occur while flexing more apparent.

If three or more turns had been used, the result is still a spiral stack for which the turns can be grouped together and flexed. But two turns is enough. Had triangles been used, three turns would be required, which is why the explanation of this construction wasn't based on triangles. Interestingly enough, the first flexagon to be discovered historically is more complicated than the ones which came after, although it is much easier to construct and fold up.

Once the idea behind this construction has been grasped, it is easy to work with any (convex) polygon, so it is worth trying it out with squares, regular hexagons, and so on. For the "so on" it is better to begin with regular polygons, bringing up the question of how to make a regular heptagon, for instance. Also, beyond hexagons, the reflected polygons overlap which means slicing the overlap, or cutting them out in pairs individually, then pasting. But that all comes later on; one should make an effort to understand the pentagons first.

4 Folding and Flexing

Flexagons are widely known because of Martin Gardner's article in Scientific American from around 1960 [1], and although few people or even libraries have copies of the magazine from so long ago, the articles have been reprinted in several of his collections and can still be bought. I think I have seen a recent announcement a new collection, titled something like "the best of the best." On the other hand, what appeared in those articles was merely an introduction, mainly because of the general level of the magazine but also because there have been later developments.

Acutally Gardner got his ideas from general folklore which had been circulating, mostly in the form of annual demonstrations given during the ceremonies of the Westinghouse National Science Talent Search held in Washington, D. C. each year. Something of a forerunner of the Science Fairs which have been popular in recent years.

One of the interesting tasks of a historian is to trace the origins of concepts, of which mathematical puzzles and recreations is one. Chess, checkers, and other board games have a more extensive and documented history, as do card games. From all I know or have been able to find out by reading the scarce literature which exists is that they were indeed recognized for the first time by some students at Princeton University in the late 1930's; mention of which is found in Gardner's article. It seems strange to me that nobody ever noticed them before, although some rudimentary variants such as Jacob's Ladder and the bar room hinge were known.

Even taking the Flexagon Committee's work as a starting point, it doesn't seem that they ever wrote down much, and it is an interesting challenge to figure out exactly how much or how little they knew about flexagons. There are two main ways to regard flexagons - either as friezes or as spiral polygon stacks. The latter is more insightful, the former more historical. That is because flexagons originated from folding leftover strips of paper when someone was trimming notebook paper and got to playing with the debris. It is also true that rolls of adding machine tape or similar objects are a good source of material to play with.

However, using straight strips becomes a limitation when joining polygons into a strip seems to be a better vastage point than making polygons out of a predetermined strip. Also, there comes a moment when using them to make spirals in three dimensions rather than sticking to one or two seems to give a better perspective. The main thing about the spirals is that when they consist of two or more turns, edges where they can be folded become parallel rather than blocking each other, and the spiral can be laid out into two pieces alongside each other to make a plane figure. With triangles, three turns are required; the reason for this can be seen in the requirement for the sum of angles around a common center must sum to 360 degrees if the polygons are all to fit together in a nice rosette.

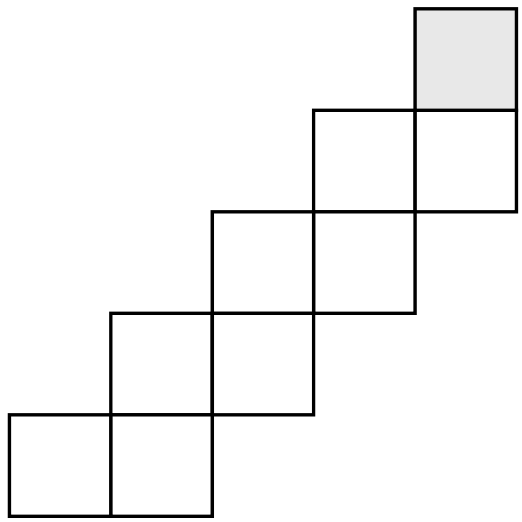
So two of the basic rules of flexagons are:

- any polygon can be flexagonned
- the angles must add to 360 degrees.

Of course, such generalities have to be qualified. The polygon should be convex, otherwise using an edge as a hinge doesn't work too well. And it is an immediate question — what angles? Obviously the ones at the point where common vertices meet when the flexagon is laid out on a tabletop, but which ones are they on the frieze?

Even without knowing the answer to that, it seems to be a good idea to work with polygons whose angles are divisors of 360 degrees, which seems to suggest triangles, squares, and hexagons. But if the polygon is not regular, still smaller angles can be used. For triangles, there is not only the equilateral 60 - 60 - 60 degree triangle (note these are the internal angles), but also the isosceles right 45 - 90 - 45 triangle, or the scalene 30 - 120 - 30. Even tinier angles like 15 degrees can be used, and there is no requirement that the triangle be isosceles. For example, the 30 - 60 - 90 right triangle could be used.

Going on to squares, they can be used to build what Martin Gardner called tetraflexagons. Actually, I don't find his nomenclature too satisfying. You get hexaflexagons, hexahexaflexagons and so following in a series which is neither the only one nor necesarily the most interesting one. But anyway, the basic square-based flexagon is not folded up from a straight piece of paper, which is why I have avoided mentioning it until now. Either it runs off at a diagonal, or it has nicks in it depending on the angle at which you want to view it.



To make the basic square flexagon, make a strip of five squares, so that the fifth could be laid over the first and pasted. That is one turn of the spiral, which is never enough. So take two such friezes, join the second to the first to get nine squares, then fanfold the frieze, open it out in the middle, and join the ninth to the first. Faces can now be painted, the flexagon flexed, more faces painted, and one has the basic tetraflexagon.

Besides squares, rhombi could be used, with angles of 60 - 120 degrees rather than just 90 degrees. Two turns will not be enough because of the 60 degree angle but three will work just as it does for the original triangle flexagon.

Eventually it turns out that there are far more polygons that can be turned into flexagons than anyone would care to work with. Still, when

5 Primary Flexagons

One of the best ways to work with square flexagons is to make up a good collection of dominoes - that is, 2 x 1 rectangles folded in the middle. The idea works with any polygon. Even though it requires pasting every polygon on top of another one, folding the structure is manageable enough with the advantage that the frieze can wander around every which way and even overlap itself. Note that if the polygon is not regular, different dominoes may have to be prepared, according to the edge that will be folded or have been reflected while constructing the frieze. Multiple dominoes can be made simultaneously by tracing outlines on a sheet of paper and then cutting through several sheets placed on top of one another. Up to four seems to still cut nicely.

By now, fairly short strips made with several different polygons should have been made and tested, with emphasis on strips just long enough to make two turns for a polygon stack, although triangles may require three turns to get good results. More turns are possible, leading to rosettes whose blades will have a single turn, and they are all interesting. In the beginning, strips should still be kept short. It should also have been noticed that when an extra polygon has been included in the stack, it can be pasted onto the top of the stack rather than being left on the bottom. This is supposing that we look at the stack from top down, which is convenient.

If the stack is not too high, the bottom can be pasted to the top while keeping a single stack, but the stack can be opened in the middle leaving two half-stacks lying side by side and the extra polygon in a favorable position to mate it with the first and glue them. Being able to do this is the reason that two turns of the spiral were called for rather than just one. There is no difficulty in seeing how to connect multiple turns if a longer frieze was initially prepared.

It should be emphasized that the whole stack should have been prepared by fanfolding. That is hardly the only way the stack could have been folded, but it is the one that we want to start with. Fanfolding the minimal number of turns gives a primary (or first level) flexagon, of which several kiinds should have been constructed and experimented with.

The experimentation consists in noticing the behavior of the pair of stacks in three dimensions. For some flexagons, additional hinges will have become unblocked and four stacks can be laid out side by side in a plane, surrounding a central point. This is the point where the rule about angles summing to 360 degrees comes into play. If they do not, the sum may still be approximate and one can work with a puckered flexagon. Such is the case when using regular pentagons, for example.

Another possibility is to begin with more turns in the original polygon stack; a necessity when using regular triangles, for example.

Without reference to whether the figure is planar or not, there is always enough flexibility to take the top polygon off one side and make it the top polygon on the other side, which has been turned over. Except for the turning over, it would have been the bottom polygon. The overall result of this operation is that the place where the single spiral has opened out to become double has advanced. Repeating the maneuver will advance any polygon that one wants to the top of the stack and make it visible. If all the polygons had been painted with different colors, the effect would be one of running through all the colors in sequence.

There is a nice diagram which summarizes this result, as well as taking the color into account. It is easier to number the faces of the polygons than to remember colors. Consider using regular pentagons with five sides. The diagram reads

$$\begin{array}{cccccc|c} 1 & . & 3 & . & 5 & . & \mathrm{x} \\ \hline . & 2 & . & 4 & . & 1 & . \end{array}$$

The reason that it has this strange form (two rows separated by a horizontal line) is that before folding the faces of the polygons in the frieze would be numbered 1, 2, 3, 4, 5, x on the top side of the frieze. The x is for connecting the first turn to the second turn. But after fanfolding, every second number goes to the bottom (as seen by the person doing the folding) of the visible polygon rather than the top. So, the reason for two rows is to show both what would be on the top and what would be on the bottom of each polygon, and that is why the numbers alternate from top to bottom.

What numbers should be written on the opposite sides of each polygon? That depends on the fact that two turns are needed to get a stack that can be unfolded, and after the two turns have been laid out side by side and there ends joined, we want to see the same color on the top of each spiral. Or on the bottom, if we look at it from below. But these pairs are consecutive polygons in the starting frieze and so should have consecutive numbers. Thus x should be what follows 1, or 2. In similar manner, 1 should be added to each number to get the number written on the other side of the dividing line. In other words,

$$\begin{array}{ccccc|c} 1 & 3 & 3 & 5 & 5 & 7 \\ \hline 2 & 2 & 4 & 4 & 6 & 6 \end{array}$$

This means that x would be 7, but since one turn of a spiral has five polygons, numbering should start over again with the sixth to give the sequence 1, 2, 3, 4, 5, 1, 2, ... That means x = 2, not 7. With this correction, the table reads

$$\begin{array}{ccccc|cc} 1 & 3 & 3 & 5 & 5 & 2 & \cdots \\ \hline 2 & 2 & 4 & 4 & 1 & 1 & \cdots \end{array}$$

The table could be continued for as many turns as needed, but it is customary to show it for a single turn with that vertical line marking where the cycle ends; plus the next polygon's label for convenience.

All of this probably seems terribly simple and obvious which, for first level flexagons, it is. Almost. Using two rows is important, and adding 1 is important. Even though the flexagon is rudimentary, these details should be checked out and verified until one feels comfortable with them.

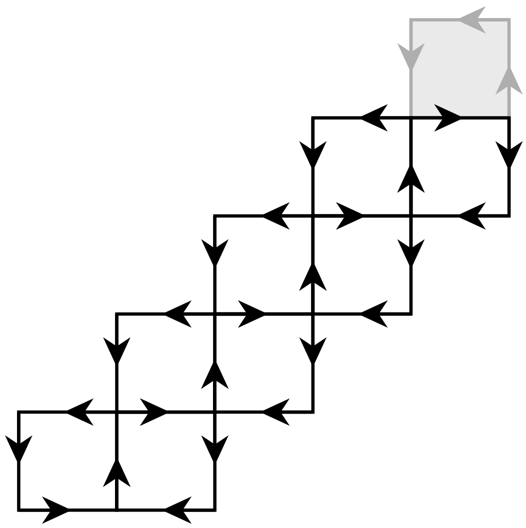
6 Representations

The diagrams for square flexagons are probably the only ones which can be described on a teletype using ASCII, but that should be alright. The only real drawback is that a fanfolded square stack may not look as much like a spiral as a pentagon stack, but it is still better than trying to think of a triangle stack in those terms. Once the idea is clear, it really doesn't matter which polygons are used. Typesetting in TEX, however, allows the insertion of line drawings, but there are still advantages to remaining with squares, so we will not change over.

The fundamental relation in the description of a flexagon is the one between the sequence of polygons laid out along a line in two dimensions (the tabletop for example) and their sequence in the polygon stack. For fanfolded polygons the sequences are pretty much the same, except for taking taking into account that the polygon has two sides (top and bottom) in three dimensions and that they get turned over with each fold. That, and the desire to record all the colors, both top and bottom, is why it is a good idea to make two lists, one for each side.

For nine squares (the ninth for pasting over the first to make a cycle of eight) the layout would be the following:

To the extent that this can be done with keyboard symbols, the squares have been marked with arrows to orient them, from which it will be seen that the cycles run in opposite directions in adjacent squares. This is consistent even where four squares meet in a common corner, and defines



what is meant by the "next" square when following along several squares in a sequence where they are connected by having common edges.

The same scheme can be followed for any prototype convex polygon. It is not necessary to embed the strip in a regular lattice. With equilateral triangles, six will meet in a corner, with squares, four, but with regular pentagons there is no regular lattice. At first, working with triangles, it seemed natural and so also with squares. That meant layout sheets could be prepared and copied, which were a great convenience. But eventually the realization came that they were not essential.

Whatever; what is required is knowing the orientation of the rim of the polygon and accordingly what are "next" and "last" edges. And eventually what it means to skip over so many edges in a given direction.

The list of nine faces will be

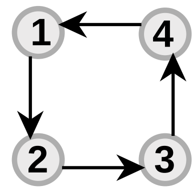
1	.	3	.	5	.	7	.	9
.	2	.	4	.	6	.	8	.

If account is taken that colors will repeat after one turn of the spiral and that the whole collection of faces exposed to view at one time should have the same color, the list becomes

1	3	3	1	1	3	3	1	1
2	2	4	4	2	2	4	4	2

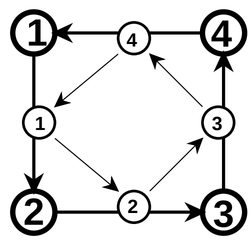
Now it is time for two more diagrams, which take into account the fact that flexing the stack, once it has been doubled (or trebled or whatever) exposes the faces of the polygon in cyclic order, one after the other.

Some arrows have been typed in the edges to indicate a preferred order; of course it is possible to run against it once one knows what the usual order is. The diagram reflects the fact that the colors must be exposed in a definite order, and that skipping is not possible. As it turns out, turning the flexagon over reverses the direction in which the colors are exposed, but that is the only alternative. Of course, one should avoid dogmatic statements; if the flexagon were made of



hexagons it might well be possible to skip ahead two colors at once, treating the hexagon as though it were two triangles, Star of David style.

The second of these two diagrams looks pretty much like the first, except that it is drawn inside the first, and the two are drawn together to make their relationship clear.



The relationship of the inner diamond is that it is the dual of the outer square. What is meant by this is that if each edge of the outer square is marked by a dot at its midpoint, then some pairs of these dots could be connected to show that their lines meet in a corner. The duality is that pairs of lines meet in points, pairs of points make lines. This still does not explain why the same numbers were chosen as labels, just shifted forward around the edges.

Well, if the square shows the colors visible at any moment on the top and on the bottom of the flexagon, that central dot represents *two* colors and so gives a better idea of what's in the flexagon than just one color. Flexing moves colors from top to bottom (or the reverse) which says that if we have colors (a, b) now, you will get (b, c) next because b has moved from the bottom of the stack up to the top. To see what happens after that, it is necesary to consult the rest of the diagram, either one of which will spell out the sequence of colors that turn up.

For the moment there is nothing particularly surprising or complicated here, raising the question of, why bother with it all? The answer comes when one decides to replace one polygon with an equivalent stack, and wonders how to describe the consequences. In the meantime one should have become thoroughly familiar with the workings of primary flexagons, of all shapes and sizes.

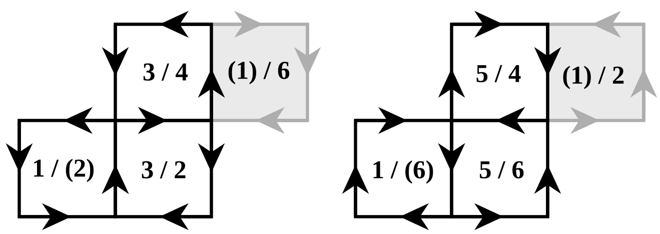
7 Binary Flexagon

There is any amount of play and experimentation possible with fanfolded copies of identical polygons connected in various ways along their edges, although choosing an orientation and taking successive edges is the best way to begin. The properties of these stacks depend upon the fact that although the fanfold may be opened at any point, only certain depths of stack allow joining the top polygon to the bottom one and still being able to open the stack. Concretely, an integral number of turns of the spiral is required at which points the stack may be opened and for which the conflicting angles between the hinges keep the rest from opening.

Making structures in two and three dimensions results in additional constraints. Turns can always be laid out side to side, with so many on one side and so many more on the other side. With many turns, they may be grouped into blades and found to be arranged around a common axis in three dimensions. With equilateral triangles, and generally where 120 degree angles are involved, three blades may be laid flat in the plane to produce a large polygon made up of the little polygons out of which the full stack was constructed. With triangles this makes a hexagon which is the "flexible hexagon" from which flexagons take their name. With sqyares, two blades is enough and you just get a bigger square. A pair of pentagon blades almost lies flat but not quite, and so on.

What matters in the spiral polygon stack is the angle between successive hinges, which lie along edges of the polygons. It is possible to get the angle between successive edges in two different ways (at least, but we are olny going to look at these two). One is to just take the polygon with n sides as it is. The other is to make a stack of n-1 polygons and turn it over. That way you get the angle between hinges by going the long way around, but it's the same angle.

To illustrate this with squares,



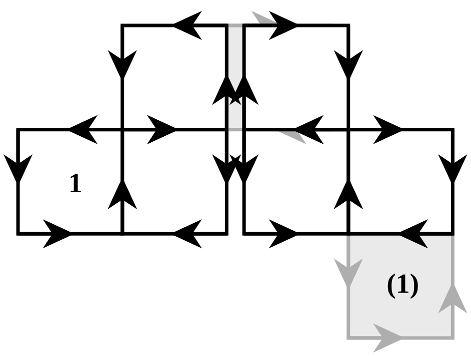
There are eight squares shown here, in two groups of four, which are going to make just one of two blades of the flexagon. So it will be necessary to do everything twice, even though the description will be run through just once.

The marking a/b means that the number (or color) a is to be written (or painted) on top of the square, b on the bottom. when it is enclosed in parentheses, as in (1), that face of the square will be covered up later so there is no point to paint the square, but a faint marking should still be left in place so as to know what to join later on. So although there are eight squares, there will only be six different faces, the other two being lost on account of pasting over them.

Also note that the arrows in these two groups run in opposite directions, because one of them is going to be turned over. The right group should be fan folded as always, but if only three are folded and the fourth left sticking out as a tab it will make joining the results easier. So the wad of three folded squares can be secured with a paper clip (they are not going to be pasted) it is easy to see that the wad meets the requirement of turning forward 90 degrees instead of backwards 270 degrees, once it has been turned over.

After all this, one will have a package of four "squares" (one is that triple which has been clipped together) which is the same as the one used in constructing the primary square flexagon. Two packages with all edges joined make a flexagon whose only difference from the ordinary one is that wad of three squares where one would have been.

Although this construction has the purpose of making the substitution quite clear, there is no reason to make the wad right away, and a group of seven squares could have been made up from the beginning.



Which in turn is half the flexagon.

From the point of view of listing the colors in order along the strip, it would now be written

1	.	3	.	5	.	1
.	2	.	6	.	4	.

which can be verified in two ways. First, by looking at the strip which was constructed above after making the indicated replacement in a primary strip. Second, by deleting the 4 in a primary strip, and inserting 4, 5, 6 in reverse order in its place and continuing the list. Adding 1 works as it did before, but it is convenient to place a notation which reminds whwether the stack has been turned over or not.

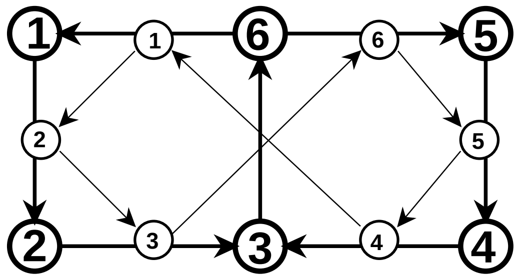
+	+	+	-	-	-	+
1	3	3	1	5	5	1
2	2	4	6	6	4	2

That accounts for the row of plusses and minuses which has been placed on top of the other rows. A + means "go to the next edge" whereas a - means "use the previous edge" while laying out the frieze of squares. Likewise the sequence of signs defines the sense of rotation of the spiral while reading down the polygon stack.

When it comes to mapping out the cycle of faces in the order in which they will be exposed when flexing the flexagon, it is only necessary to think that one polygon has been deleted, only to be replaced by a cycle running in the opposite direction with one point less. But there is no need to take the deleted point out of the diagram; simply skip over it, to get

The outer squares still tell the sequence of colors which can be turned up while flexing, while the diamonds are related to the order of appearance of the faces while reading down the stack. Note that the point where the two diamonds touch conforms to the construction, which is to replace the 4 which would have been there by the sequence 6-5-4 running around in the opposite direction, each implying a rotation of -90 degrees.

All of this discussion leads to what could be called binary flexagons, namely those in which one single polygon in a primary flexagon has been replaced by a counterspiral of equal polygons so as to preserve the overall rotation of the spiral. If the counterspiral is closed off, say by lightly pasting



it (or better by using clips), there hasn't been any change in the flexagon except for making one of those polygons fatter and giving it an internal structure.

But the interesting obnservation is that the relationship is mutual. If the flexagon is flexed to a point where the wad is on top and visible, there will be a similar wad in the other blade (or repetitions in each blade if there are several), and then the rest of the polygons will be laid out alongside of it. Well, if those are grabbed and stuck together by moving the clip, and the original wad released — the result will be exactly a mirror image. Hence the reason for calling it a binary flexagon. The same symmetry is apparent in the map, where two equal subdiagrams are stuck together as mirror images of one another.

This is an extremely important construction, and should be practiced with all the primary flexagons which have so far been constructed. As usual, the original model of a flexagon, Arthur Stone's triangle strip, fits rather awkwardly into this scheme of affairs. That is because the next logical step, to replace ALL polygons by their counterstack, is a very natural step for triangular flexagons, but not quite so obvious for other kinds of polygons. So it is easy to skip over the binary triangular flexagon, in spite of its importance as a first logical stem in making more complicated flexagons. So much for trying to build a general theory from special cases!

8 Second Level Flexagon

Although flexagons were discovered by folding long strips of paper into closed loops, it turns out that the best way to think of them is in terms of stacks of identical polygons folded back and forth over each other and then closing the ends together so that they will fold out in three dimensional space. In other words, polygon stacks versus folded friezes; it is interesting how a change in viewpoint can make all the difference in interpretation. Or at least in visualizing the proces.

At first the stack is fanfolded from polygons connected edge-to-edge in succession. Before trying out skipping edges or occasional scroll folding, it is interesting to replace a polygon by a counterstack, which consists of a spiral turned over so that the angle between edges is the same as it was before. By experimenting it can be verified that clumping the counterspiral together as a single entity changes nothing except that a fatter polygon with internal structure has been used. It could be glued together on all faces and that would be quite literally true. Conversely it could be thought that the first polygon had been sliced up salami fashion, except the pieces were still partially joined.

In fact, that is how we thought of complicating Stone's original triangle flexagon, except that it didn't occur to us that with squares you needed three slices rather than two and that the connecting edges had to rotate around the square. Later the idea caught more attention, but apparently never got elaborated and formalized to the extent that it should have. Besides, it is helpful to transfer the idea to the maps and face lists, but we only knew about those as recipes that worked and weren't too sure how they were derived. Since the Flexagon Committee never wrote up a report, doubts remain as to how they finally evolved the process that was discussed here and there.

Even now, the concept of replacing a point on the cycle map by a tangent cycle running in the opposite direction seems to be something which is more justified by the fact that it seems to work than by any rigorous geometrical deduction.

This is a distinction which is often missed in geometry classes and by people who are either too rigorous or insufficiently rigorous. To say that geometry is a collection of "obvious" facts misses what seems to have been Euclid's objective, which was to show the interrelationship of all those facts, and to show that they could be built up in an orderly progression starting from relatively few basic assumptions. So just as the geometry teacher who won't let the students draw pictures is depriving them of a helpful visualization and memory aid, those students who feel that by drawing a picture, they have proved something, have erred in the other direction. One needs to know, Why?

With flexagons the situation isn't so drastic, because a whole branch of science doesn't depend on it. Still, what one is looking for is a model which, while not as complicated as the object it is describing, captures its essential properties, describes it, and allows predictions. So it would be possible to do flexagons by analytic geometry, assigning all the points coordinates, and writing formulas for reflections, rotations, and the other operations. People do something pretty much like this when the model structures in civil engineering, and the people at Industrial Light and Magic do it when they sumulate dinosaurs running across the field and jumping over logs.

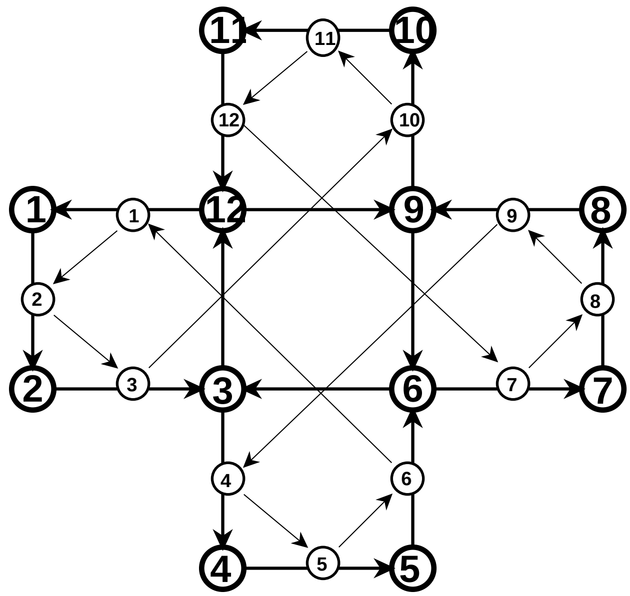
Even without the algebra and computation, we have had several people try their talents as artists in drawing flexagons and showing clearly the operations. It is more complicated than one would think; all the more so before the best line of reasoning was apparent. Now that we have nice video cameras, we have been thinking of showing the process by actual example without the drawing. We may actually get around to doing it.

In the meantime, "what has been done once, can be done again," as the explanation of mathematical induction goes. Having understood the principle of substitution, there is no reason not to go on and on, substituting here and there. Some sequences of substitution produce interesting designs, and much of our early work was dedicated to making them, in hope of grasping a general principle. But it is still worth the effort to advance in an orderly fashion, an example of which would be to replace every polygon in a primary spiral by a counterspiral, to get a second level flexagon. Replacing each of those in turn would lead to the third level, and so on.

For squares, and again working with a single blade of the flexagon while understanding that in the end several (say two) will be required, each of four squares will be replaced by three, which means a total of twelve. The two interleaved maps, of visible colors and of the polygon positions, will look like this:

In the ASCII script, only the diamonds were labelled, and since there aren't any diagonal arrows in ASCII, they would have to be drawn in over the printed page. Note that all turns are in the same direction, which can be counterclockwise, so that the three line list of labels for the squares will read

+	+	+	+	+	+	+	+	+	+	+	+	+
1	.	3	.	11	.	7	.	9	.	5	.	1
.	2	.	10	.	12	.	8	.	4	.	6	.



before adding 1 to the numbers to get the number opposite. That can also be done on a printed copy. The required frieze will be made from squares joined along consecutive edges, so it will look like the strip we have used before, only longer.

This is half the flexagon; another copy is needed, to be joined with this one.

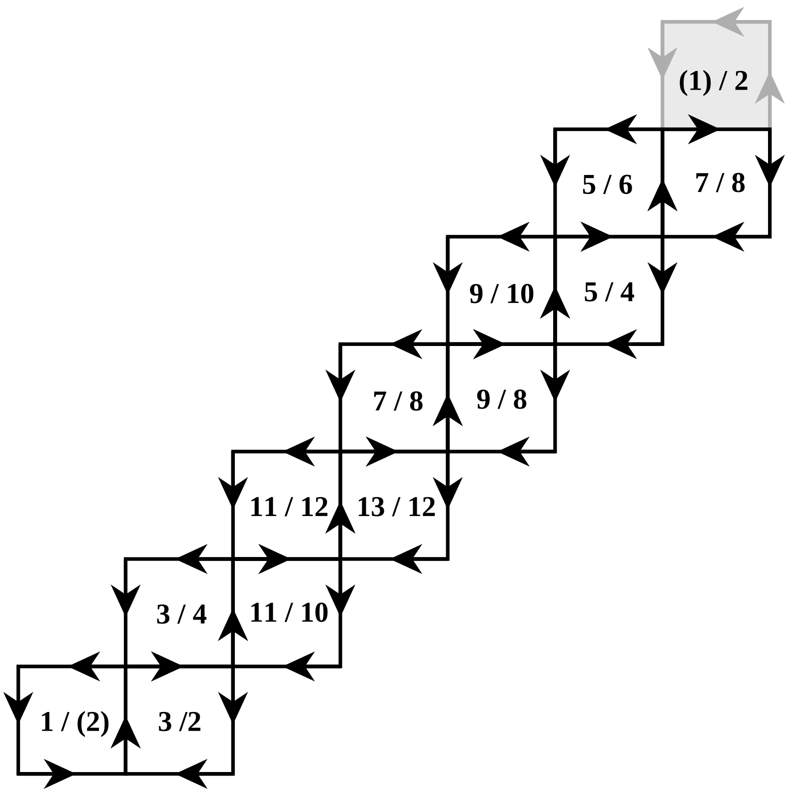
As a general proposition, all second level flexagons will look like the first level flexagons, except for the fact that some segments could be folded up first to get a strip which looks exactly like the first level strip, according to which the remainder of the folding would be the same as before.

For triangles, there is a natural temptation to give additional twists to the triangle strip before closing it into a hexagon, which is how Arthur Stone discovered them right away. For squares, the strip isn't quite as natural, and it is necessary to pause after folding three squares to fold the next three, and then fanfold those before going on to the next group. Pentagons work the same way, save that the secondary groups have four pentagons; and so on for any other polygon with more sides.

All the variety will doubtless seem very confusing, which is why it is mportant to organize the work in stages and procede step by step.

9 Tuckerman Tree

Several different graphs, and kinds of graphs, turn up while describing flexagons and trying to organize their properties. To begin with, there is the layout of the flexagon, for which there are two points of view. One is that is a polygon stack, members of which are joined by edges, yet forming a single sequence. That means each polygon is joined to a successor, and that a predecessor was joined to it, along a separate edge. No other polygons are connected to it than these two. It took a while to see that this configuration was essential to the definition of a flexagon.



The other view that one gets is that of a long chain of polygons meandering around in a plane. There may be subconscious objections to letting it cross over itself but these are easily overcome. It just complicates constructing the flexagon because then it has to be laid out in layers. Such chains are called friezes, which is an architectural word referring to the stripe usually placed in a room to separate the wallpaper covering the walls from that covering the ceiling; but also to the space between the walls and roof in a temple or building. In that context the overlapping never comes up, of course.

Laying out a frieze, there is no reason to work from a single polygon prototype nor to obtain its neighbors by reflecting in an edge, although it is clear enough that the resulting stack is uniformly built from the same polygon, turned over if need be, and connected by two different edges to its neighbors.

There is no reason to think that a frieze has to be overlaid on a lattice, and in fact squares and equilateral triangles are about the only polygons from which a grid can be formed. Nevertheless, those two polygons are so frequently used to make flexagons that it is often convenient to prepare paper on which the friezes can be drawn. This is true whether the frieze is to be cut out from the drawing, or the drawing is to be filed away to preserve the shape of the frieze for future reference.

Such graphs are concerned with physical construction of the flexagon. Others summarize their properties abstractly, but are nevertheless worthwhile for just that reason; they are maps of the flexagon's behavior. Again we have seen two different graphs, and how one is nestled inside the other. Someone given a flexagon and told how to manipulate it would probably want to diagram the possibilities. The most visible property of the flexagon is the set of faces which are visible when it is held in the hand or laid down on a tabletop. To distinguish one set from another, they can be marked. Painting all the exposed surfaces is an artistic way to do this, whereas writing letters or numbers will result in a better mathematical description, especially the use of numbers.

The basic operation is that of flexing, which means moving some constitutent polygons from one part of the figure to another. As an entirely new set of surfaces will become visible, they ought to have a new color or number as the circumstances dictate. The main choice in flexing lies in making valley folds or in making mountain folds. That gives two choices for exposing the colored sets, which we may well just call faces. That is reminiscent of the two neighbors seen while looking at the frieze, but these are abstract neighbors which can be marked as points on a map and connected by lines according to whether they can be exibited relative to one another by flexing.

Instgead of talking about mountain and valley folds, it is possible to concentrate on one kind of fold and change to the other by rotating the flexagon. That means that instead of linking points by "valley" and "mountain" they would be linked by "flex" and "rotate," taking into account that "rotate" doesn't change any colors. This can all get fairly confusing in the lack of practice and expecially in the absence of exact definitions.

Nevertheless by persisting it is gradually observed that some operations on the flexagon run through cycles, and that other operations result in changing the cycles, although you can't change cycles forever, and that they can only be changed in certain sequences. That is what makes flexagons good entertainment devices - having to figure out all the relationships. Not in the construction, which is another issue, but in their usage and their external appearance.

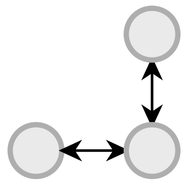
Close observation relates the internal structure to the external structure. The cycles correspond to three dimensional movements of the polygon stack, especially of removing a polygon from the top of one stack and placing it on the bottom of the neighboring stack, recalling that the stack is arranged as turns of a spiral and that two or more of these turns get laid out flat to form the visible flexagon.

The choice of fold types becomes possible when individual polygons are replaced by whole structures - single counter turns or the result of a whole series of such substitutions - and the unfolding and transfer is made in one of these structures rather than in the original spiral. That is the recursive step which allows one to call a flexagon a "fractal polygon stack" and compare it with other objects built up by similar procedures.

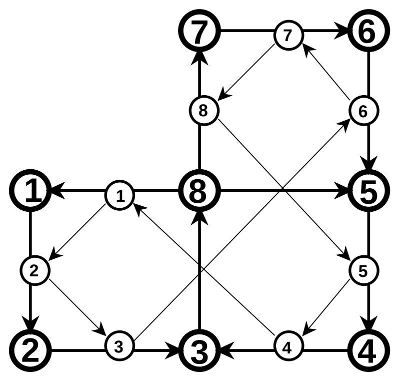
The sequence of substitutions can be carried out in varying degrees in any flexagon to get another, new, flexagon, and considerable effort has gone into cataloging all the possibilities. Or at least as many of them as practicality and patience permits. When we were first studying flexagons there was an effort to build as many different ones as possible so as to be able to compare them and try to capture the essence underlying them. Now that the interplay of fundamental cycles and substitution is better understood, interest lies more in finding nice illustrative examples or in especially educational combinations. In any event, placing large numbers of polygons on top of one another places mechanical limitations on an otherwise limitless construction.

The third diagram is the first one which we constructed, although we never knew the sequence followed by this Princeton Flexagon Committee except for the fact that Stone came across the use of multiple doubling right away, according to Martin Gardner. It is a natural enough discovery; one wonders when tripling was first revealed as the essential operation for square flexagons.

The diagram uses points for cycles, and connects cycles which have faces in common. Precisely, the color on top of the flexagon and the color on the bottom. It may or may not be possible to switch cycles, but if it is it is by folding "up" or "down," which is equivalent to "mountain" or"valley." To make an example, suppose that we want three cycles in a certain kind of square flexagon, and that they should be connected according to the following diagram:



The best way to procede mathematically would be to draw the diamonds, although historically squares representing the cycles would have been drawn first. To shorten the time to make the drawing, they can both be drawn at the same time.



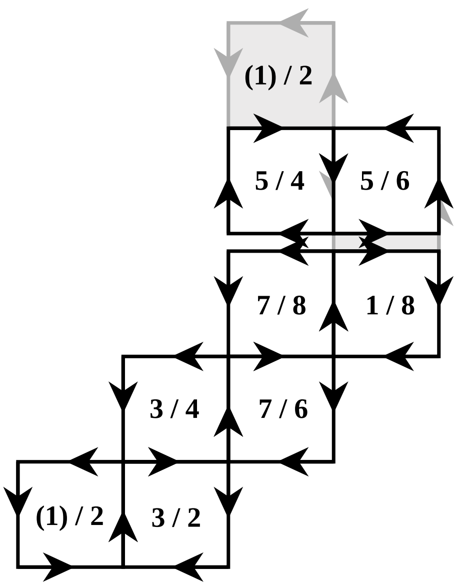
That will result in a list of eight labels (with a ninth for connecting)

+	+	+	+	+	+	-	-	+
1	.	3	.	7	.	5	.	1
.	2	.	6	.	8	.	4	.

which would be completed by adding 1's,

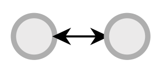
+	+	+	+	+	+	-	-	+
1	3	3	7	7	1	5	5	1
2	2	4	6	8	8	6	4	2

Finally the frieze, or rather half of what is needed, would be

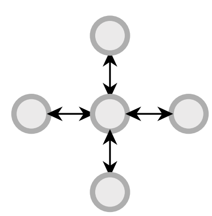


This is an example of a process which can be carried out in great variety, and which should be done with enough examples that one is fairly sure of how the construction goes.

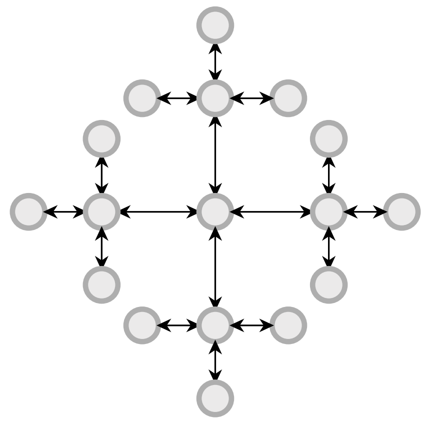
The map for the binary flexagon would simply be



whereas for the second level flexagon it becomes



To get to the third level,



although I don't think we have ever made one of these and gotten it to flex, the folded paper having gotten so thick that operation became unmanageable. More levels keep adding little crosses to the arms of the crosses which are already there, but the congestion would be all that much worse if anyone tried to make the flexagon.

10 Unbranched Sequences in the Tuckerman Tree

There is now sufficient material to think of flexagons as fractal polygon stacks. This word fractal seems to be a fairly recent invention of Benoit Mandelbrot, and for a few years when microcomputers were still a novelty, they were used to graph a kind of fractal gotten by repeating a quadratic polynomial over and over again. By the novelty has fairly well worn off, but when it was rampant you could see copies of "Julia Sets," or more commonly, "Mandelbrot Sets," on the covers of magazines and in science oriented articles everywhere.

What he meant by fractal was something, usually spectacularly irregular, that was gotten by repeated substitution, usually at smaller scale, of a whole structure or design into selected parts of itself. But that is just the process by which flexagons can be built up; replacing individual polygons by spiral stacks insertable in place of the original polygon. It took quite a while to come to this realization, but there is no reason that it had to wait for Mandelbrot; he simply gave a name to what had previously been known as recursion or induction. But the application is more exciting than the places where induction is usually used, which is to prove mathematical theorems.

To get a description of flexagons we have worked through three stages. First, some simple but nevertheless interesting geometric structures were exhibited. There is a historical account of how they were first found by folding strips of paper, although the formation of spiral stacks and unrolling them against each other came quite a bit later than just making wandering friezes and folding them back and forth.

An induction needs both a base and a rule of advancement. These spirals form the base, especially the shortest ones which will both close but also lay out flat in a plane as part of their manipulation. So the second stage, which reveals the continuation principle, is to replace an individual polygon by an equivalent, albeit more complicated, structure. The essential ingredient in the spiral is the turning angle, which is the same by going forward one step or backwards n-1 steps, n being the number of vertices (and hence edges where folding is done) of the polygon.

The third stage can be as complicated as one needs or desires; it just consists in making the substitutions over and over again. For all this it is convenient to make a map that will show where, and how many, replacements have taken place as well as their relation to one another. But mapmaking also breaks down into three parts; rather, the same map can be drawn in three different styles, each of which reveals some properties of the flexagon better than the others do.

10.1 Mapping the flexagon construction

Given a flexagon, the middle map level is the one that would probably get drawn up, because the flexagon sprawled out on the tabletop is the first thing you would see, unless you were already holding it in your hands. And even then, whatever was visible would look like a "face" and would be the thing you would want to keep track of. Coloring would aid this considerably, although setting down numbers would be more readily available and appeal to the mathematically minded: face 1, face 2, ...

Writing the first number - say 1 - on a piece of paper would be the way to start the map. Doing whatever can be done to the flexagon will make another face, and give the chance to write another number, both on the flexagon itself and on the map. On the map it could be joined to the other number by an arrow, to remember which came first. If the flexagon can be adjusted in several ways, several numbers will result, all connected to the first by arrows, but not (yet, anyway) to each other.

The flexagon can be abused, and by beginners often is. That is to say, changes in its layout can be forced, it can even be torn or ripped; so some judgement has to be used as to what is a natural movement and what is not. For example, with pentagonal flexagons, a certain amount of coercion is required, but is is very minor in comparison to breaking the flexagon. And even if no stretching or tearing is involved, ploygons can sometimes be slipped sideways relative to one another, to get something which looks and feels reasonable, even though it is more congested. Such have been called "slipagons" and discussed in articles and patents.

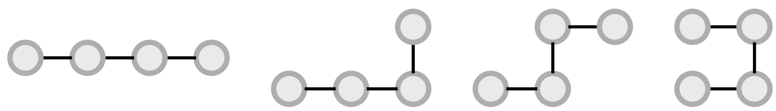
Laying out the map can continue, by setting down new points with their labels and connecting them to the previous points. Two things can happen during this process; no way to continue, or arrival at an earlier place. A place where no continuation is possible is called terminal, whereas coming back to an earlier point makes a loop (or cycle). For flexagons it turns out that there are only loops, no terminals, as a consequence of their recursive construction. But for maps in general, there can be points at which there are no arrivals as well as there being no points without continuations. There can also be groups of points internally connected without connections to each other. But with flexagons, patience will lead from each face to any other.

Just drawing the map with reference moving the flexagon around, lines may cross over but that would be the result of a first attempt. Redrawing the map can eliminate crossings, but that is characteristic of flexagons; there are maps for which it is not possible. Try connecting three points with each of three others.

Once it is apparent that there is a map of mutually tangent loops, it is time to draw the higher level map where the cycles are replaced by points with lines connecting cycles which have edges in common. For historical reasons this map, or graph, is called a Tuckerman Tree. Even though using it gives a good overall picture, there are still rules for connecting the cycles together, which are a little more complicated than just drawing linking lines, or even lines with arrows on them. That is because the middle level has arrows, and they have to be respected. So a number can belong to several cycles, but the only allowed cycle switching is one which respects the arrows.

The map which comes the closest to the actual polygons is the third, or lowest, level map. That one takes into account what is visible both on top of and underneath the flexagon, and is the one which reflects both the base of a cycle representing a polygon stack and the recursive step of removing o polygon and replacing it by its counterspiral. That is the diagram made by diamonds in the examples.

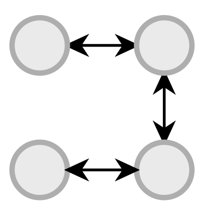
Although there are maps which help with designing and understanding flexagons, it is still necessary to pay attention to them and learn the details of their use. For example, to get a flexagon for which it is possible to run through a sequence of colors without repetitions, there are four different ways to put four cycles in line:



Actually there are more than four, but the new ones that you can get by rotation or reflection don't lead to essentially different flexagons. That is, the results are also either rotated or reflected. So what *is* important is the angle between the links, because that decides which portion of the loop has been extended, so that it has another tangent loop. Flexing is accordingly different in maps where the angles are different.

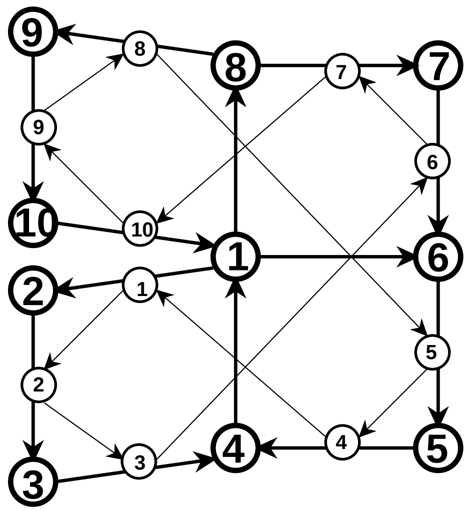
10.2 Spiral makes turnover flexagon

With these ideas it is possible to go ahead and make flexagons. But there are so many of them that the idea soon gets boring, in spite of all the time and effort which went into deducing the description in the first place. Given that one of the uses of flexagons is in magic shows (that is how Martin Gardner became involved with them) it is worthwhile looking for some simple yet striking combinations. One of them has a Tuckerman Tree which is a spiral; with four nodes it has the form



which could continue for still more turns but then would be hard to draw in ASCII. Anyway four is enough to talk about and see how it works.

The middle diagram of (square) cycles would be In the ASCII drawing the line of ='s implies



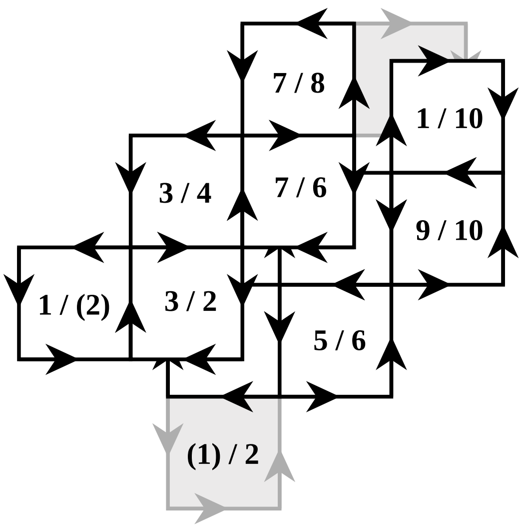
that the top square is not joined to the bottom square, which is also why the Tuckerman Tree does not connect to itself on the left (a Tree has no cycles, which is another consequence of the way flexagons are formed). For that reason points 10 and 2 are distinct from each other; it is easy to separate them in the line drawing.

The detail which makes this class of flexagons, which are called spirals, interesting is that the point 1 is a part of every square, and so it is possible to return to it over and over as the flexagon is run through its paces. In fact, the construction is so nice that extremely long spirals can be constructed. By flexing, then turning over, the square packets fanfold and avoid the bulkiness of scrolling. The turning over is because of the arrows, which can only be respected by turning over.

The list of turns and faces would then be

+	+	+	+	+	-	-	-	-	-	+
1	3	3	7	7	1	9	9	5	5	1
2	2	4	6	8	10	10	8	6	4	2

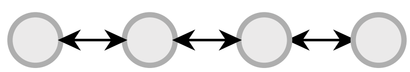
from which the attendant frieze would be



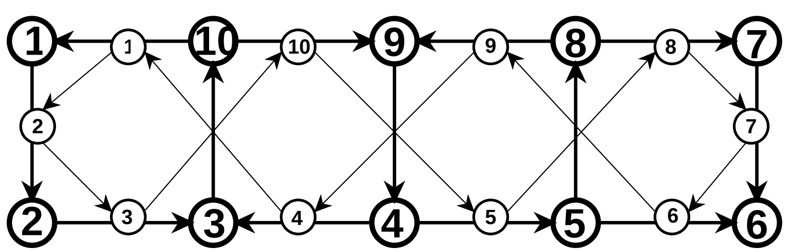
or rather, this is half of it.

10.3 Chain flexagon

Other attempts can be made to find a sequence of faces which will show up in a linear or quasilinear order. A simple enough example would be take a line as a Tuckerman Tree,



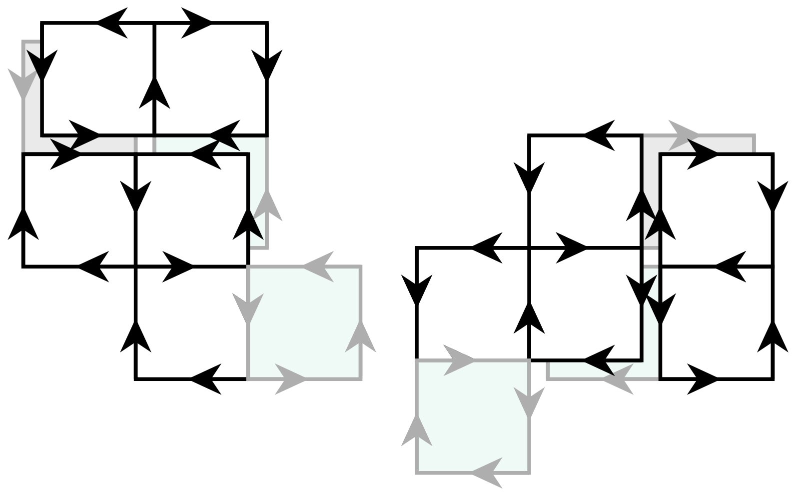
with middle and Tukey Square diagrams



with polygon list

+	+	+	-	+	-	-	-	+	-	+
1	3	3	1	5	9	7	7	9	5	1
2	2	4	10	6	8	8	6	10	4	2

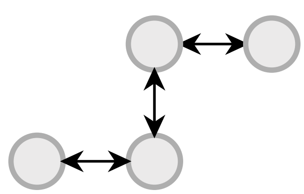
and frieze



With a longer line as a Tuckerman Tree, the terminal $+++$'s will remain but there will be more $+-$ alternations and depending on parity, $+++$ can become $---$. The $+-$ sequence tends to make the frieze curl up into a spiral, so the drawing ends up in two parts.

There is an interesting duality in flexagon construction, that spiral Tuckerman Trees tend to produce linear friezes, whereas linear trees tend to produce spiral friezes. It is not hard to figure out why, but it is curious nevertheless.

10.4 Scrolling flexagon



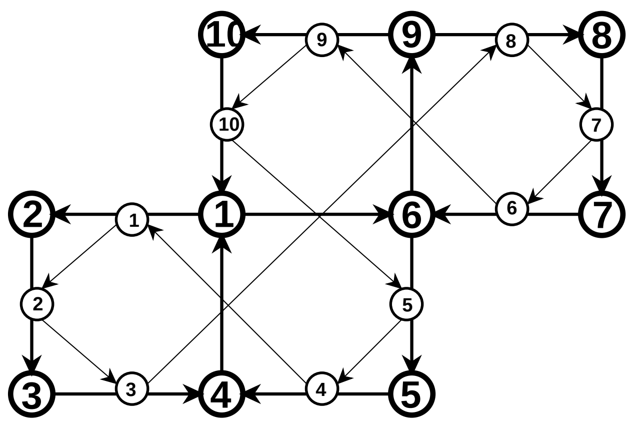
The last of these maps was discussed already, being a nice example of a design which can be extended indefinitely. In other words you just wind the map around and around its center, which means that the squares in the middle map overlap and the Tukey squares have to be figured out carefully. On the other hand, once it is clear how to make the polygon list and place its signs, it is no longer necessary to draw the Tukey squares in detail.

It also has the nature that by turning the flexagon over after each flex, the segments can be built up in fanfold style which means that many more colors can be incorporated into the flexagon than otherwise possible. Of course, if the precaution of turning the flexagon over is not taken, the result may be so bulky that no more folding is possible, and the mistake will have to be undone before proceding. Turning the flexagon over reverses all arrows in the map; that way it is possible to pass through junctions which otherwise would be blocked.

The first of the four maps, the straight line, was also mentioned. It is indeed possible to work through all the colors by judicious turning over, but the flexagon it defines is not particularly elegant. Still, the sequence works.

Making a flexagon from the second map is a useful exercise, but the result doesn't show any noteworthy symmetry. On the other hand, the third map, which was not yet discussed, leads to a scrolling flexagon. Making one of several turns leads to a thorougly unmanageable wad of paper, but the principle is most definitely interesting. And it works well enough for the map shown.

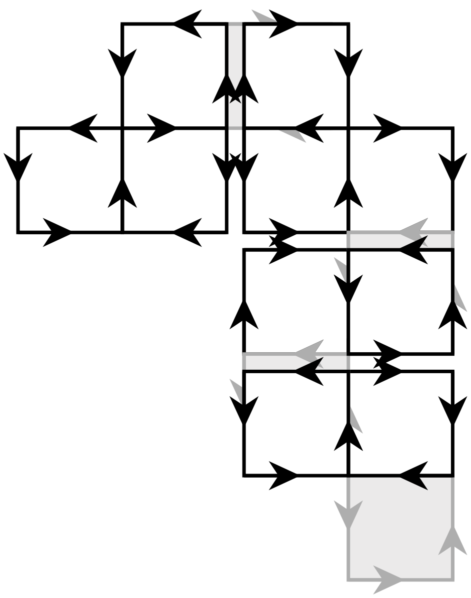
The Tukey squares and their duals are shown in the following diagram:



with polygon list

+	+	+	-	-	-	+	+	-	-	+
1	3	3	9	7	7	9	1	5	5	1
2	2	4	8	8	6	10	10	6	4	2

In turn, the frieze for one of the two required segments is



To get scrolling, not all the colors are exposed, but rather six of the ten are taken in the sequence 3 - 4 - 1 - 6 - 9 - 8. With that restriction they can just be rolled out in sequence.

It is interesting to observe the duality which persists through all the flexagons made with differing polygons - triangles, squares, pentagons, and so on - that a frieze which runs along at a natural angle folds up into a flexagon which has a complete tree, but if the map runs along at this natural angle, the flexagon meanders back and forth before it is folded up, and scrolls. But if the map runs around in a circle, the frieze is still made of natural segments, but they join at a kink. And the flexagon can be made to flex by fanfolding.

With experience, one acquires a repertoire of folds and maps, which can be used to get combinations of fanfolds, scrolls, and full behavior from some particular level. All this is particularly striking for triangle flexagons, but they are hard to type in ASCII. But the square flexagons perform just as well, and with patience so do the pentagonal flexagons. There is nothing wrong with higher polygons, but they begin to approximate circles, which makes them both bulky and harder to manipulate.

Since these special cases are so regular (of course, that's why they have been chosen) it is not overly difficult to work out simple mathematical formulas to express their properties. For example, with the spiral Tuckerman Tree we have

length	plus	minus
4	5	5
5	8	5
6	5	7
7	10	6,

a pattern which can be checked by looking at the Tukey squares.

The best way to get this information is to construct some examples, make several members of the family, and compare them to see how going from one member of the family to another changes both the map and the frieze.

11 And Then Some

What started fairly simply has grown into something which can be quite complicated. It can be taken as it is, comprising a reasonably complete theory of natural flexagons. On the other hand, several assumptions were made along the way which could be reconsidered. For example, the primitive polygon stack was gotten by fanfolding, but that is not the only way a string of polygons can be doubled up. Another assumption was that consecutive, adjoining edges of the polygon were used as hinges, so that none were skipped.

What still remains constant is that one single polygon, regular or not, is used in the construction, and that where polygons are joined, it is done by reflection in the edge where they are joined. That ensures that the stack really is a stack; that is, all the edges are parallel and there is a common margin for the whole cylinder. Later it may be opened up to get parallel stacks, or several blades in a leafed structure or whatever, but that comes later on.

It also remains constant that the flexagon is a frieze, namely that it is an unbranched string of polygons joined along their edges (not just by vertices, for example), although it may meander around and even cross over itself due to the angles between the hinges. So the only real question remaining is how it is to be folded from a frieze into a stack. After that, its folding properties in three dimensions have to be examined, following which it may be seen that some friezes work and others don't. But the first thing to be done is to figure out how to describe the layout of the frieze and how it has been folded.

Since the frieze is laid out by joining polygons, the only variation which has to be described is the order in which to take the edges as hinges; so far the order has been assumed to be consecutive, the only variation being in whether to run around clockwise or counterclockwise, and then only because the sense is reversed when the polygon is turned over as it might have been in the recursive step.

Some people like to lay down the polygon and think of external compass directions, such as North, South, East, West for square flexagons. But internal directions can be used; orient the polygon, choose a positive sense and mark that on the polygon along the rim so it doesn't matter whether it has been turned over or not. Then `+`, `++`, ... , `-`, `--`, ... can be used to say how many edges to skip over and in which direction. If there get to be too many plusses, say n+ as in 3+. We have already used this notation in the three-line polygon list; where the choices in the first line were confined to `+` or `-`. So that notation will be kept.

Where the polygon is located on the stack is the other thing which has to be specified. The first polygon goes on top. The second polygon will go somewhere lower down, exactly where will only be known when after the folding has been done. But when it is finished, it is possible to count down the stack and see where it ended up. In a fanfold, it will be the second polygon, turned over. But even in flexagons built recursively from fanfolds, some scrolling will take place, so there is no guarantee of where the second polygon will sit in the stack. Nor for the third and so on, but they have to sit somewhere. No two sit in the same place, and there are no repetitions untill all have been used.

That is the information stored in the second and third lines of the polygon table. Two lines were used to recognize that the polygon has two sides, only one is visible on top but both have colors (or better, numbers) The principal numbers alternate between the two lines and their counterpart is gotten by adding 1 (signifying "next color to be seen" but neither "next on the frieze" nor "next on the stack" unless it happens that way).

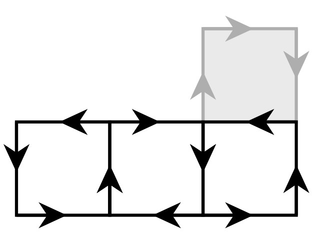
Well, this correspondence between two lists is called a permutation, and that is what decides that the p^{th} polygon on the frieze sits in the q^{th} site in the stack, upside down or not. Altogether there is enough information to describe the flexagon, and with adequate care, to build it. The recursive process which has been described up to now is just one way of building a flexagon, with the advantage that it is quite clear where everything goes without first having to put the polygons in place and then having to peer into the flexagon to decide just what ended up where. Or conversely, having to stick your fingers or twigs or something between polygons to save place for one which is going to be inserted later on.

The main difference in this general point of view and the graphs heretofore described is that the graph isn't necessarily the same one. Now the preferred graph describes the permutation. For the fractal polygon stack kind of flexagon, that is the sequence of labels you get by following the Tukey triangles (in the flexagon paper this is called the Tuckerman Traverse) and so the permutation graph and the one already worked out are the same. But skipping edges and skipping position in the stack need a reinterpretation of the graph, which is probably best omitted. Rather, use the three-rowed polygon list directly.

To see how this works, think of a flexagon made up from two squares (and then continued into more segments if necessary). A basic requirement for all flexagons is that the initial edge and the final edge be parallel, which means that the total sum of angles is 360 degrees (remember the dictum, the sum of the angles is 360 degrees?). If the angle in a square flexagon is 90 degrees, that means `+ + + +` or else `++ ++`, or the analogue with `-`'s. But we have just two squares so it is `++` and they are joined at opposite edges. You don't get anything at all!

That isn't totally true; a string of squares joined at opposite edges is something you can buy at the post office, namely a roll of stamps. One of the things mathematicians have studied is, how many ways can they be folded into a stack? The result is called a "Catalan Number" and you can look it up on the Internet with Google. But that isn't something that you would want to call a flexagon.

The next possibility is to take three squares, for which the only frieze summing up to a rotation is `+ ++ +`, as in the following figure:



As usual an additional square has been included as a tab for pasting. The heavy lines (with =’s) should match when joining two of these friezes and then connecting their ends.

The flexagon actually works, and can be recognized as a pair of barroom hinges opening in opposite directions and placed one above the other. This was pretty hard to get using natural flexagons.

Thinking of four squares, there are combinations like `(+ + + +)`, `(+ ++ - --)`, `(+ - ++ --)`, etc. Of course, `++` is the same as `--`, implying opposite edges, and a sum of 0 is as good as 360 degrees.

Trying `(++ ++ ++ ++)` comes back to stamp folding, so we ignore it. As for the permutations, it is agreed that 1 matches 1, but 2 can become 2, 3, or 4 which gives six possibilities, and then 3 can become one of the two leftovers. That leaves no choice for the last, but otherwise there are twelve possibilitiers. To be checked, one by one, against three or more possible friezes! Quite a bit of work.

A while ago we worked out a few of these combinations. The permutation (1 2 3 4) is fanfolding, the sequence (1 2 4 3) fans two and scrolls two; or in other words the fourth square is placed on top of the third square, rather than under it. The following table shows the results:

		sign sequence		
permutation	+ + + +	+ + − −	+ ++ −− −	+ ++ − −−
(1 2 3 4)	natural	flaps	flap-scroll	Good f'gon
(1 2 4 3)	tubulates	tubulating	scrolling	crossed

As the variety of adjectives indicates, we have a collection of mixed results. Tubulating refers to the fact that the flexagon may not respond to the normal flexing movements in some positions, because the hinges are not at right angles to one another, but rather parallel at opposite ends just as happens in the stamp folding exercise. But the combination makes a square cylinder - the “tube” - which can be opened and closed with the internal hinges now on the far ends, and the reverse for what were formerly widely separated. After making this change it may be possible to continue flexing in the accustomed manner, revealling new faces. This switchover is called tubulating, and is a requirement in some flexagons for viewing all the faces.

Careful examination of the flexagon shows that the switch from fan to scroll while changing the permutation was responsible. That means that there are indeed other primitive flexagons besides the fanfolded ones, even though they arise from exactly the same frieze. Just the scheme of folding changes.

Other parts of the table show what happens when squares are joined on opposite edges into a strip rather than by adjacent edges into a zigzag. One of the most common results is to get back to stamp folding, which is ambiguous from the point of flexagons. That is, a flexagon should be held in place by the angles between fold axes except when everything has been carefully aligned while flexing. But the strips often have to be positioned by hand and carefully maneuvered to avoid disturbing the layout. So while they are technically good flexagons, they are usually excluded from systematic lists.

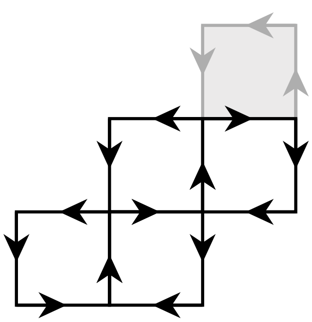
Another consequence of folding willy-nilly is that the frieze may be doubled into a hard knot which cannot be undone. So in contrast to the natural flexagons wherein the recursive process assures that everything will work out, the general designs have to be checked. What that means is really that some more complicated rules need to be derived for taking them into the category of flexagons. Or that the properties of recursion based on another primitive cannot be anticipated, allowing the other primitive to be used interchangeably with the fanfold.

It would appear that fairly random things can be flexagons, and it could be there that a systematic exposition of the theory ends. Looking through the books we have accumulated and offerings on the Internet, and of course such journal articles as exist, only flexagons based on equilateral triangles have gotten any mathematical treatment at all. There are several designs based on squares which don't go much beyond showing some examples. No doubt there are knowledgeable people here and there, but if they haven't tried to disseminate their knowledge is some publicly accessible form, there is no way to know what it is.

On the other hand, the recognition that fractal polygon stacks comprise an extensive and well structured assortment of flexagons, and that still other classes can be found could be taken as a good stopping point. One of these other classes consists of the tubulating flexagons; those for which more complicated spatial arrangements exist but which still have phases which lie flat in a plane. We have not looked too far into the degree to which the fractal philosophy could be applied to them, but something is surely possible. Meanwhile, there are some simple families such as scrolls which can be examined with little effort.

Just setting down permutation and turning sequences gives an enormous number of possibilities, not all of which fold up neatly and deserve to be called flexagons. Part of that neatness is that there shouldn't be loose groups which can move back and forth without special positioning of the rest of the flexagon. Another would be that the frieze should'nt be tied into a knot which can't be undone by rotations, but only by sliding the frieze into position. The recursive construction assures the behavior of the flexagon, but that is just one way of making one.

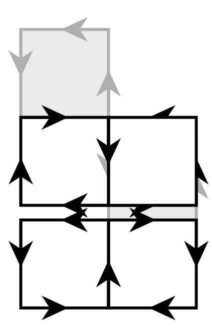
To better visualize the partial list showing some four-square flexagons, the following diagrams repeat the table but includes the friezes.



	+	+	+	+	+
natural	1	3	3	1	1
	2	2	4	4	2

	+	+	+	+	+
tubular	1	3	4	4	1
	2	2	1	3	2

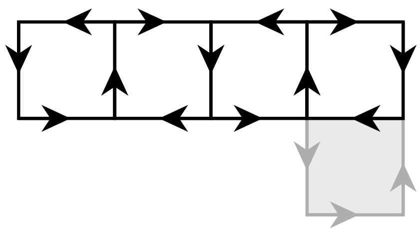
The same frieze, because of the sequence + + + + for both examples, folds two ways according to the two different permutations. The first is the primitive flexagon made from squares, the second will not fold in a cycle but given parallel hinges at the extreme ends of a pair of squares will make a cylinder which can be folded back along the complementary hinges and opened out to lie flat. That is the operation of tubulation, following which different colored faces have become visible. Both flexing and tubulating are needed to see all the faces.



	+	+	-	-	+
natural	1	3	3	1	1
	2	2	4	4	2

	+	+	-	-	+
tubular	1	3	4	4	1
	2	2	1	3	2

The sign sequence + + - - typically leads to a frieze in which the squares move back and forth connected to form a snake. When used as the basis of a flexagon as short as this the normal form sprouts flaps whereas the tubular form tubulates. The sequence + - + - would generate a spiral, which is not shown.



	+	++	--	-	+
natural	1	3	3	1	1
	2	2	4	4	2

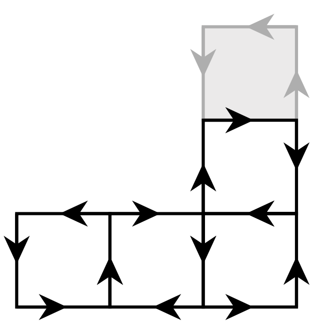
	+	++	--	-	+
tubular	1	3	4	4	1
	2	2	1	3	2

The normal form of this long strip leads to flaps, which is to be expected because of the hinges on opposite sides of the squares rather than on adjacent sides where they cab block movement. But in the tubular form, where a little bit of scrolling alreasdy exists, the final result is a pair of contrarotating scrolls. The combination is bulky and only allows running through a sequence in which all hinges are parallel, but it works out. Indeed, it is the prototype for general scrolls.

The polygon list for a general scroll would read

+	++	++	++	++	++	++	...	+
1	.	2	.	3	.	4	...	1
.	n	.	n-1	.	n-2	.	...	.

Depending on whether n is even or odd, the sequence has two endings. If n is even, there is an even number of ++'s, the last entry in the list would be -, giving an overall sum of zero. When n is odd, the odd number of ++'s has to be balanced by an initial and final + to again get a zero sum. This says that one frieze of n squares will connect to the next at the top edge of the last square or at the bottom edge according to parity. That turns out to be just what is needed to make a scroll.



	+	++	-	--	+
natural	1	3	3	1	1
	2	2	4	4	2

	+	++	-	--	+
tubular	1	3	4	4	1
	2	2	1	3	2

Surprisingly, the normal form (identity permutation) with the turning sequence + ++ - -- generates a good flexagon although its layout is slightly curious. The tubular form is also interesting, although not quite a flexagon. Laid out in planar form with two squares at each corner, folding along a horizontal axis runs through three colors while from the same position and folding by the vertical axis runs through three other colors (two of which are the same in both positions). That is why the flexagon was described as "crossed."

So there it is, "A Quick Flexagon Survey."

The construction of flexagons as fractal polygon stacks gives a nice theory with beautiful ramifications, but just making up a stack without any particular order seems to produce vastly more flexagons. The problem lies in deciding which stacks will fold nicely, which is readily apparent with the fractal process based on fanfolded cycles of regular polygons. There must be other starting points, but it seems that nobody has carried flexagon studies that far.

With squares, building arbitrarily long scrolls has already been mentioned, although there are practical limits due to the thickness of the folds if half a dozen or more squares are included in the base stack. But in looking at that list of some flexagons made with four squares, there is a vestige of a braided stack. They are made from long strips just like the scrolls, but the strip is folded at a right angle in the middle and the two halves fanfolded back and forth across each other to form a braid.

Actually, if just two strips are crossed and folded this way, and the loose ends joined, but for the two strips separately, something which Tony Conrad called a bregdoid results; they have interesting folding properties of their own, but they remain largely unexplored. However, having fanfolded a single bent strip, there are two loose ends which can be connected to a similar stack to get an interesting flexagon which will not fold indefinitely. Nevertheless, turning it over after each flex, the braid can be transferred from one diagonal of the square planar form to the other. As such, very much longer strips can be folded, although there comes a point at which nothing particularly new is gained through doing so.

Both this form and the scrolling form can be created with the natural flexagons; the difference is that they can be made with a single thickness of paper from which every face can be exposed whereas the other version uses more complicated folds and some faces are never exposed although it is theoretically possible to do so. Again, paper thickness is the governing factor. We once built up a fanfolding triangle flexagon with several hundred faces, just to prove that it could be done.

Here are the first few flexagons in this series:

+	++	-	++	+
1	3	4	4	1
2	2	1	3	2

+	++	+	++	++	+
1	3	4	1	3	2
2	2	5	5	4	1

+	++	++	-	++	++	+
1	3	4	1	5	4	1
2	2	5	6	6	3	2

+	++	++	+	++	++	++	+
1	3	4	7	7	6	3	2
2	2	5	6	1	5	4	1

The pattern repeats by going on to use more squares, so it will not be written out further here. The differences obviously arise from the different ways the tails of the braids have to be joined, and how to write the sequences beginning with 1 each time. between even and odd total lengths, there may be an odd square left over and the other braid will have to be turned over to make the join. But then the middle will have to be described differently, and the choice of + or - in the middle shows that the two braids have to be oriented differently there too.

Allowing for the four cases, the general pattern is more or less

$$
\begin{array}{cccccccc|c}
+ & ++ & ++ & ++ & ... & ++ & ++ & ++ & + \\
1 & . & 4 & . & ... & . & 5 & .. & 1 \\
\hline
2 & 2 & . & 6 & ... & 7 & . & 3 & .
\end{array}
$$

The ++'s grow on the right, and the 1 after the vertical bar (signifying whether or not to turn the second copy over to join it), alternates between top and bottom. If the proper number of positions (n, that is) has been provided in the list, working in from the ends will decide the middle. Altogether there are four cases, not only as n is odd or even but on its remainder modulo 4. Don't forget the rule to add 1 to the number on the opposite side to fill in the single dots.

The resulting flexagons are nice enough that they are worth adding to the list of easily made examples. Looking at them for a little while shows that scrolling can be combined with braiding (or fanfolding) to get mixed results; indeed inadvertently moving the +'s or ++'s around in the construction list can already make the change for you.

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[3] Martin Gardner, Chapter 1: Hexaflexagons (pp. 1 - 14), *The Scientific American Book of Mathematical Puzzles & Diversions*, Simon and Schuster, New York, 1959 (ISBN 0-671-63652-9 Pbk.).

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