Julia Curves, Mandelbrot Set

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Abstract

Julia curves, and their relation to the Mandelbrot set, are explained.

1 Introduction

The solution of equations by iteration has a long history; once upon a time the main emphasis was on finding good initial approximations and convergence criteria, but nowadays computer experiments have shown that there are intricate global properties related to nonconvergence and the stability of convergence which are equally interesting and important.

The behavior of quadratic functions, the simplest of all nonlinear mappings, combines ease of calculation with sufficient generality to illustrate most of the abstract properties of iterations. Just as using complex variables often clarifies the properties of functions of a real variable, studying complex iterations can be expected to generalize and illuminate real nonlinear mappings as well.

Julia curves and Fatou sets are very intricate fixed sets belonging to complex mappings, whose structure summarizes many of the properties of real mappings. In turn the Mandelbrot set provides a map of the Julia curves, by use of which their appearance can be classified.

Several very interesting books of pictures have been published relating to these topics, some of which are listed in the bibliography.

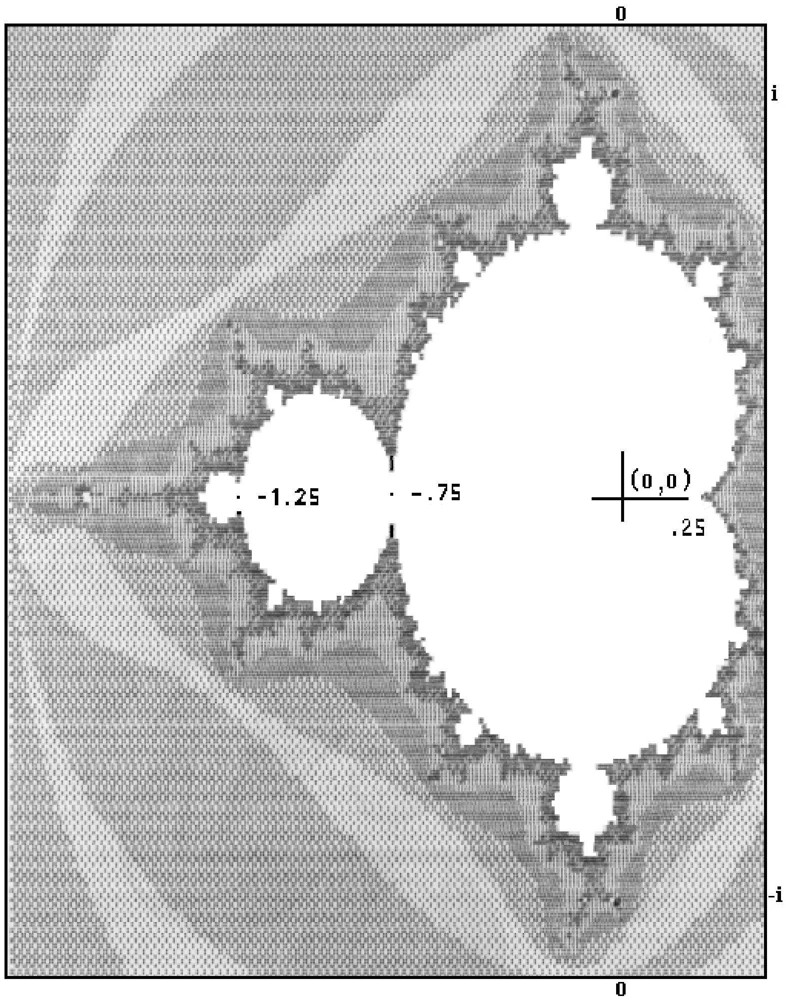

Figure 1: the Mandelbrot set

Figure 1 shows the Mandelbrot set, which serves as an overall guide to the Julia curves. The main lobe is a cardioid, running from -0.75 to 0.25, so that the origin is about 3/4 of the way across the figure. The tangent blobs are actually circles; additionally there are filamentary tentacles with copies of the complete Mandelbrot figure strung along them. They are not particularly evident at the resolution of the figure, but are responsible for considering the Mandelbrot set to be fractal.

Since the screen dump distorts the vertical scale, unit distance is almost the full height of the figure. The point i sits out on one tentacle, and corresponds to a limiting Julia curve which is a mere skeleton without an interior.

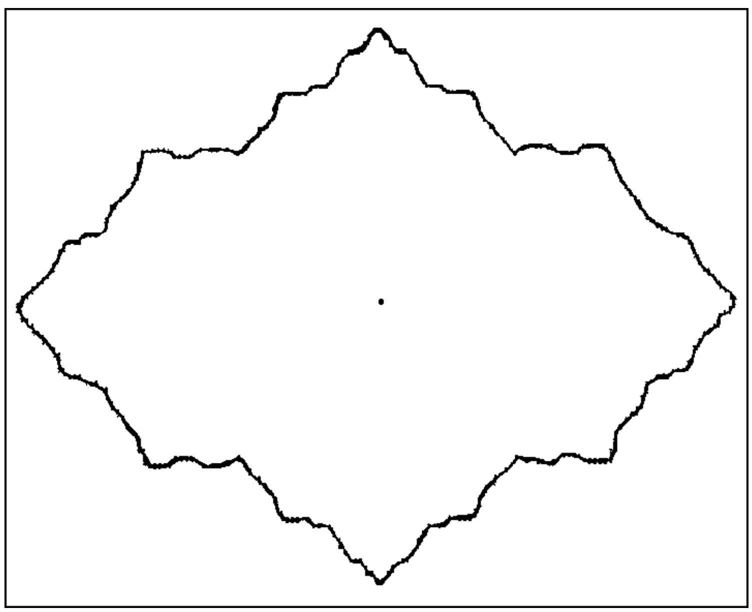

Figure 2: Julia curve for $c = -0.5$

Figure 2 shows a Julia curve taken from the main blob. All Julia curves taken from the main blob are similar to a circle, which appears more and more pinched as its parameter approaches the boundary of the blob. The number of pinches will depend on the direction of the boundary point as seen from the origin. What we are actually dealing with is a fixed point which is stable. Therefore an iterative trajectory will approach it unless it is strictly periodic, or unless it converges to zero or infinity. Out in the tangent bubbles, the fixed point is unstable, but it has stable iterates. Just which iterates are stable depends upon the bubble in question, but their order is higher, the further out from the main bubble they are located.

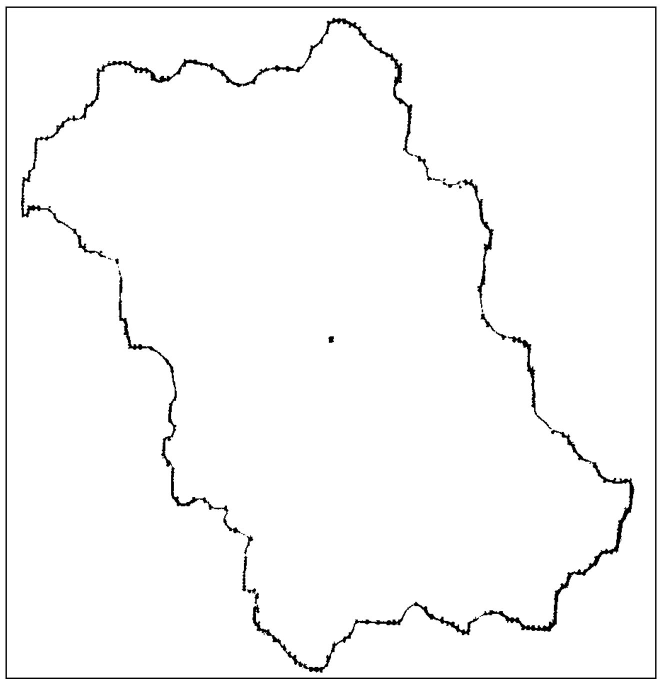

Figure 3: Julia curve for $c = 0.5i$

Figure 3 shows another curve. The parameter of the first curve was taken from the negative real axis; that of the second from the vicinity of the imaginary axis. The curve arising from the complex conjugate of a parameter is the complex conjugate of the curve arising from the original parameter. Thus the curves arising from a real parameter have a symmetry by reflection in the real axis. Along the real axis, positive parameters lead to a squared up and vertically exaggerated circle;

along the negative axis is is stretched horizontally and compressed vertically.

2 Real iteration

Even though the fixed point equation for a quadratic function is one of the simplest that there is, there is already considerable ambiguity about how it should be written. In its most general form, a second degree polynomial can be written in the form

$$p(x) = ax^2 + bx + c,$$

with three parameters. However the invertible mapping

$$r(x) = \alpha x + \beta$$

can be composed with $p(x)$ to produce any desired change of origin and scale, since

$$p(r(x)) = (a\alpha^2)x^2 + \alpha(2a\beta + b)x + \alpha\beta^2 + b\beta + c.$$

Although α is effectively determined by a, β can be used to remove the linear term by solving a linear equation, the constant term by solving a quadratic equation, or to fulfill some other requirement on the coefficients. Knowing the geometric effect of the substitution and realizing that f represents a parabola, it is not difficult to visualize the effect of the transformation. One of the three parameters remains to make distinctions which cannot be accounted for by such a simple change of axis.

Even so, different canonical forms of the second degree fixed point equation

$$f(x) = x$$

are convenient in their own contexts. If emphasis is placed on mapping the unit interval into itself, the form

$$f(x) = \mu x(1 - x)$$

is preferred. It is readily deduced that zero is always one fixed point and that the other is

$$x = \frac{\mu - 1}{\mu}.$$

That $x = 1$ is approached asymptotically from below as a fixed point is a consequence of the zero of f at $x = 1$; for larger μ the parabola rises more and more steeply through this point.

Stability of the fixed point is the fundamental concern of the theory of iterations; according to whether the magnitude of the derivative of f is smaller than 1 or not, iteration in its vicinity will be stable. In the present case,

$$f'(x) = \mu(1-2x)$$

whose value at the fixed point is

$$f'(1-\frac{1}{\mu}) = 2-\mu.$$

Therefore the fixed point is stable only in the region $1 \leq \mu \leq 3$.

At one time the discussion would have finished with this observation; letting curiosity motivate some inquiry just above this interval will show that successive iterates may oscillate about the fixed point, settling towards consistent underestimates followed by consistent overestimates. Effectively, there is a range within which two stable fixed points of the iterated function

$$g(x) = f(f(x))$$

bracket the unstable fixed point of f itself.

According to the chain rule of differentiation,

$$g'(x) = f'(f(x)) \times f'(x),$$

so it is quite possible that the absolute value of this quantity does not reach 1 slightly away from the unstable fixed point. Nevertheless it is likely to close to 1, passing 1 if μ increased. In that case the fixed point of the iterate would also lose its stability, but with the possibility that still newer fixed points of

$$h(x) = g(g(x))$$

would be stable. Such reasoning quickly inspires the conjecture that there might be an infinite sequence of fixed points, progressively destabilizing as μ increases while creating a new pair which promptly splits away from it.

Less evident, but still understandable, is the fact that it is possible to run through the whole infinite series of splittings within a finite range of the parameter mu, leaving open the interesting question of what kinds of transfinite behavior to expect beyond such a limit.

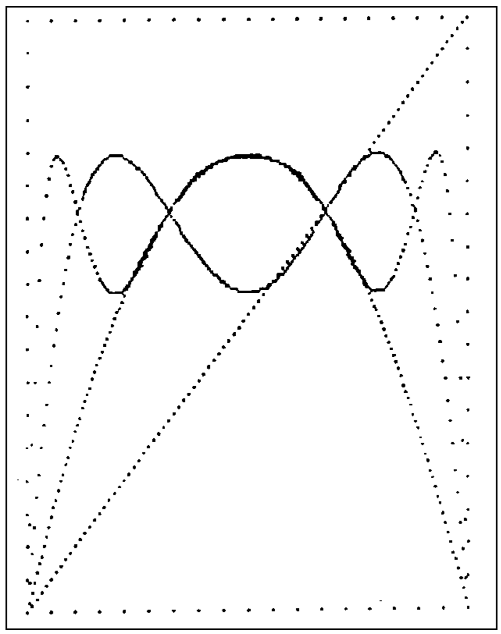

Figure 4: three iterates of f(x) for $\mu = 3.1$

Figure 4 shows the first three iterates of $f(x)$ for the value $\mu = 3.1$, which lies just beyond the value of μ for which the fixed point becomes unstable; it is clear that the second and higher iterates still have stable fixed points.

Examining the graph confirms two useful properties of iterates. First, the fixed points of any function are automatically fixed points of its iterates; clearly $f(a) = a$ implies $f(f(a)) = a$. Second, critical points of a function are also critical points of its iterates; by the chain rule if $f'(a) = 0$ then $(f(f(a))' = f'(f(a))f'(a) = 0$.

At the fixed points, derivatives of the iterates are powers of the derivative of the function, making the stability of their common fixed points consistent for all the iterates.

Further examination of Figure 4 suggests that the higher iterates become squarer and squarer, their flat portions coinciding with the two fixed points of the second iterate. The suggestion is confirmed both by calculating the higher iterates, and by increasing μ slightly, causing the flatness to appear with earlier iterates. Indeed, reducing μ ever so slightly would make the second iterate more closely tangent to the diagonal, indicating that the onset of the instability of the original fixed point, and the rapid separation of the two fixed points of the iterate, depend sensitively on the parameter μ.

It would seem that a second graph, showing the location of the fixed points as a function of the parameter μ, would serve as a map giving an overall view of the process, playing the same role that the Mandelbrot set plays for complex iterations.

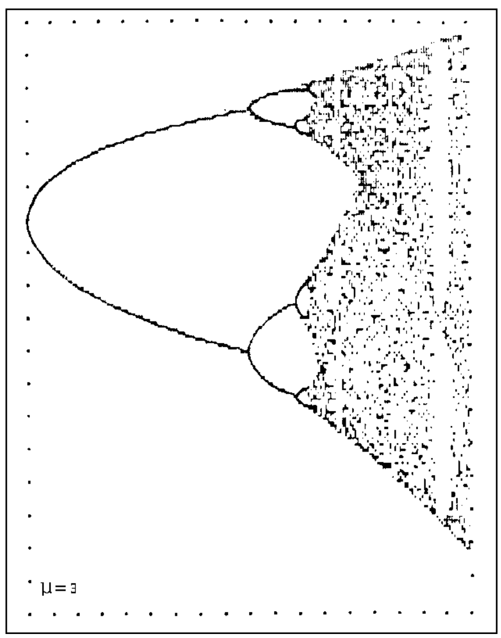

Figure 5: fixed points as a function of μ

Figure 5 shows such a graph, which was obtained experimentally by calculating a very high iterate of f, and then forming a histogram of the values of successive iterates. All values which had a sufficiently high frequency were marked on the graph.

One of the striking features of Figure 5 is the fact that the proliferation of fixed points for higher iterations of f has reached an infinite limit at a finite value of μ, and that the graph has further structure on beyond such a limit. Much of the recent work on the theory of iterations has been devoted to understanding this region of the diagram.

3 Complex iteration

Since the discovery of the Mandelbrot set the preferred formula for complex iteration has been

$$w = z^2 + c,$$

wherein the quadratic polynomial has been reduced to canonical form by scaling to a unit leading coefficient and suppressing the linear term. According to this interpretation the fixed points are

$$z = \frac{1 \pm \sqrt{1-4c}}{2}.$$

Much insight into the process can be obtained by choosing $c = 0$, for which iteration is the simple process of squaring.

The fixed points are then 0 and 1, although ∞ should be considered as well, given its natural role in complex variable theory. The stable fixed points are 0 and ∞, while 1 is unstable. Working with absolute values these conclusions seem to be reasonable; the square of a small number is even smaller, the square of a large number is larger still; only the absolute value 1 will be conserved.

The complex numbers occupying the unit circle exemplify one of the differences between real iteration and complex iteration; the unit circle is a fixed set even though the only fixed point which it contains is 1. While all the rest of the complex plane is drawn towards one of the stable fixed points, the unit circle will wrap around itself; the modulus of each point gets doubled.

Since a second degree polynomial has two roots, quadratic iteration will usually map two distinct points into a common image; in our case

$$z = \pm\sqrt{w - c}$$

shows that +1 and -1 both map into 1, both +i and -i map into -1, both $\frac{1+i}{\sqrt{2}}$ and $\frac{-1-i}{\sqrt{2}}$ map into +i, $\frac{1-i}{\sqrt{2}}$ and $\frac{-1+i}{\sqrt{2}}$ map into -i, and so on. There is an infinite number of counterimages to be arranged into a binary tree, the set of whose nodes is representative of the unit circle. Similar trees exist for other values of c; only for $c = 0$ do the nodes lie on the unit circle.

The Julia curves of Figures 2 and 3 were constructed by this process; the $\pm$ sign in the formula for counterimages explains their symmetry by 180 degree rotation. There are other sets which sample the Julia curve; even if not fixed themselves, they can still outline the curve.

The binary tree constructed from the sign sequence through which the counterimages are obtained gives an invariant representation of the Julia curve, valid even when $c \neq 0$. Its nodes can be expressed as binary numbers on the range $0 \leq x \leq 1$ by writing a decimal point followed by zeroes or ones according to whether the plus sign or the minus sign was used to form each succeeding counterimage.

For example, .0 represents the positive square root of the unstable fixed point 1, which is 1 itself, while .1 represents -1, its negative square root. Continuing, .01 would represent i, .11 $-i$, and .001 $\frac{1+i}{\sqrt{2}}$. Writing the digit representing the most recent sign change on the left, just after the decimal point, ensures that the order of the counterimages along the Julia curve is the same as the order of the angles in the exponential representation of the unit circle. This is because the positive square root tends to halve the angle of the point, while placing the negative square root in the diametrically opposite position. The notation works out well because an arbitrary string of zeroes still represents 1, or the fixed point when $c \neq 0$.

With respect to this binary notation, iteration is nothing but multiplication by 2 modulo 2; the formation of counterimages is division by 2, followed by the adjunction of a "sign" bit to distinguish the two possible counterimages. To the extent that the entire Julia curve is accurately approximated by the counterimage tree, the remarkable amount of self symmetry of the Julia curves is summarized by this correspondence.

Take any small arc, say the portion for which the first k binary digits are fixed. Iteration magnifies the arc, doubling the angle which it subtends from the origin, then translates it by c, which does not change its shape. But this is just the arc that would have been constructed with one less binary digit. Inspection of Figures 2 and 3 shows this effect quite clearly, especially if the radial peaks or valleys in the curves are used as reference points.

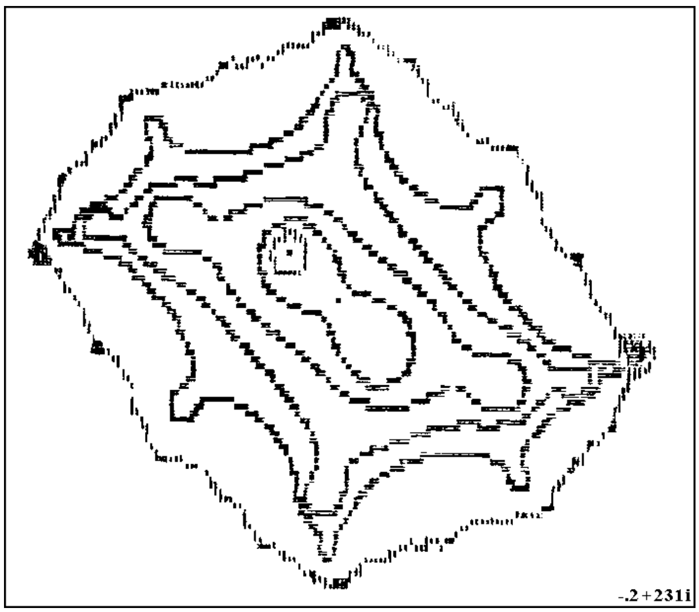

Figure 6: counterimages approaching a Julia curve

Figure 6 shows how the Julia curve can be approached as a series of counterimages of the fixed point in its interior. It was constructed by drawing a small circle around the fixed point, then calculating four levels of counterimages for the points on the circle; a number which fits nicely within the diagram without crowding.

The unstable fixed point sits on the point of the figure at the far right. It is its own counterimage, together with the diametrically opposite point. As successive counterimages are adjoined to the figure, rough squares, octagons, and so on, may be perceived. Evidently the vertices of these polygons move more rapidly toward the fixed point than the edges do, with the vertices of higher order polygons being more reluctant than those of lower order.

It is not surprising that a similar differentiation is to be found in the interior of the Julia curve, accounting for the doubling of the number of protuberances on the counterimage "circles" at each stage, producing curves with recognizable even harmonics in their polar coordinate Fourier transform. With increased deviation from the circular form the curves begin to pinch together and even to break up into disconnected sections.

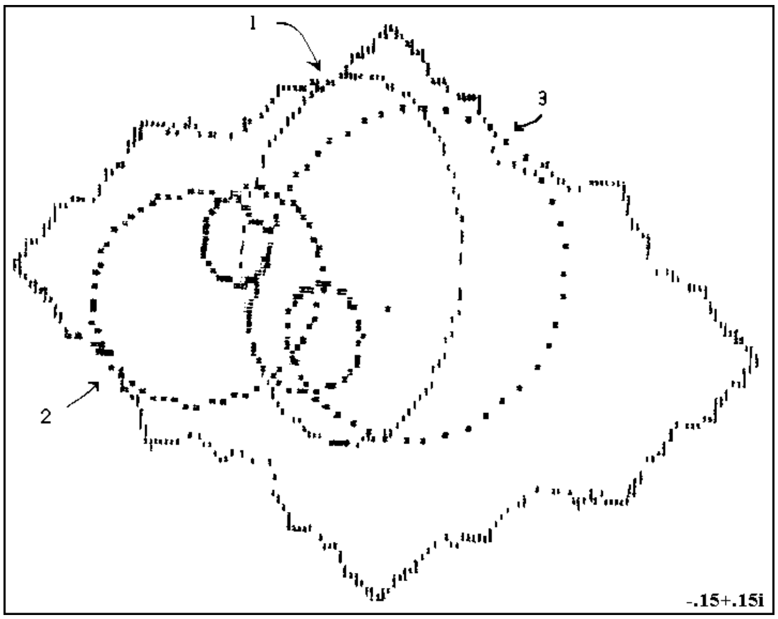

Figure 7: ellipse with two of its successive images

Another viewpoint, which is shown in Figure 7, is to take a fairly large curve which still fits within the Julia curve, and calculate its images according to the mapping. In principle they converge to the fixed point; nevertheless the form of successive iterates gives some idea of the speed of convergence and reveals other details of the process. If the starting curve is inadvertently made too big, so that it intersects the Julia curve, it is possible to see a sampling of three types of behavior.

The portion inside converges to the fixed point, the portion outside diverges to infinity, and the points of intersection with the curve itself remain upon the curve. This detail can be seen in Figure 7, where the ellipse touches the curve; the point of tangency has rotated in successive images because the angle of its polar decomposition has doubled each time.

4 Scaling

The representation of a Julia curve as a binary tree of counterimages of the unstable fixed point accounts for its fractal nature, because the arc at any stage of the construction resembles a doubled image of the arc at the next stage. A polygonal approximation to the Julia curve results from connecting successive points around its circumference by straight lines. If the Julia curve were a circle the lengths of each chord would be the same, but of course they are not.

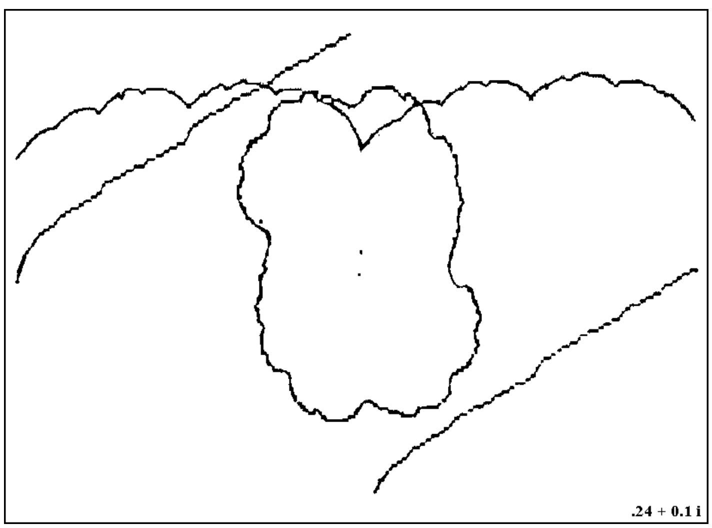

Figure 8: angle and radius in invariant coordinates

As the number of points in the approximating polygon is doubled, the length of the chords would be halved; since the curve is not a circle, the statistics of the fluctuations in the ratio should reflect some of the structure of the curve. Figure 8 shows the absolute value and complex angle of each of the vertices of the Julia curve, plotted as a function of the invariant binary representation of the points.

Superimposed on the figure is the Julia curve itself, for purposes of comparison. The complex argument runs out to 180 degrees in both directions, causing the discontinuity at middiagram in the graph representing the angle. Fractal structure is especially evident in the curve representing the radius.

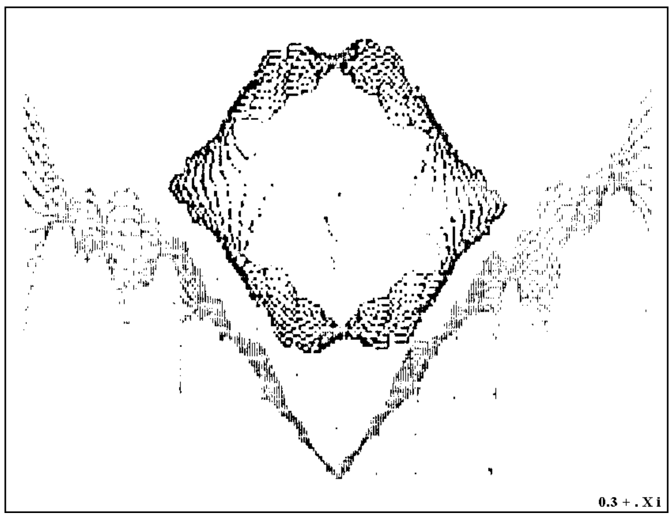

Figure 9: ratio of chord lengths for two successive generations

Figure 9 in turn shows a plot of the ratios of successive chord lengths. The chords are not those forming the 2^n-gon approximating the Julia curve, but rather the ratio of the differences of two successive iterates. Consequently they are taken from a varying assortment of locations. If ω is a root of unity, then for $c = 0$ the ratio is

$$\frac{\omega^4 - \omega^2}{\omega^2 - \omega}$$

which simplifies to $2|\cos\frac{t}{2}|$, where the t in $\omega = e^{2\pi i t}$ is the invariant coordinate. To facilitate comparison, Figure 9 actually shows a family of values of c, together with the associated Julia curves.

5 Questions and extensions

- why are the critical points so important for the theory?
- develop a program to display the Julia curves in more detail.
- include an option in the program to explore the Mandelbrot set.
- for some ranges of c the contours shown in Figure 6 pinch off and become separated. Is there any particular significance to this phenomenon, or of the region in the Mandelbrot set in which it does not occur?
- the curves in Figures 8 and 9 represent only part of the theme developed in the reference to Jensen, Kadanoff and Procaccia. Modify the program to calculate the ratios of the peripheral chords according to this article; investigate the validity of the correlations implied by their two stage de Bruijn matrix.
- quadratic iterations give a comprehensive introduction to the general theory, but further varieties of designs can be obtained by varying the iteration equations.
- the diagram of fixed points as a function of μ for real iterations would be much more informative if relative frequencies were shown in the chaotic bands, which can be done nicely by coloring the display.
- the windows between bands of the μ vs fixed-point curves correspond to the nodules surrounding the Mandelbrot set. Examining both diagrams at greater magnification should reveal details of this correspondence.
- fixed points of the higher iterates correspond to cycles of doubled period, but this is no longer true once the chaotic region has been reached. Look up some of the literature concerning possible periods and how to determine them.
- modify the program to follow iterates of points from various regions of the complex plane and designated parameters from the Mandelbrot set. What limitations on the trajectory are implied by the finiteness of the fixed point representation, and thus on the details which the program can explore?
- use the contour program in <**PLOT**> to build up the Mandelbrot set by finding where iterates of $z^2 - c$ have unit derivative.

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