Section 8 alludes to "One-Sided Shifts," a subject of considerable historical as well as practical importance. The ladder operators in quantum mechanics, for instance. It would hardly do to omit "Shifts with a Countable Alphabet," particularly if one were a mathematician interested in generality. Pushing just a little bit further, one might encounter coupled map lattices and even probabilistic automata.

The concluding tenth section recognizes "Higher Dimensional Shifts," about which practically nothing is known and yet everyone would like to know something.

Very well, everyone has their favorite wish list. This reviewer's main concern lies in having seen how far an approach which he does not particularly like has advanced. For that, the book has been extremely enlightening, in spite of one's having had most of its references in hand for several years and having studied them with some diligence. Since theorems tend to be true or false independently of whether one likes them or not, any "improved" theory is going to have to account for all these results, even the conjectures, one way or another.

Fortunately, the quibble is not so much with the results, although one would hope to see fewer of them and better organized. Rather, having seen this book, less time needs to be spent in trying to figure out what all those diverse authors really meant; instead effort can go into juggling the little pieces into a better picture.

That, at least, is a personal viewpoint. In the meantime, until another book comes along (there is one on the horizon - "Theory and Applications in Additive Cellular Automata" announced for February 1997 by the IEEE Computer Society) - we will let this series rest for a while, content to repeat the admonition that although the book by Lind and Marcus is not about cellular automata, it comes so close to being so that nobody who is interested in celllular automata should ignore it (Nor, either, those who would just like to learn about coding theory and symbolic dynamics).

viewpoint, the zeta function depends on a shift's periodicity, but calculated with a particular weighting. No matter; the section still is chock full of algebraic number theory.

The third section, "Pure Subgroups of Dimension Groups," continues the trend from dominant eigenvalue to non-zero spectrum by involving the dimension groups; the question is mostly: "What can be said about a chain of dimension subgroups?" The answer is of some use in the next chapter.

The twelfth, which is the final substantive chapter and entitled "Equal Entropy Factors," ties off a few remaining loose ends. Or, which the authors assert, the loose ends are almost tied down, making the chapter a suitable transition piece into the final prospective. In any event, some more algebraic number theory peeks around the corner.

The sections of the twelfth chapter are:

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12.1 Right-Closing Factors
12.2 Eventual Factors of Equal Entropy
12.3 Ideal Classes
12.4 Sufficiency of the Ideal Class Condition
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Our reaction to all this is nothing short of amazement: Can resolving the question of two graphs containing the same labelled paths (alternatively, what pairs of paths can be superposed?) really be all this complicated? As consolation, we could recall the demonstrable insolubility of Post's Correspondence Principle and appreciate our good luck.

In their final Chapter 13, "Guide to Advanced Topics," the authors thoughtfully remind us that the show is not yet over, although they do not go quite so far as inviting us back for the second act.

The first section, titled "More on Shifts of Finite Type and Sofic Shifts," lists several additional topics for consideration:

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the core matrix
constraints on degrees of finite-to-one factor codes
renewal systems
almost finite type shifts
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One of the constraints in the second item is imposed by the multiplicative nature of the Welch indices.

The second section continues with "Automorphisms of Shifts of Finite Type," a whole area to which the monograph "Textile Systems" by Nasu, which has previously been mentioned, is dedicated. Also worthy of note are the papers of Wagoner interpreting the conjugacy relations, especially those depending on the factorization of the connection matrix, as constructs from algebraic topology.

The third section is dedicated to "Symbolic Dynamics and Stationary Processes," It and the fourth, "Symbolic Dynamics and Ergodic Theory," elaborate on the connections hinted at in Chapter 9, that symbolic dynamics can be built up from measure theory as well as from topology.

Section 5 recognizes that there are other shifts, under the heading "Sofic-Like Shifts;" and then there are "Continuous Flows," the topic of Section 6. Section 7 returns to "Minimal Shifts," an historical antecedent which occupied Hedlund, Gottschalk, Birkhoff, Morse, and others. Involved periodicity classifications with which we are still involved date from those times.

What is left is spread out into three sections, the first of which is called "The Embedding Theorem." It asserts that a strict embedding is possible when there is a lessening of entropy and cyclage (distribution of cycles), requiring fourteen pages for its proof, and introducing a refinement of the cycle count wherein multiplicity counts. Those who actually work with zeta functions know that there are tricky distinctions amongst the ways cycles can be enumerated.

The second section, "The Masking Lemma," is short and concerns the behavior of the graphs defining shifts of finite type consequent to embedment of the shifts themselves.

The third section, "Lower Entropy Factor Codes," covers similar ground for surjections rather than injections.

The book still has three more chapters, one of which contains still more technical detail, one of which surveys the extent to which all the theory shows up in actual practice, and the conclusion which tells us of all the things which remain to be done and possible directions which the theory could take.

new books and old articles (19)

Douglas Lind and Brian Marcus, in their book "An Introduction to Symbolic Dynamics and Coding," published last year by Cambridge University Press, have given us a comprehensive treatment of shifts of finite type and sofic shifts, an area which has seen considerable activity since the sixties; one in which they have taken an active part. Maybe a little long for its intended use as a text book, it nevertheless encompasses the literature of recent years in a masterly fashion, unifying a heretofore scattered literature.

Shifts both are and aren't cellular automata. The rules governing shifts make their shift-commuting continuous mappings into cellular automata; Particular shifts are best regarded as restricted classes of configurations, whose behavior under cellular automaton mappings is the topic of interest. Coding theory concerns possible functions between pairs of configurations whereas automata theory foresees the consequences of iterating a function which has already been chosen. Evidently the two activities are intimately related.

We have been summarizing the thirteen chapters of the book, two by two, and so have nearly finished the book, having arrived at Chapter 11. But there are still more than a hundred pages to go. If we hurry, we can finish them up.

"Realization," the title of Chapter 11, reckons with the handful of invariants which have been adduced for shifts - entropy, cyclage, dimension group, and so on. There is a converse question, consisting in knowing their arbitrariness. There are concepts, like prime numbers, which only mathematicians appreciate, but which still cannot be ignored by the rest of the populace. So we find here that there are actually distinctions between entropies which are integers and those not, some are rational yet others not, algebraic or transcendental and so on. Esoteric distinctions at first sight, but vital in applications.

Why this should be so is perhaps not too difficult to perceive: entropies refer to rates of proliferation amongst paths, so integers might well be associated with uniform out-degrees, or with the authors' road coloring problem. Maybe the association isn't quite that simple, but it is plausible; knowing for sure comes back to an exercise in algebra. So that is the content of the first section, "Realization of Entropies."

The second section, "Realization of Zeta Functions," goes a step farther, given that the zeta function depends on traces rather than the dominant eigenvalue alone. From a less algebraic still others without coming to an entirely conclusive termination. All this makes its appearance in Chapter 9, "Degrees of Codes and Almost Conjugacy." One time more, Hedlund's influence is recognized in that chapter's historical notes.

Welch's index theory is pretty definitive with respect to the left and right indices, which have already made their presence felt in the previous chapters through the concepts of closings, coverings, resolutions and all their variants. But the last has not yet been heard from the middle index, the actual surviving multiplicity, "All endomorphisms have the same multiplicity but some of them are more obtrusive than others," to paraphrase a famous quotation; Hedlund expressed it by saying that "almost all endomorphisms have the same number of counterimages."

The first section of Chapter 9, The Degree of a Finite-to-One Code, defines the degree of a code and introduces such concepts as a magic word (could that be the vertex at which a path enters or leaves the Pair Diagram on the one-way trip required by the bubble theorem?).

The second section, Almost Invertible Codes, begins with "Definition 9.2.1. A Factor Code (phi) is ¡almost invertible¿ if every doubly transitive point has exactly one preimage, i.e., d(sub)(phi)=1." Could we be asking whether or not the nuclei of extradiagonal subsets of the Pair Diagram encompass a sufficiently representative set of vertices?

The third section, Almost Conjugacy, goes on: "in this section we strengthen the relation of finite equivalence by requiring the legs to be almost invertible," and we meet the non-wandering sets.

The fourth and final section of the chapter is entitled: Typical Points According to Probability." This is an interesting way to confront those endomorphisms whose failure to be reversible is due to a few mavericks; they are simply banished to a set of measure zero.

As we know, there are essentially three views of symbolic dynamics: combinatorial, topological, and probabilistic. The first would probably answer all questions were it to be pursued diligently. In the worst case, topological or measure theoretic arguments could simply be followed out without mentioning the specialized vocabulary of these disciplines. So symbolic dynamics must have some unique features, not found in all topological spaces or in all measure algebras, which give it a unique personality. Certainly, commutation with the shift is one of them, and its derivation from finite sequences must be another.

It is nice that topological continuity enters the picture, and much is to be gained from applying knowledge of topological spaces which is already available within a body of knowledge which has been studied for many years. As this final section illustrates, measure theory encompasses a vocabulary and style of reasoning which can be equally illuminating, when used with discretion and applied with caution.

Had the authors not used restraint, the book could easily have acquired another chapter or two, because there has been no lack of interest and publlications on the probabilistic aspects of symbolic dynamics. Here is just enough of an introduction to rationalize the lukewarm endomorphisms which aren't invertible, but only weakly so.

Moving on to Chapter 10, Embeddings and Factor Codes, the complement to surjection, which is injection, comes to the fore. In both cases, concern lies with the circumstances under which all the invariants, equivalences, and partial invariants can be used to compare two shifts. As expected, emphasis is given to shifts of finite type. Also, the subject matter of the chapter splits into three parts, one of which was already presented Chapter 7. Thus the chapter is confined to strict embeddings, and strict entropy change; entropy conservation is reserved for Chapter 12.

There is no lack of advanced and sophisticated concepts in these later chapters. As well as more familiar concepts: the Pair Diagram makes a brief appearance in Section 3 under the appelation "fiber product," so we know that we are getting close to the arena in which all the distinctions between endomorphisms and automorphisms are played out. There are also hints of the Third Great Theorem, but let it emerge in the remaining chapters.

Still it is hard not to wonder if the presentation could have taken another direction, with another orientation. We are only beginning to appreciate the publisher's understatement of the situation: ... "Mathematical prerequisites are relatively modest [...], especially for the first half of the book." ...

new books and old articles (18)

Our perusal of "An Introduction to Symbolic Dynamics and Coding" by Douglas Lind and Brian Marcus, published by Cambridge University Press, is gradually advancing, eight of the thirteen chapters having been scrutinized. Although the book is promoted as a textbook dedicated to symbolic dynamics, especially coding theory, such characterizations have to be taken with a grain of salt.

As an introduction, with possible application as a textbook, it is something which has been lacking for the longest time - a single, coherent account of a whole generation of research on these topics, taking up roughly where G. A. Hedlund's often cited paper on automorphisms and endomorphisms of the shift left off. It is filled with interesting exercises - hundreds of them - thus traditionally incoporating still more material into an exposition, providing thereby sidelights and additional details which would otherwise make the text even bulkier.

It also follows the tradition of placing historical notes at the ends of the chapters; what else could we desire except maybe some biography of assorted investigators, with a flow chart showing who studied under whom, who met at what conference, and other nice, gossipy details? In other words, an even longer book?

What kind of students are going to assimilate a five hundred page book, filled with the most advanced and abstract of modern mathematics, even in a full year course? That is another matter. Especially when it turns out that still more time is going to be required for even the experts to assimilate and organize all the fascinating ideas which are just barely peeking out from beneath a horrendous maze of facts. We are reminded of that kindly elephant, she who provided gainful employment to a whole coterie of wise mem.

Close inspection shows vestiges of Hedlund's magnum opus surfacing here and there in the book; mention has already been made of the way that continuity and shift invariance influence the kinds of mappings; for the authors, they define the sliding block codes, even as they define the cellular automata for others.

Allusion to Hedlund's encounter with uniform multiplicity finds its way into the historical notes concluding Chapter 8, although the chapter's presentation is slightly weighted toward the effects of non-uniformity as seen in Moore's Garden of Eden Theorem. From the outset the orientation of coding theory is toward surjective mappings, which means that it is very much concerned with multiplicities, uniqueness and decodability, and would take anything else as an aberration.

The third leg of Hedlund's trinity (your reviewer's interpretation, not Hedlund's) quantifies exact details of the loss of limiting multiplicity, via the Welsh Indices. Hedlund's treatment relies on quite a few supplementary notions, such as periodicity, transitivity, recurrence, and is used several times in the text.

The first section estasblished the Decomposition Theorem, which follows readily enough given the previous preparation; it requires a conjugacy to factor into specific components, that resemble duality. The second section gives the dual tower an algebraic form, very reminiscent of the LR scheme for diagonalizing matrices. The third section introduces a variant process which makes two matrices equivalent when they have a common factorization but with the factors in the opposite orders.

Although the usual course on matrix algebra may not emphasize these relationships, they are implicit in such standard sources as Halmos' Introduction to Finite Dimensional Hilbert Spaces. The context of graph theory replaces the field of complex coefficients with the positive integers, which aren't even a ring, making the results more complicated and requiring fresh proofs.

Example 7.3.12 answers the question posed in the introduction to the chapter by exhibiting a tower of length 7, commenting that there is a whole family of matrix pairs whose relationship is still unknown.

The fourth section surveys the results from linear algebra which retain their validity, while introducing some group theory of its own. The Jordan form remains, as well as the Smith form for integral matrices. The fifth section introduces an entirely new concept, which is called the dimension group. Basically, it seems to result from giving positive (or nonnegative) matrices a place to work out their idiosyncracies by asking which vectors they eventually place on an integer lattice after multiple iteration, and which of those are positive.

The authors have done a good job of explaining these ideas; hitherto it has been necessary to cull them from a mass of journal articles. Even if is hard to grasp the deeper significance of all the definitions, at least they are laid out in a form where they can be examined. One gets the idea that somehow iteration is being used to turn the carrier space of the connection matrix into a shift sequence maybe akin to the original.

The chapter closes with a flow chart relating all the different theorems and the different kinds of equivalence which they embody.

In Chapter 8 we find out that if we had been talking about cellular automata, we would have been talking about reversible cellular automata, and that they are just about the hardest ones to deal with. Next in line of succession comes surjective automata, which are not reversible because of multiplicity. If some way could be found to defeat multiplicity, reversibility would prevail. One approach is topological; Hedlund's version. Yet another is statistical, since it sometimes happens that the set of sequences which fails Hedlund's criterion has measure zero. But we are getting ahead of the game.

The title of Chapter 8 is "Finite-to-One Codes and Finite Equivalence," with four sections: 1. Finite-to-One Codes, defining the concept, and relating it to right resolution, which would give it a high Welch R Index. Theorem 8.1.16 about bubbles could be recognized as Moore's Garden-of-Eden Theorem, and the general discussion as deriving from Hedlund's Second Great Theorem, namely the Uniform Multiplicity Theorem.

Section 2 takes up the case of Right-Resolving Codes, with section 3 devoted to Finite Equivalence, an interesting concept from Universal Algebra having to do with ordering functions according to the equivalence relation induced by counterimages, and finding the least upper bound of a pair of functions. Section 4 combines the previous two titles under the heading "Right Resolving Finite Equivalence," all of which is done for the sake of obtaining a lesser, but more tractable (computable), kind of equivalence.

new books and old articles (17)

"An Introduction to Symbolic Dynamics and Coding" by Douglas Lind and Brian Marcus, published by Cambridge University Press, is a book which means what it says, in the sense that it is a thoroughgoing exposition of material to be found in the literature of symbolic dynamics over the past thirty years. Much of this is research was performed by the authors themselves, but the book goes much farther than that, making it a very readable presentation of the whole entire subject.

Although cellular automata receive the most extremely casual notice, the fact remains that the treatment has the most detailled possible applications to cellular automata, particularly with respect to the borderline between reversible automata and the others. The book takes great pains to introduce topology and to explain it, a feature conspicuously lacking in the journal articles on which it is based. Nevertheless, the reader whose grounding lies in automata theory will find an entirely different point of view, with its own traditions, language, and priorities.

For the most part, cellular automata theory starts with the definition of its working materials - states, lattice, rule of evolution - and goes on to examine all kinds of particular examples and classify them before finally turning to an organized theory. With symbolic dynamics, structure seems to have the highest priority, sometimes to the extent that examples are never contemplated; at most maybe to provide a counterexample or two. There is another important difference: whereas reversibility is one of many properties of an automaton which merit attention, it is first, foremost, and seemingly the only characteristic of dynamical systems which seems to garner interest.

It is in keeping with the systems dynamical viewpoint that the first part of the book explains concepts like the full shift, subshifts of finite type, and sofic systems, in considerable detail, relating them to specific kinds of graphs; even including a respectable introduction to the theory of graphs (directed graphs), About the only thing missing is the definition of a dual graph, but it soon becomes apparent that such operations as state splitting and the introduction of division matrices and amalgamation matrices are very much involved with the calculation of duals.

Sometimes a dual graph is called an edge graph, which is a term the authors use liberally. However, the essence of the dual transformation consists in one specific factorization and reassembly of the connection matrix when the factors are multiplied in the opposite order. Repetition produces a dual tower, but it is not always possible to run the process backwards; certain supplementary conditions must be met. Not to be overlooked is the similarity of the construction of the dual tower to the formation of the higher block presentations, both of which are intimately related to the presence of paths in the basic graph.

Bearing in mind the relationship that sometimes exists between the sequences in a shift space, paths through a graph, and the algebraic properties of its connection matrix, it is time to examine Chapter 7, Conjugacy. Several levels of relationship have to be taken into account because algebra, topology, and indexing are all there. The basic concept is that a function f between two systems A and B, themselves with structures a and b, should satisfy a f = f b, something which is often depicted in a commuting category theory diagram. The trick lies in giving the symbols concrete meaning.

The chapter opens with an innocuous pair of 2x2 integer matrices, and the question of whether they are conjugate? It is a good example to bear in mind as the theory unfolds, and of sliding block codes.

Only about a third of the chapter is dedicated to the strict business of topological definition; an equal portion describes the zeta function, with a third dedicated to a specialty of one of the authors, the Markov partitions.

Both of these latter concepts are vaguely related to the idea of representing a function as a matrix. Practical difficulty arises because it would require infinite dimensional matrices to represent most mappings, especially to represent the mapping point by point. If the representation were possible, the matrix could be interpreted as the connectivity matrix of a diagram, which would then be analyzed in terms of paths, roots, cycles, leaves and the rest of the vocabulary. In particular, traces connection matrix powers define loops if their points are weighted to make the correspondence work.

Then, an identity relating the determinant of a matrix exponential to the exponential of its trace relates the characteristic polynomial of the matrix to the cycle structure. The exponential of the trace is the zeta function, whose properties derive from the matrix identity. By defining the zeta function directly in terms of cycle counts, difficulties with infinite matrices are avoided.

The matrix identity has been known since the beginning of the century, but the zeta function seems to be an artifact introduced somewhere around the middle of the century, perhaps by Weil or Artin. There are many ways in which the bibliographical notes in the book could be extended, to trace more little historical details such as this one.

Markov partitions derive their origin from trying to dissect a space into subsets, and then to describe the mapping between subsets in matrix form. A laudable objective, it does not always work out in practice. Sometimes a dissection can be shown to be compatible with the given function, to which the section on Markov partitions is dedicated. Linear mappings of integer lattices have provided many practical examples for study. Of course, no mention is made of any matricial representation, setting the discussion directly in the realm of dynamical systems.

From Chapter 7 onward, the book enters another realm.

That might make one wonder, what there is left to mention in Chapter 13, which is entitled "Guide to Advanced Topics?" Mainly, Chapters 7 through 12 could be characterized in terms of what has actually been done in the past quarter century, more or less in the interval since Hedlund wrote his paper on automorphisms and endomorphisms. With that, glimpses of what might remain are available.

Entitled "Conjugacy," Chapter 7 starts out by asking just that: "When are two shifts of finite type conjugate?" Sofic shifts have not even entered the picture just yet. The answer that would be desired would be that, inasmuch as shifts of finite type depend on a connectivity matrix, they would be conjugate whenever the matrices had the same canonical structure.

There are classical results: the work of Frobenius and Perron on positive and non-negative matrices is available, and has been expounded in at least a half dozen widely available and popular textbooks. Linear algebra has had a complete and exhaustive treatment in Halmos' book on "Finite Dimensional Vector Spaces," particular parts on polar forms and square roots and such.

However, applications to Symbolic Dynamics require matrices with integer, and even non-negative integer, elements, which creates inordinate complexity, only gradually and only recently being resolved.

the same path. In essence, that is coding theory (sometimes caled transducer theory), the title of Chapter 5.

This is the chapter where the authors expound their "state-splitting" theory, whose role is sufficiently important that one of its diagrams occupies a place of honor on the cover of the book. Basically the problem is that although Symbolic Dynamics works with sequences, the best representation of sequences seem to be paths through a diagram. But the question is: "What diagram?" No one diagram fits all requirements, bringing on the study of equivalence and interchangeability between graphs.

Supposing that it is possible to reduce the study of sequences and mappings between sequences to the study of diagrams (directed graphs, to be precise), problems have only been transferred to another arena; the only justification for the substitution would have to be that it is a more familiar arena. But diagrams are well described by their connectivity matrices, so the arena is linear algebra, albeit over the integers or rationals, and in any event with respect to positive matrices.

The discussion of the Perron-Frobenius theory in the earlier chapters results from this agenda, but there is more to come. Although that theory encompasses individual matrices, their eigenvectors, eigenvalues, and canonical forms, there remain questions of factorization, equivalence, transformation and membership in families. Subsequent chapters get to deal with such annoyances. For the moment, Chapter 5 gives procedures which can be used to relate two sequences to one another, and conditions under which success can be expected.

Chapter 6 introduces some of the rudiments of point set topology, then goes on to topologize dynamical systems and discuss the importance of topological relationships. Important among these are the invariants, which typically include periodicity, and the whole family of periodicities subsumed in the zeta function. Sometimes the zeta function is an adequate characterization of topological properties, but often it is not, leaving the remainder of the book to discuss what else is needed.

It is remarkable how much of a semester course on topology has been hidden away here in a little less than fifty pages. Let us look at the cover burb once again: ... "Mathematical prerequisites are relatively modest (mainly linear algebra at the undergraduate level), especially for the first half of the book" ... The problem with topology, really, is not with its list of definitions nor its list of fundamental theorems. Rather, it lies with knowing why they should be used, and gaining experience in consequences. There is the Hardy "Pure Mathematics" approach with deltas and epsilons, and a Bourbaki fondness for theorems as axioms and definitions as theorems.

Case in point: "Lemma 6.2.8. Let M and N be compact metric spaces, and (th):M (arrow) N be continuous, one-to-one, and onto. Then $(th)^(-1)$:N (arrow) M is also continuous."

Our authors strike a happy balance between Hardy and Bourbaki: the theorem avoids using words like "surjective" and "injective," but as we read on we had better know the relationship of compactness to convergent subsequences and something about the uniqueness of limits.

This is an excellent book and it is eminently readable. It is just that the computer science student who had the notion that it was not necessary to take the calculus courses had better choose another book (or else go back and fill in some gaps). Facility in thinking topologically is well worth acquiring.

Anyway, in the next paragraph we discover that Hedlund's First Great Theorem means different things to different people. Those who thought that it laid the foundations of cellular automata theory, disabused of their misconception, will find that it actually authorizes the use

The chapters that follow are much more dependent on the topology of symbolic dynamics, and the relationship between topological mappings and algebraic mappings of the connectivity matrices behind sofic shifts and shifts of finite type. However, the relationship between the topology and the matrices does not seem to be as clearly defined as it should be, with the consequence that there may be much preoccupation with things which are not really issues. At the same time, of course, there are some real substantive issues.

The book does not go into these details, but before continuing with the review it may be worthwhile to expound a point of view.

First, although many benefits flow from the use of topology, and there are excellent prospects of discovering many more, it is also true that topology can be a great distraction. One of the principal topological results is Hedlund's (Curtis-Lyndon-Hedlund)'s theorem that continuous, shift-commuting maps of the shift dynamical system are cellular automata. The practical consequence is the use of local maps to define global maps, or in coding theorists use of sliding block maps.

Once the cellular automaton approach has been settled upon, labelled de Bruijn diagrams provide the machinery for discovering their properties; different labellings for different characteristics. They are NOT exclusion matrices for shifts of finite type, but ARE essentially connection matrices for sofic shifts. There is a duality involved, and mostly it is a historical accident that shifts of finite type held the predominant role.

Surjectivity is decided by the subset diagrams of the de Bruijn diagrams, which is also where the influence of the Welch Indices is felt. Uniform multiplicity results from surjectivity, the second important Hedlund theorem.

Amongst other things, pair diagrams derived from de Bruijn diagrams mediate the multiplicities of surjective mappings, isolate reversible mappings, and perform other servives. There is a counterpart of Moore's Garden-of-Eden theorem, but the simplest test of reversibility is "no loops outside the diagonal."

These "other services" have wrought havoc with interpretations of topology. In reading Hedlund's article, be it noted how often (n-1)-separability arises as a concept (basically, the nodes in the de Bruijn diagram), the preoccupation with transitivity, recurrence, periodicity, and a host of other requirements. In Lind and Marcus's book, note the preoccupation with magic words, road coloring problems, left and right resolution, and so on.

The fundamental question is: what shall we do with loops outside the diagonal? If they do not intersect it, they may be harmless, implying a multiplicity of counterimages. If the connection is one-way we have the resolvings, with memory or with anticipation, depending on the direction of the connection.

new books and old articles (16)

We have come up to chapter 5 in a review of the new book by Douglas Lind and Brian Marcus, "An Introduction to Symbolic Dynamics and Coding" recently published by Cambridge University Press. Although not dedicated to cellular automata, most of its content is thoroughly relevant, with direct applications to automata theory.

Taking the viewpoint that the de Bruijn diagram is the fundamental structure for cellular automata, we find that Chapter 5 is not so much concerned with how paths through the diagram describe the evolution of configurations, as how different descriptions can be given to or the paradoxes which led to quantum theory. The "solution" of the three body problem via the appearance of Poincare's "New Methods" seems to have opened a new chapter just as much as it closed an old one. The new theory seems to have prospered in some environments just as much as it stagnated in others.

Whatever went on in the twenties and thirties, even as late as the forties and fifties, there seems little reason to dispute the authors' assertion that shifts of finite type began to be studied as such by Parry in the late sixties or by Smale at about the same time. Smale's motivation, at least, still seems to have had connections with the theory of nonlinear differential equations.

Whatever the motives for singling out shifts of finite type, whether it was pure convenience or whether that was the nature of the examples which had been taken as models up until that time, dissatisfactions arose. Credit for the initiative in adopting more general systems seems to go to B. Weiss in the early seventies, some five years later. The route followed by these newly baptized "sofic systems," first based on abstract semigroups, was tortuous and labored. There were intimations that they had something to do with paths in graphs; also that regular expressions were involved.

In Chapter 3 sofic shifts have finally reached a degree of respectability, so that a whole chapter can be devoted to defining them, characterizing them, and even proving theorems pertaining to their combinations.

The title of Chapter 4, "Entropy," would seem to take it out of the smooth flow of chapter titles. But that is because the word entropy means different things to different people, sometimes many things to the same person, few of them related to classical thermodynamics. What we are talking about here is the largest eigenvalue of the connectivity diagram of the matrix which has finally been settled upon. Such a discussion could logically follow the presentation of the matrix itself, so the chapter is not out of place.

By and large, Chapter 4 is a standard recapitulation of the Perron-Frobenius theory of non-negative matrices. At one time, knowledge of this theory was a rarity, but that is no longer so for anyone who works in communication theory, biology, economics, or a variety of other fields. Given that we are reviewing a text book, we are pleased to have found a good place for practicioners of coding theory and other readers to acquire the basic information. It is all here - positivity, eventual positivity, irreducibility, cyclic structure, and even entropy as a proliferation factor.

The fundamental conclusions of the Perron-Frobenius theory are the existence of the dominant eiganvalue, its associated eigenvector, and the alternatives for matrices which are not strictly positive. Dealing with matrix families or equivalence between matrices is a more advanced topic, the preoccupation of some of the later chapters.

Chapter 5 goes into one or two detailed schemes for constructing codes, by which is understood a mapping from the full shift to a subshift. The full shift implies an arbitrary data stream whilst subshifts must meet physical restraints, such as the idiosyncracies of a transmission or storage medium. According to the authors, "This chapter uses most of the concepts discussed in the previous four chapters, and it provides concrete and computable solutions to a variety of coding problems."

Given that information would not be encoded if it could not eventually be recovered, it is easy to appreciate that coding theorists are primarily concerned with reversible or conditionally reversible mappings, whereas cellular automatists see them as special cases, if at all. To each his own, but the difference in emphasis has introduced significant distortions into the two approaches.

that the authors are clearly laying out the theory in the way that many prople understand and practice it.

For the purposes of automata theory, probably the most important item to be emphasized is that shifts of finite type have little to do with automata. They represent a means of defining a certain class of configurations, whose evolution automata theory would want to examine. But there are alternative classes of configurations, such as those defined by one sort of formal language or another, which are equally worthy of consideration; indeed recent literature contains specific examples.

The chickens finally begin to gather about the roost in Chapter 3, which is devoted to sofic systems. Sofic systems have a great deal more to do with automata theory than shifts of finite type, but it apparently took quite a while for the connection to become evident. Just what, if anything, is or was the problem with shifts of finite type? Basically, if one takes pure symbolic dynamics as outlined by Hedlund, the continuous, shift-commuting mappings (which are coextensive with the class of cellular automata) are defined by a countable list of exclusions.

Well, finite is an instance of countable, and representable by that exclusion matrix, so it was taken up as an object worthy of study. Notwithstanding the fact that early examples were actually based on finite exclusions, the Notes which close each chapter suggest that the explicit term was created by Stephen Smale and cite a reference. Well, it couldn't have been as simple as that, but for those far from the halls of Berkeley, such statements and citations have to suffice.

Going ahead with cellular automata, the result is that de Bruijn diagrams contain the keys to their evolution, although the information gets extracted by labelling the diagram. Hence one wants to talk about labelled graphs, and the natural entity is actually the dual of the connection matrix of the shifts of finite type. But not every graph is a dual, and thereby hangs a tale.

new books and old articles (15)

The new book now being discussed is "An Introduction to Symbolic Dynamics and Coding" by Douglas Lind and Brian Marcus, recently published by Cambridge University Press. It is intended as a textbook, with numerous exercises and a comprehensible level of presentation, but it is no less a reference work, with its extensive treatment of a wide variety of topics for which a systematic treatment is difficult or impossible to find elsewhere.

Cellular automata are implicitly included in the subject matter; although the whole width and breadth of the subject of cellular automata is hardly present, that is only because the authors' interests and specializations confine them to the reversible and conditionally reversible automata. For that reason, automata theorists will find a wealth of material whose existence they barely suspected.

The previous posting listed the table of contents and summarized the first two chapters. The third chapter is dedicated to sofic systems, where we continue.

Symbolic Dynamics seems to have had a strange history, some of which can be gleaned from the historical notes at the ends of the chapters, and more of which has to be surmised from personal experience or alternative reading. Not having much of the source material readily at hand doesn't help matters either.

The end of the last century saw the culmination of some fields of study, such as complex variable theory or classical mechanics, as well as the emergence of new themes such as relativity origin, in fact), the result is far more illuminating than oppressive, and stands in marked contrast to Hedlund's well known style on the one hand, and thousand-page calculus texts which can be found on the market, at the other extreme. .

Wolfram, in his World Scientific reprint collection, introduction to the bibliography, remarks: "The cellular automaton literature is diverse not only in content but in origins. Cellular automata have been invented independently many times, and in many cases, independent parts of the literature have developed."

Many people see a tradition of cellular automata extending back to von Neumann, but this book by Lind and Marcus belongs to the lore of mathematical communications theory as exemplified by the work of Shannon, and related to such electrical engineering topics as coding, error detection and correction, and to an extent, cryptography. These areas have been heavily influenced by Hedlund's exposition of the Shift Dynamical System, which in turn has its roots in the work of Poincare, Birkhoff, and such things as the Ergodic Theorem.

Symbolic dynamicists have rarely resisted the temptation to assert that cellular automata theory is just a special case (thanks to Hedlund's first great theorem) and cellular automatists have adopted the practice of citing Hedlund's article sight unseen, either out of a feeling that it is the right thing to do, or because it makes a good marker for Citation Index. So we can welcome this new book for laying the cards out on the table, where the possible relationships can be examined and evaluated.

The first chapter introduces the basic definitions, such as a sequence, a sequence space, and the relevance of the shift. But even at this very earliest stage, the idea of forbidden blocks and a shift of finite type is injected straight into the theory. That brings up the concept of languages and grammars, and pictorial representations of sequences. De Bruijn diagrams are here, under the guise of "higher block shifts" as are the "higher power shifts" where the blocks are contiguous rather than overlapping. The "local map" of cellular automaton theory makes its appearance as a "sliding block code" together with such variants as expressing the mapping as a polynomial in a finite field, and the subclass of convolution codes. It is apparent from the outset that there will be a premium placed on manageable mappings, meaning reversible, error correctible, and the like.

The second chapter is devoted to the all-important concept of a "shift of finite type." At first sight, it is both a very simple concept and a very natural one. We haven't actually seen Hadamard's use of shifts of finite type to define orbits in a space of constant negative curvature, but the mathematical literature is not lacking in examples of a similar nature. If memory serves, higher geometry encounters a similar construction with pedal triangles. Even the prototypical Cantor Set is defined by excluding 1's from the trinary expansion of numbers which would be sequences taken from the set (0, 1, 2).

A matrix definition can be given for a shift of finite type, so it is not surprising that an enormous effort has been expended on relating properties of the defining matrix to properties of its dynamical system. But the second chapter is only just defining the concept; most of its content consists in discussing properties of graphs, connectivity matrices, and transformations to which they can be subjected.

"State splitting," which underlies much of the authors' publications and their applications of coding theory, is introduced here. However, the concept of a dual graph does not seem to be mentioned, so there may be some roundabout presentations and inconveniences as the treatment advances. Premonitions of what is to come are found as distinctions begin to be made between vertex graphs and edge graphs. Nevertheless, let us not lose sight of the fact where new exclusions are more numerous than continuations of old exclusions, in a way entirely consistent with the observation that the periodic repetitions of shorter configurations generate the larger basins of attraction.

To summarize, now that we have some new tables of data, and another point of view (commutation and composition relations among rules), as expressed in the book we were reviewing, there seems to be motivation to go back and reexamine some of these older results, especially in the context provided by having studied Hedlund's ideas more carefully than they otherwise may have been. On the other hand. to complete this series, we need to examine the subshifts in more detail.

new books and old articles (14)

There is another new book which should be of considerable interest to anyone working with celllular automata theory, even though the title and orientation of the book are totally toward symbolic dynamics and coding.

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Douglas Lind and Brian Marcus
An Introduction to Symbolic Dynamics and Coding
Cambridge University Press 1995
ISBN 0-521-55900-6 (paperback)
ISBN 0-521-55124-2 (hardback)
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Chapter 1.  Shift Spaces...............................23 pages
Chapter 2.  Shifts of Finite Type.                     30 pages
Chapter 3.  Sofic Shifts.                              22 pages
Chapter 4.  Entropy....................................26 pages
Chapter 5.  Finite-State Codes.                        28 pages
Chapter 6.  Shifts as Dynamical Systems.               30 pages
Chapter 7.  Conjugacy..................................35 pages
Chapter 8.  Finite-to-one Codes and Finite Equivalence. 30 pages
Chapter 9.  Degrees of Codes and Almost Conjugacy.     27 pages
Chapter 10. Embeddings and Factor Codes.................21 pages
Chapter 11. Realization.                               29 pages
Chapter 12. Equal Entropy Factors.                     22 pages
Chapter 13. Guide to Advanced Topics...................35 pages
Bibliography                        15 pages (approx 380 items)
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Cellular automata receive little more than passing references, the most recent of which are to the proceedings of the first Los Alamos conference and the proceedings edited by Demongeot, Goles and Tchuente (for which they at least spelled the name of the first author correctly). Small wonder, the book being oriented towards endomorphisms and automorphisms of the Shift, the topic of Hedlund's classical paper.

In spite of that, it really IS a book about cellular automata, and one which ought to be read by anyone seriously interested in the subject. Not the least of the factors reccommending it is its expository style; while it is intended as a textbook with numerous examples (its actual is worse, to first approximation, these eigenvalues are fairly well determined by the counterimage imbalance in the automaton's neighborhood, which is to say, by Langton's parameter.

What seems to have been overlooked in our previous discussions of this subject is the applicability of Hedlund's third great theorem, even for mappings which are not surjective. Generally speaking, if the mapping f has a certain multiplicity, and g has another, the composite fg ought to have the product of the two multiplicities. For surjective mappings, the multiplicity is uniform making the Welch indices truly multiplicative, and even guaranteeing that both fg and gf have the same multiplicities. Likewise for the equivalence fh = hg, it would follow that f and g would have the same multiplicity, at least with nonzero h.

Combinatorially, things don't work out too exactly, as may be seen from the way Rule 18 factors. The boolean function OR has the forbidden word 010, but XOR has none. Rule 252 excludes 010 because the second mapping couldn't make this sequence, no matter what the first mapping may have done. The other way around Rule 18 has the forbidden word 111 (amongst others); although OR's immediate deficiencies might have been compensated, new ones seem to arise. Necessarily so, because one of Hedlund's results is that if a composite mapping is surjective, the same must be true of the factors, something at which OR fails.

Note that the common sense result would allow the second mapping to heal deficiencies in the first, which means that either the special environment of cellular automata or the topological properties of a continuous mapping of a shift conspire to prove the theorem. In either event, the strong result exists that if a composite is surjective, so are the factors; we need something weaker for merely continuous maps.

One intermediate possibility is to look at the composite of a surjective map with its uniform multiplicity and a merely continuous map, with its spectrun of multiplicities. Checking on (2,1) Rule 18, for example, and coaxing the T-matrices involved a little bit, yields estimates for the top eigenvalues of the T-matrices for Rule 18 and the (2,1/2) OR which are close enoough to suggest a relationship.

Does anyone happen to recall the rule for compounding variances for composite distributions?

Somehow trying to explain a result by both topology and combinatorics seems to end up without an explanation from either end. Combinatorially, the basic result is not exactly the uniform multiplicity theorem; instead it is the process by which the theorem is proven, namely once having counted out all possibilities, remarking that an average cannot surpass a maximum replaces the usual cardinality argument for finite sets.

Hedlund's topological proof depends somehow upon the conservation of closed sets by continuous mappings, so that if the image of a subset is the whole space, it had better have been the whole subspace in the first place. Working that out with deltas and epsilons ought to bring us back to the combinatorial line of argument.

In the end, considering multiplicity as a multiplicative factor characterizing an automaton rule, we find that it only works with precision when multiplicity is uniform. When the rule is not surjective, the example of Rule 18 (or, in another context, the Chaté-Manneville rules) shows that the dynamics can be much more complicated, as when ease of production of favorite configurations promptly eats up all the raw material from which they arise.

Continuing to speculate on the temporal versus spatial proliferation of ancestors, observe that since the subset matrix has a constant row-sum, the number of ancestorless configurations multiplies by the state sum with each length increment, so an interesting quantity has to be the ultimate percentage of orphans. This ratio is first established for short configurations, originally present will seem to survive. Thus "kinks" (parity shifts in the cummulative length of gaps) will annihilate in pairs at the tops of the large inverted vacant triangles so typical of the evolution of binary automata, with a frequency which depends on the likelihood of forming such triangles. Peter Grassberger published many results on the statistical properties of this particular automaton, beginning for example with

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P. Grassberger
"Chaos and Diffusion in Deterministic Cellular Automata,"
Physica 10 D 52-58 (1984),
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whose mechanism was elucidated and generalized by Erica Jen

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Erica Jen,
"Aperiodicity in One-Dimensional Cellular Automata,"
Physica D 45 3-18 (1990).
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Even at the time of the first Cellular Automaton Conference, Douglas Lind described the relation of Rule 18 to a Sofic System in

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D. A. Lind,
"Application of Ergodic Theory and Sofic Systems to Cellular Automata,"
Physica 10 D 36-44 (1984),
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although being able to do so required some knowledge of the actual workings of the automaton, just as did Jen's analysis.

In the last installment, the formations of membranes was mentioned. A good account of this and the usage of de Bruijn diagrams in greneral can be found in the article:

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Erica Jen,
"Invariant Strings and Pattern-Recognizing
Properties of One-Dimensional Cellular Automata,"
Journal of Statistical Physics 43 243-265 (1986).
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There have been innumerable efforts to interrelate the properties of automata, often with the intention of finding a member of a class which is in some sense "soluble," from which to extrapolate the other members. These attempts include finding simple transformations as well as trying to work up a perturbation theory, according to which automata with similar rules will evolve similarly. Perturbation works best when it conserves as much as possible of the loop structure of the de Bruijn diagram, failing more severely according to the number of cycles which are broken.

Concentrating more on generalities than on particular mechanisms, the principal characteristic of cellular automata is their basins of attraction, which is to say, the behavior which they exhibit after a long time, and the route by which that becomes the preferred mode of operation. Basins of attraction depend for the most part on the distribution of ancestors for single-generation evolution, configurations with many ancestors being the most likely to become attractors.

The number of ancestors is something which can be estimated from the de Bruijn fragments, mainly by looking at the comparitive eigenvalues of products of the de Bruijn fragments. What are spectrum-conserving operations on the moment matrices. For reflection, say, we are not saying that moment matrices are symmetric, only that we get equivalent matrices to go with equivalent configurations.

What should be seen as a much more interesting possibility is that there are other equivalences, such as the one Voorehees uncovered as the background for the Ito relations. This may not be too hard to prove algebraically, but in the meantime it is possible to compare the eigenvalues of the T matrix for Ito pairs. A spot check has not revealed any contradictions, although there is an interesting contrast in the prototype $(18, 252)$ pair in that the pair diagram of Rule 18 has two connected components yet Rule 252 shows just one component.

Nasu's mapping theorem refers to just one of these components, but the T matrix involves both. The point needs further contemplation.

new books and old articles (13)

In the process of preparing the way for more abstruse theories of equivalence, perhaps it is possible to lose sight of some of their simpler aspects. The general definition is that two systems are equivalent if they work the same way both before and after the first has been mapped into the second.

When the working is described by a matrix, such as the sequence matrix in subshifts of finite type or the de Bruijn matrix for the evolutiuon of an automaton, and the mapping is characterized by another matrix, the algebraic formulation of the requirement is that A X = X B, with A and B as system matrices mapped by X. If X is invertible, the equation can be rewritten in a more familiar form. If the relationship can not so easily be described by matrices, then rewrite the equation by composing more abstract functions, creating one of category theory's commutative diagrams.

Some sleight of hand reveals one possible origin of such an equivalence, namely that A and B have a mutual factorization wherein A = P Q and B = Q P, for some pair of matrices P and Q. Just this situation has turned up in one of the books under review, wherein the (2,1) automaton Rule 18 is a composite of (2,1/2) rules XOR and OR, whilst Rule 252 is the same composite in reverse order. This is yet another variant on the Ito relation, wherein (126) (252) = (252)(18), although these are but two of a multitude of transformations serving to relate the two rules.

That such an intensely studied rule as Number 18 has factors is not so mysterious; that's the rule that turned up when the automaton was built that way. Although the OR-XOR combination has been used in several contexts, with the intention that the OR proliferate live cells (ones in a field of zeroes) following which the XOR curtails them, leaving only boundaries.

For (2,1) automata, there is an interesting effect, which is built up by observing several details of the de Bruijn diagram. There are only two links labelled "1" in the diagram, and they connect two different connected subgraphs which generate only zeroes. One of them is the self-loop centered on a pair of zeroes, reflecting the rule's quiescence. The other has a loop centered on a pair of ones, but it can never be used after the initial generation because the diagram admits at most two ones in sequence. That leaves a subgraph generating zeroes which can only create an even number of zeroes, possibly by destroying an isolated pair of ones.

The only way a pair of ones can arise is for a string with an even number of zeroes to shrink by two cells each generation until all are gone; if that interval itself was generated by cancelling ones rather than by shrinking a longer interval, only one of whatever pairs were

were not closed. There was a fair amount of historical evolution and reassessment in their definition and interpretation, which in the end makes them depend on a dual of the kind of matrix which defines subshifts of finite type, and allows their characterization as the set of all (bi-infinite) paths through a labelled graph. This places the theory of sofic systems close to the theory of cellular automata; in effect we still need to relate the de Bruijn matrix of the cellular automaton to the path matrix of the shft of finite type whose homeomorphic image is the sofic system.

From the viewpoint of cellular automaton theory, we need to have a single matrix with nonnegative integer entries (preferably zeroes and ones) associated with each automaton, such that equivalence (properly defined) between matrices guarantees equivalence between configurations, and conversely. Until it is labelled by the evolution, there is only one de Bruijn diagram for each class (k,r) fo cellular automata, leaving its connection matrix inadequate to the purpose.

Labelling creates the de Bruijn fragments, say A and B for a binary automaton, but then there are two matrices, not one. There is also an entire family of moment matrices:

$D = A + B$

from which the first moment, the average number of ancestors, may be obtained. For D itself, the number of ancestors is constant and uniform, but products of A's and B's occurring in the non-commutative binomial expansion of $(A + B)^n$ tell how many ancestors each configuration of length n has.

$T = A \otimes A + B \otimes B$

wherein $\otimes$ is the tensor product is the second moment, from which the variance may be derived. As with the first moment, individual terms of T^n describe corresponding configurations, but unlike D, T can distinguish between automata of the same Wolfram indices (k, r).

$S = A \otimes A \otimes A + B \otimes B \otimes B$ and so on.

The higher moments are required to fully distinguish automata, but if surjectivity is the only information required, it is already provided by T when the variance is zero, giving another criterion for surjectivity of automata.

Since T is the connection matrix of the pair diagram derived from the de Bruijn diagram, it can also be used to distinguish injectivity from the different degrees of surjectivity, although not necessarily by its eigenvalues alone. We should also bear in mind that Nasu's bundle mapping theorem based on Hedlund's third great theorem only directs us to some level of the subset diagram, not necessarily the second, and not necessarily the same for the left diagram as for the right diagram. If the corresponding matrix is chosen from the moment hierarchy, it should be a better representative of morphisms than any of the other matrices.

One thing which is easy enough to do is check whether the known symmetries preserve the moment matrices. Reflection transposes the de Bruijn matrix as well as its fragments, and transposition is respected by the tensor product. Simultaneous permutation of the arguments and values of the evolution function, that is, relabelling the states, is also a symmetry of the de Bruijn matrix, its fragments, and their tensor powers. That leaves the cluster-defining operation, permutation of the values alone, to be accounted for. But that merely permutes the summands in the sums of tensor powers, and so we find that all of the ordinary morphisms

Wolfram surveyed linear cellular automata looking for characteristic features of their evolution, summarizing part of his insight by describing four Classes. He also noticed that many automata formed membranes, bounding macrocells within which the evolution had to fall into cycles because of their finite extent and fixed boundary conditions. There are shifting macrocells as well as semipermeable membranes, altogether leaving another specialized class of automata whose particular properties could be studied in greater detail.

Just as de Bruijn diagrams can be used to calculate periodic configurations of a cellular automaton by looking for the loops, it is also possible to look for unbranching paths which are bounded by saturated nodes, from which the membranes can be read off.

That is the result of labelling the paths in the de Bruijn diagram by some boolean function of the neighborhood cells, such as repetition or shifting. When labelling is done according to evolution, the usual application of a de Bruijn diagram, the result leads to the Welch indices. In other words, if multiple continuations from a fixed node all lead to the same evolution, they define a compatible extension. This can be refined into a maximum compatible extension and so on, as spelled out both by Hedlund and by Nasu.

Hedlund's third great theorem relates the product LMR to the number of nodes in the de Bruijn diagram, under conditions of surjectivity, but the same ideas still have meaning for arbitrary automata. When the automaton is supposed to have just two neighbors, and that number is prime, only one of the triple L, M, R can differ from 1. If it is M, every configuration has M fully (that is, $(n-1)$-separated) distinct ancestors. If it is L or R, the automaton is of a shifting type, but M may still exceed 1 if distinct ancestors nevertheless coalesce when followed out to the extreme right or extreme left.

Suppose that $k^{(}n-1)$ is composite, for example 4 given a $(4, 1/2)$ automatom. Maybe then $L = 2$, $R = 2$, still leaving $4 \geq M \geq 1$; such a possibility is included in Eloranta's partial mermutivity, and one might even venture to say that is the essence of the concept. Relative to the previous discussion, it would be the equivalent of partial membranes, taking the form of moving kinks capable of mutual annihilation or transformation; just the eventualities comprising Eloranta's second paper.

To continue with the book reviews, the role of subshifts of finite type or of sofic systems and their relationship to cellular automata needs to be examined. The definitions are easy enough and it is not hard to guess their motivation. Once Hedlund's first great theorem is examined from the axiomatic topological viewpoint, continuous functions require sets of sequences derived from a countable list of exclusions. Such a list is readily obtained from the other characterization of continuous functions, as limits of shifts of extensions of block maps. In particular, it is the list of paths connecting the full set to the empty set in a subset diagram derived from the evolution-labelled de Bruijn diagram.

Independently of this, perhaps guided by another class of applications, someone decided to concentrate on finite lists of exclusions, generated with the help of an auxiliary matrix, the connection matrix for a distinctly-node- labelled graph. With an emphasis on endomorphisms and automorphisme as well as merely continuous functions, the natural consequence was to relate mappings between sets of sequences to mappings between their defining matrix, for which the principal source of information is the series of articles of R. F. Williams which have already been mentioned. Much of Williams' work concerns finding an axiomatic topological definition for shifts of finite type arising solely from the shift and intersection properties of the open sets of the topology.

Sofic systems met the objection that, in a theory of morphisms, subshifts of finite type central concern is different from cellular automata.

To the extent that they overlap, that is certainly well worth knowing about.

new books and old articles (12)

Date: Sun, 5 May 1996 20:24
To: ca@think.com

To create a background against which to review the new books, we hope to be forgiven for including certain articles future and articles present to supplement those articles past which have already been discussed, or whose existence might require mention.

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Cristopher Moore <moore@gila.santafe.edu> has announced:

> ...
> \title{Quasi-Linear Cellular Automata}         [April 26, 1996]
> ...
> and the other one:
>
> \title{Non-Abelian Cellular Automata}          [September 29, 1995]

as well as another which seems to be available at the same place:

\title{Algebraic Properties of the Block
       Transformation on Cellular Automata}      [October 3, 1995]
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Although the abstracts of (at least two of) these articles profess a concern with computational complexity, examination of the articles themselves would disclose a large variety of schemes for the symbolic realization of cellular automaton evolution, principally via the exhibition of numerous algebraic systems for which relationships like the binomial theorem hold. Anyway, they contain a nice exposition of these ideas.

All three of Moore's preprints refer to the following two articles:

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Kari Eloranta
Partially permutive cellular automata
Nonlinearity 6 1009-1023 (1993)

Kari Eloranta
Random walks in cellular automata
Nonlinearity 6 1025-1036 (1993)
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which are in reality one long article split into shorter pieces. Besides the fact that Eloranta also uses blocking transformations to reduce the analysis to two-neighbor automata, and makes some use of the fact that the resultant binary mapping includes many algebraic systems of interest, our attention is drawn to the relationship between "partial permutivity" and Welch's indices. The first paper is combinarital and introduces subalphabets; the second is concerned with the statistics of kink diffusion..

procure Voorhees' book (as well as Wolfram's) and we feel that it will be just about right if they manage to master its contents.

Well, everyone will have their own environment and background from which to decide whether to use the book and at what level, so suffice it to say that it could be a textbook as well as a reference book.

As a reference work, both the author's own studies and Erica Jen's analyses occupy a good part of the volume, and invite attention for the details upon which they cast light as well as the opportunity they offer to generalize to other automata with diferent state sets and neighborhoods.

One example is the classification of automata according to the generators of the Garden of Eden, namely whether the number of poison words is zero, finite, linear, or (presumably) exponential relative to the length of the onfiguration. The first category corresponds to endomorphisms of the full shift, the second to the Subshifts of Finite Type, and the last two to Sofic Systems. As far as we know, this viewpoint relative to symbolic dynamics is not emphasized in the book nor elsewhere, although it is a logical extension of any study of Hedlund's point of view. If memory serves, the original motivation had to do with Feigenbaum's scaling theory, and involves still further considerations such as observing the rates of growth of those ancestors which actually exist when various kinds of boundary conditions are imposed.

When it comes to calculating ancestors, various techniques exist and have likewise had varying approaches and reasons for studying them. Regular Algebra offers some interesting symbolic calculations, not to mention the theory of factors and considerably more machinery. There are also numerical matrices, which form semigroups, with a whole well developed lore of semigroup theory waiting to be exploited. Voorhees has collected several semigroup tables, and has called attention to the possibility of having them further illuminate the properties of their cellular automata.

In practice, after getting considerable attention in the late fifties, including Samuel Eilenberg's treatises on monoids, semigroup theory has pretty much faded away from automata theory. To begin with, it is much more complicated than group theory, which in itself doesn't offer much engouragement to anyone who wants to evaluate long group products. But our recent reading of Hedlund suggests that there is much more to be gained by applying Welch's theory of indices, particularly when one wants to enumerate the reversible automata. Likewise Williams' theories of strong and weak equivalence, which has an impact on Voorhees' study of commutation and rule factorization, should open up a much greater range of equivalences for cellular automata .

At the time that inquiries were being made about the commutation of rules, there were also questions from other participants in the cellular automaton discussion group about whether it would be worthwhile to examine quadratic rules of evolution, considering the success which the study of linear (more properly, first degree) rules. This is another line of inquiry which has probably still not been exhausted. Voorhees was simply content to split rules into two parts, going on to concentrate on the linear part, although the splitting provides nice colorful diagrams. This is a subject which might bear reexamining, due consideration having been given to the new ideas which have been brought forward.

It remains to discuss the other new book, but to do this properly seems to be going to require some additional preparation. There is no question that it is an advanced research monograph, and although applications to cellular automata are promised in its promotional literature, there seems to be little doubt that the study of subshifts is an entire field whose long path in the pair diagram has to enter the diagonal; a similar definition embodies the other handedness.

Definiteness allows loops in the pair diagram which do not intersect the diagonal; this is where totally $(n-1)$-separated, bilaterally transitive, and such adjectives enter Hedlund's and Nasu's theorems. Injectivity says NO loops outside the diagonal, intersecting or not, which serves to fix the multiplicity M; from there on there should be no difficulty in applying Hedlund's third great theorem. Note that other authors sometimes use the adjectives "separating" and "closing."

In conclusion, surjectivity admits a variety of multiplicities running from injectivity to a full complement of counterimages, all with a wide variety of intermediates.

In summary, we have given a variety of arguments favoring the book under review, pointing out that there are few others treating cellular automata at a comparable theoretical leval. At the same time, it can hardly be overstated, that so far there has lacked a really elegant overview of cellular automata theory; for some years this placed it on the videogame level, where the attitude was "look at this nice fractal pattern I've just now generated!" But we also tend to get lost on another level, where there are elaborate formulas and theories to list the nice fractals, but the feeling of much computation (or theorem proving) for little practical gain, or understanding, remains.

Still, the book is a worthwhile contribution to furthering this understanding and we appreciate the sentiment involved with "My hope is that in describing the little that I have been able to see, it will encourage others to go further." Maybe that rascal in the front row; maybe the author himself; let us wait and see.

new books and old articles (11)

Date: Thu, 25 Apr 1996 23:53
To: ca@think.com

Substantially, our review of Burton Voorhees' new book, "Computational Analysis of One-Dimensional Cellular Automata" published by World Scientific in Singapore, with sales offices around the world, has been concluded. Amongst its other virtues, there are few other books primarily devoted to an exposition of the theory of cellular automata. Others have been conference proceedings, collections of articles, software or hardware manuals, and so on, and all have had their merits, but until now there has not been a textbook.

The author proposes it for a Junior level course; we had a discussion of this in our regular staff meeting a few days ago. For many years here in Puebla, there was a Sophomore course called Fortran III, dedicated to graphinf and graphics procedures that we used as a pretext to develop the theory of cellular automata (and the lcau programs). However, the students were never expected to do more than know some of the properties of cellular automata and prepare (or at least understand) the graphics displays. The following years there were courses on formal language theory and compiler construction in which things like regular expressions, finite automata, Chomsky's hierarchy, and the like were presented, giving an opportunity to look at cellular automata once again. Minsky's book, "Finite and Infinite Machines," or Aho and Ullman's dragon books were texts at this level.

In the meantime, we have a graduate level course on cellular automata theory in which all those things are supposed to be known, and the students have been told that they ought to as it is influenced by a topology in which locally distinct counterimages coalesce in the limit. At least one will survive, maybe more, and possibly all. This is governed by Hedlund's third great theorem and the Welch indices. Of these, the left and right indices behave sedately and multiply under composition. If left and right subset diagrams are constructed, the indices identify the strata on which the ergodic set of the subset diagram is to be found. Nasu's theorem, an interpretation of Hedlund's result, assures this.

If the automaton is not surjective, the ergodic set permeates the subset diagram, without necessarily encompassing all of it. The transients are also important and interesting. Even though Welch's indices describe the ergodic stratum, the acute observer will notice that Nasu's theorem describes an inequality, and that the statement of Hedlund's theorem has a proviso (as it turns out, resulting from and amply justifying all the preliminary attention to degrees of periodicity, recurrence, and so on).

For observing these details, it is useful to work with something more elaborate than Wolfram's (or Voorhees's) (2,1) automata; the Amoroso-Patt example with $(2, 3/2)$ automata gives the smallest binary instance, although there are some nice examples using $(3, 1/2)$ automata.

Consider the $(3, 1/2)$ automaton 14433, whose rule table is

	0	1	2
0	0	2	1
1	0	1	2
2	1	0	2

which is quiescent in all three states, but for which 1* has (02)* and (20)* as counterimages also. The de Bruijn fragments are row stochastic, so Welch's R index is 1. The reflected automaton follows rule 18369, whose subset diagram reveals that L is 3, a level to which the maximum compatible extensions arrive in two steps, with some dithering. This is also consistent with the two trees of height 2 rooted on states 0 and 2.

Were $LMR = 3$, we would conclude that $M = 1$; but by Nasu's theorem that is only a lower bound for the multiplicity.

If we now go on to construct the pair diagram and the triple diagram, we find some interesting features. The triple diagram exhibits exactly one loop, which means that the quiescent 1 background has three ancestors, and that it is the only such configuration, which is hardly uniform multiplicity.

In the pair diagram, there is a subdiagram containing internal loops, from which the diagonal has only incoming links. That means that there are some configurations (but not all) which have two ancestors, but there are crucial junctures following which both ancestors must coincide and continue to coindide thereafter, while running onwards to the right. That means that there are many configurations with two distinct $(n-1)$-separated (according to Hedlund) ancestors, and still others agreeing to the rightwards of some juncture.

If we ask what our dynamical systems theorests have to say about this, we will find the concepts of mergibility and definiteness.

With respect to the pair diagram, mergible means that the diagonal has no exits to loops. Thus any sufficiently long path (pair of paths in the de Bruijn diagram) has to have originated within the diagonal; the minimul such length is an index of mergibility. Alternatively, running in the other direction, long enough forces the beginning into the diagonal.

Definitiveness is a closely related concept; it says that there have been no entrances into the diagonal from loops. Thus (unless there are some completely exterior loops) any sufficiently a concentration on binary, first neighbor, automata; the same ones Stephen Wolfram studied so intensively (and let us not lose sight of the fact that Wolfram was one of the first to observe ALL the automata of a given class rather than just the one which served the moment's purpose). Additionally, the book gravitates around linear automata, those whose rule of evolution can be expressed via a multinomial of the first degree in variables representing the cells of the neighborhood.

Within these margins the book contains a wealth of techniques, procedures, and collected data, this latter dispersed in tables throughout the text and six additional appendices. Between most chapters there are collections of color plates embodying several of the author's techniques for visualizing the properties of automata; it is all quite colorful, and even possessed of a certain abstract beauty.

The reason authors, including Guan and He, or Martin, Odlyzko and Wolfram, amongst many others, have chosen to work with additive rules, was undoubtedly the perception that such rules would be susceptible to mathematical analysis while other rules might not be so easily treated. Nor is there much doubt that they actually obtained a large collection of results; if formulas for periods, transient lengths, sizes of basins of attraction, did not yield nice algebraic formulas, they did give closed expressions in terms of Euler's phi function, the Moebius inversion formula, or what not. Nice number theoretical results.

Unfortunately all of this activity has taken place against a background which has been better understood by some persons than by others, apparently without there having any single comprehensive picture. Even now, we can wonder whether it will be necessary to bring differential algebra (or finite difference algebra) in the style of Ritt and Kolchin into the picture, but the large body of knowledge implied by the work of Hedlund and his successors was not widely disseminated amongst automata theorists, and even those who were aware of the work may not have been fully aware of its implications.

Speaking more personally, it is only lately that we can even claim to be able to read Hedlund's articles, and that only as a result of longtime efforts and a more intensive seminar lasting months. Obviously others have had advance warning or better preparation, but there is evidence that even those concepts which have reached across from one field to another have been as fully appreciated as they might have been. Two or three years ago, when the cellular automaton discussion group was approached by Prof. Voorhees seeking information about the commutativity of cellular automaton rules, did anyone suspect the relation to Williams' equivalences, or even think to cite Rhodes' paper? We might have saved him a bit of trouble if we'd have known these things.

Well, to flog a dead horse just a little bit more, there is still this question about the number of counterimages of a surjective cellular automaton map. There are several answers, depending upon the level at which they are asked. At the local mapping level, there are always $k^(n-1)$, for k states an neighborhood content n. Mirroring this result in de Bruijn fragments is another story, but the result is clear and definitive enough.

An immediate complication is the fact that the counterimages need not have the same period as their sequences, although there are limits and divisibility relationships. Therefore it one works with basins of attraction for periodic automata the fact that these larger counterimages exist may go unnoticed; the proper interpretation of a Garden-of-Eden result for such automata should take this into account, just as it should note that there is only a finite number of the wider counterimages.

Another complication lies in the transition from local mapping to global mapping, especially out, we can decide whether rules compose from the left (category diagrams) or the right (matrix products). A truly delightful experience awaits the unwary, or at least suffice it to say that lcau21 produces a composition table quite unlike the book's Table 4.3.

The reason for noticing this discrepancy results from trying to exploit the fact that Grassberger's Rule 18 is a composite of the (2,1/2) or and xor, and that many studies of its behavior depend on that fact. Rules 126 and 252 also factor, so it is tempting to pick up commuting triplets by associating terms in a composite (which is not necessarily the only way commutation can arise).

Generally speaking, textbooks which provide ample tables listing such items as composite rules and their factors, edge-insensitive, permutive, edge-linear, or other specialized automata, all serve to aid whomever is searching for patterns and relationships amongst automata. So we can only praise our author for the tabular information he has provided us. To balance the compliment with a quibble: "compliment" is when you say something nice to or about a person; "complement" is when you fill in the rest of the rule table, or finish some other undone task, or in electronics, invert the sign of the signal you're processing (Just as those who live in glass houses shouldn't throw stones, those who never learned touch typing shouldn't criticize authors' penmanship, but it's still a temptation).

To summarize today's discussion, chapter 4 describes idempotence, Ito's relationships, and existence (including symmetry properties) of constellations, something far greater than clusters, constructable using the author's generalization of Ito's, Grassberger's, and similar, observations. By their formulation, constellations tend to share invariants such as entropies (or whatever appropriate interpretation of the Perron eigenvalue) which in turn creates similarities in their patterns of evolution.

new books and old articles (10)

Date: Mon, 22 Apr 1996 23:33:17
To: ca@think.com

Someone looking for a textbook on cellular automata theory, which treats the theory on its own merits wherein the underlying lattice is a fundamental part of the topic rather than appendage to the theory of finite automata, would do well to consider Burton Voorhees' "Computational Analysis of One-Dimensional Cellular Automata" which we have been discussing. As his preface says, "The actual mathematics used is not hard, and the material should be available to anyone with a junior level university background, and a certain degree of mathematical maturity." Matrices are for multiplying but not diagonalizing, the only finite field is a boolean algebra and the algebra never mentions Galois theory explicitly, but there IS a contour integral in theorem 5.13.

Text book or not, it is one of the few places where the "advanced" theory of cellular automata may be found, meaning the use of the de Bruijn diagram for deriving properties of automata, concern with ancestors, the Garden of Eden, surjectivity and injectivity, and all those aspects which go beyond the mere phenomonology of observing evolution and compiling statistics about it. Aside from the one chapter on time series and some discussion of entropy, the book makes no attempt to relate automata theory to statistical mechanics, which is a whole field for investigation in its own right.

On the other hand, it strongly reflects the research interests of its author, which accounts for

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P. Grassberger,
"New mechanism for deterministic diffusion,"
Physical Review A 28 3666-3667 (1983).
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However, the book skips over a step in this saga, because it seems to have been Voorhees himself [bibliography 5] who noticed that Ito's coding from one rule to the next could be accomplished by a cellular automaton, whose rule was 252. Anyway, the idea was born that there were constellations of cellular automaton rules with related properties. Of course, that was based on empirical studies of a variety of automata as much as from any theoretical expectations, gaining strength as more and more theoretical underpinnings were discovered.

It is also true that it is possible to think about commutation amongst rules without connecting that with the commutative diagrams of category theory as Williams has done, but the ideas cannot be very far from one another. Another Voorhees article [bibliography 13, actually Complex Systems 7 309-326 (1993)] [to also complete bibliography 14: it is now Complex Systems 8 151-159 (1994)] refers to similar concepts circulating in Hedlund's circles, originating with

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E. Coven, G. Hedlund and F. Rhodes,
"The commuting block maps problem,"
Transactions of the American Mathematical Society 249 113-138 (1979)
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Here we have an interesting cycle, in which persons interested in mathematical communication theory and dynamical systems, thinking of composite mappings as cascaded shift registers, got interested in how their order of connection could affect the final result, taking up first degree mappings, at least with respect to an edge variable, as a case susceptible to analysis. Then the book's author, having developed a symbolism adapted for first degree mappings, comes along with a new interpretation [bibliography, 13]. Where will it all end? Not until Lagrange Interpolation Theory, expecially in finite fields, subsumes all these diverse mapping techniques into a single coherent approach, following which Hedlund's procedure is brought into play once again, this time for merely continuous mappings, not only the endomprphisms and automorphisms. And even then it probably won't be all over.

We say "constellations" to distinguish them from clusters as the latter appear in the Wuensche-Lesser Atlas or Hillman's determination of reversible automata, because they represent still larger groupings, something which is possible because they are not confined to surjective or injective mappings alone.

Empirical analyses of automata are replete with enumerations of restricted configurations which receive identical treatment with respect to two or more different automata. If any of those restrictions could be realized by defining classes of subshifts, they would thereby become susceptible to the theories of symbolic dynamics. In a case which will be discussed later on in more detail, Erica Jen has classified automata according to the size of the generators of the Garden of Eden - null, finite, countable, worse - which clearly relates the Subshifts of Finite Type to the second category.

In actually working with the concepts of Chapter 4, we see how careful and consistent one must be, between authors and (surprise!) with the same author. There are Wolfram rule numbers and Perry Rule numbers, reflecting two possible orderings of the digits in a number. Then, there is the question as to whether the automaton will be referred to by its christian name (and, or, xor) or its social security number (6, 12, 14). Once this has been straightened of evolution are algebraic conditions for their solubility can be discussed algebraically, especially for linear and linear inhomogeneous rules.

Reference is made to the original determination of Amoroso and Patt, to Hedlund's theorems, and to the de Bruijn and Subset diagrams. Some interesting diagrams are shown which could be used to calculate the products of symbolic de Bruijn diagrams, the ones which originate from the "vector subset diagram."

new books and old articles (9)

Date: Sat, 20 Apr 1996 21:04:23
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In Computational Analysis of One-Dimensional Cellular Automata just published by World Scientific, Burton Voorhees undertakes to "give an introduction to the analysis of cellular automata (CA) in terms of an approach in which CA rules are viewed as elements of a non-linear operator algebra, which can be expressed in component form much as ordinary vectors are in ordinary algebra."

This is a very distinct undertaking from trying to formulate the operations and concepts of cellular automata theory in terms of matrices insofar as possible, and also from the tradition in communication and coding theory to work with finite fields, recursion relations, and generating functions. Yet that approach is entirely consistent with with the book's emphasis on rules of evolution described by first degree polynomials while regarding other rules of evolution as perturbations on this basic theme.

The book contains many tables constructed in furtherance of these ideas, of which the ones associated with the sections concerning commutators deserve some attention. Relative to the background expressed in this review's preliminaries, commutators for binary automata are closely related to Williams' topological conjugacies. In other words, if X any Y are two rules of cellular automaton evolution, there may be a third rule T for which the composite two-generation rules XT and TY are the same. The relationship is more familiar when T is invertible; even so, there is always some degree of equivalence, even at the extreme where T is zero ("everything is equivalent, almost").

The historical precursor of this relationship is

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Hiroyuki Ito,
"Intriguing Properties of Global Structure
 in some Classes of Finite Cellular Automata,"
 Physica D 31 318-338 (1988).
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which worked out basins of attraction for the (2,1) Rule 18 (whose properties had been extensively investigated by Peter Grassberger) and the Rule 126 whose evolution is closely similar, just with thicker edges on all the triangles. The resemblance was stronger than that, plots of the distribution of cycle lengths, size of basins of attraction, and so on, coinciding almost exactly for the two rules.

Ito worked out a mapping from rule 18 to rule 126, and verified the resemblance between the rules by detailed alculations of their basins of attraction, which we might do nowadays with the "A and B matrices," or Voorhees' d0 and d1. Ito credits Grassberger with the reverse mapping, published in

10. the Garden of Eden

11. time series simulation

12. surjectivity of cellular automata rules

    The final four chapters concern more general issues, still for one dimension, but not confined exclusively to additive rules. Generally speaking, it is known that there is a strong imbalance in the number of counterimages for different configurations, as well as in their propagation backwards in time. Most often, either zeroes or ones will dominate, the effect growing more and more pronounced as longer configurations are examined, and as they are followed further backward in time. Of course, this is just the counterpart of the concentration and formation of basins of attraction seen in forward evolution.

    Sometimes uniform stretches lose out, yielding their popularity to configurations of simple periood, such as alternating zeroes and ones. There is a community of rules for which simple periodicity dominates, but there are still minorities of rules in which the dominance passes to longer periods with more complicated unit cells. Finally there is a realm in which absolute democracy, in the form of Hedlund's second great theorem, the one about uniform multiplicity, prevails.

    Chapter 9 introduces another specialty of the author's, which he calls the basic matrix. Instead of creating the connectivity matrix of the de Bruijn diagram by placing a 1 where the partial neighborhoods can overlap, or of creating a symbolic matrix by placing the image cell at that location instead of a 1, this time the links tell whether there exists a neighborhood of double length, which starts with the row index, but which evolves into the column index.

    Trying to multiply such matrices would imply a tower of neighborhoods such that paths in the connection matrix would describe consecutive initial segments of the members of the tower, as successive generations evolve. But this is not the use to which these matrices are put; rather longer and longer segments are followed through a single generation, yielding a sequence of matrices with fractal structure, supposing that indices are mapped into "decimals" the same way they were in earlier chapters. Samples for fifteen different (2,1) rules are shown with the help of another collection of color plates.

    Quite a few questions can be asked about the Garden of Eden. One arises from relating periodic configurations to arbitrary configurations, even without going into all the periodicity refinements which characterize symbolic dynamics. The simplest complication arises when a counterimage has a longer period than that of the lattice of the configuration. When rules aren't surjective, there is still characterizing growth rates, and growth rates subject to a variety of constraints.

    Another possible question is: what are the rules exhibiting prescribed "poison words" or groups of poison words. This is like characterizing subshifts, and is altogether complicated by the fact that it is hard to create a subset diagram to order, in the way that it can be done for boolean properties of the de Bruijn diagram.

    Chapter 11 ventures into the arena of time sequences for an automaton, as distinguished from spatial distributions. Interesting results exist.

    The book concludes with a discussion of surjectivity, to be found in Chapter 12, a topic of interest for discovering reversible cellular automata; particularly so when the rules

Family relationships include factoring rules, enbedding them in larger neighborhoods, and restricting the neighborhood. The rule algebra of chapter 2 has the defining of rule commutativity as one of its goals, something which is also one of the author's specialties, explained at greater length in chapter 4, and which probably has additional relevant interpretation in terms of William's commutative diagrams.

The study of fixed points and cycles comprising the third chapter has an elegant formulation in terms of de Bruijn diagrams, something about which Erica Jen was amongst the first to publish. In fact there is much reference to her detailed classification of (2,1/2) and (2,1) automata, much of it in terms of recursion relations and number theory.

5. additive rules I. basic analysis

6. additive rules II. cycle structure and entropy

7. additive rules III. computation of predecessors

8. the binary difference rule
   The middle part of the book is devoted to explicit calculations, as they can be carried out over the boolean field GF(2). Thus, Shannon's canonical form for boolean algebras is replaced by the nearly identical Lagrange interpolation basis for the field, exclusive or replacing the inclusive or as the algebra's sum.

   Chapter 5 introduces a technique for visualizing the additivity, or lack thereof, as introduced in Chapter 2. It produces something similar to the plaid diagram, but it displays, for the one-sided shift, whether or not configurations i and j obey f(i+j) = f(i) + f(j). Since this all depends upon whether neighborhoods themselves obey this property, the result is a nice fractal picture, in the topology of the shift. For the (2,1) automata, fifteen color plates (the other one is identically zero) illustrate nontrivial "obstruction classes;" the ways in which rules depart from additivity.

   Periodic additive rules admit matrix representation for the evolution of rings of cells; typically by circulant matrices whose eigenvalues are described by Tchebycheff polynomials and whose eigenvectors form discrete Fourier transform matrices, although the results eventually need to be referred to a finite field. The author and others have made such analyses. Whether or not the rules are injective is always interesting, to which Hedlund's third great theorem concerning the Welch indices, is relevant. Although periodic configurations have periodic ancestors, those periods do not always coincide.

   Running forward in time lends interest to knowing the onset of periodicity, the height, leafiness, and convergence ratios of the tree of evolution, its description by such measures as entropy, and related data. It is no less interesting to run backward in time, obtaining the same information in terms of ancestors. In the process Langton's parameter lambda is encountered, not to mention all of the interesting results in Wuensche and Lesser's Atlas.

   For linear rules, the same opportunity exists to use linear algebra, recursion relations, and finite field theory to calculate ancestors that was used for the calculation of evolution. Additionally, the subset diagram and symbolic de Bruijn matrices can be used.

   Chapter 8 is dedicated to the bellwether of all additive binary rules, the exclusive or.

9. computation of preimages

product from which some coordinates have been ignored. We see that this is a necessary feature of such rules; it would be interesting if a Fredkin factorization or something similar could actually be exhibited for all reversible rules.

new books and old articles (8)

Date: Wed, 17 Apr 1996 22:33:55
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Of the two books, Voorhees' Computational Analysis of One-Dimensional Cellular Automata by World Scientific would be more suitable as a textbook, while Nasu's Textile Systems for Endomprphisms and Automorphisms of the Shift from the American Mathematical Society is essentially an advanced research monograph. Both are neatly prepared, apparently from TeX prepared by the author in Nasu's case, and something akin to ChiWriter for Voorhees' book. In either case, the mathematical text seems to have been meticulously prepared, yet both contain one or two idiosyncracies which either referees or some copy editor ought to have detected: Nasu's use of "weaved" where "woven" would probably be more gramatically correct, and Voorhees' pluralization of -x words using -icies. If there are ever second printings, electronic correction of these anomalies should be a triviality.

As described in its announcement, Voorhees' book has twelve chapters, six appendices, and a bibliography (of 89 items), all for a total of 275 pages. Part of the introduction is a bibliography of fourteen of the author's own papers, whose subject matter is consistent with the book's emphasis on one- dimensional automata of Wolfram's type type (2,1), and with additive rules at that. This is a realm for which reasonably explicit calculations are possible, from which concrete results can be used to treat the topics of the last chapters, wherein surjectivity, injectivity, and the calculation of ancestors are considered.

There are many diagrams, illustrations, and tables, including the compilation of results in the appendices. In keeping with its possible use as a textbook, each chapter has a handful (that is, ten or so) of exercises. The book breaks down more or less according to the following lines:

1. operator algebra of cellular automata

2. cellular automata arithmetic

3. fixed points and cycles

4. commutation of CA rules
   The preliminary portion of the book sets out the usual definitions for cellular automata, with explicit reference to Hedlund's first great theorem. In so doing, the author prefers to work with either the one-sided shift or with periodic sequences; even so, machinery is required to center neighborhoods, especially to facilitate describing composite rules.

   One of the author's specialties has been to work with a sort of algebra of automaton rules, which is part of more extensive efforts to deduce family relationships amongst automata. This algebra foresees giving additive (and linear homogeneous) rules a privleged position amongst all rules, in no small part due to an intention to use linear algebra over finite fields or else integers modulo n to study their evolution.

from the left and right subset diagrams (sets of subsets, no less) and counting theorems which Hedlund states and Nasu reiterates.

The counting theorems produce inequalities, which become equalities when shifts are bilaterally transitive. What does this mean? It can be spotted rather quickly in the pair diagram. Reversibility, or automorphism, once the subset diagram has verified surjectivity, requires that there be no loops in the subset diagram outside the diagonal. There may be transients, whose length and handedness can be related to Welch's indices, but no loops.

Mere surjectivity, or endomprphism, allows additional loops outside the diagonal, which correspond to Hedlund's "totally (n-1)-separated shifts," but there is another possibility wherein the diagonal connects unilaterally to an exterior graph. That would mean, say, that from some point onward, counterimages which extended uniquely to the left, could diverge to the right, even while precluded from ever rejoining. Apparently this is a combination in which the Welch product LR attains its maximum value $k^(2r)$, yet the multiplicity M is still larger than 1. So the quibbles in the theorems are actually to be taken seriously.

Someone who would like to check this out with an actual example might examine the $(4, 1/2)$ cellular automaton CBAD1670, whose rule table is

0	0	3	1
2	1	1	0
1	3	2	2
3	2	0	3

The rule has been planned to be fully quiescent in all its states; for reversible automata there is such a rule in every cluster, which need not hold for automata which are merely endomorphisms.

Examining configurations with spatial period 2, some of them lack ancestors of period 2, but rather the least period of the ancestor is 4. Examining either the pair diagram or the subset diagram shows that the phase between two such ancestors can slip exactly once, without any chance of recovering it again.

The left, or in-link, subset diagram (seen by examining the reflected rule E127B4D8) has Welch index $L = 1$, given that the unit classes are closed under in-linkage, whilst the right, or out-link, subset diagram has Welch index $R = 4$, given that the full set can be reached from the unit classes. Which means that it is a single-point image of the de Bruijn diagram.

Knowing that there is a fully quiescent rule in each cluster of a reversible rule, and using Nasu's theorem about the tree rooted on each quiescent node, the de Bruijn diagrams, or equivalently, the rule tables, for reversible $r = 1/2$ automata can almost be written down by inspection - especially when the number of states is a prime number. Endomorphisms are more complicated - think that only 0 is quiescent for (2,1) XOR - and the possible values of Welch's M must be considered - but there is much less to enumerate while searching for endomorphisms than there would be in considering all possible rules, even after insisting on uniform multiplicity.

When the state set has composite order, even when the automaton is reversible, it is possible to split the indices between the two sides. Nasu's diagrams for Amoroso and Patt's reversible (2,3/2) automaton show the factorization 8 = 4*2; programs available nowadays permit the construction of many more examples.

The importance of the indices can be seen in another respect; anyone who has used Fredkin's scheme for constructing reversible automata will have noticed that it depends on a cartesian that they have always been easy to understand. Partly this is due to their origins, in which sequences arose from other circumstances, such the iteration of a mapping or the solution of a differential equation. To bring understanding to sequences originating haphazardly, for all that was apparent, many kinds of periodicity were invented starting with strict periodicity, but eventually extending to recurrent, which meant that sometime, no matter how long it took, any given combination was bound to repeat, and then again, and again ... .

For that reason, Hedlund examines various backward and forward periodicities, with the intention of seeing how they are treated by the hierarchy of mappings which he distinguishes - continuous, surjective, reversible. Meanwhile, studies of subshifts take up a similar range of concepts, not always with the same vocabulary, and varying somewhat from author to author.

Were it not for concrete results, there wouldn't be much reason for theorists of cellular automata to pay too much attention to all the classification. But not only are there definite results, their counterparts in automata theory have often been overlooked, or were simply unknown. For example, much of the theory of reversible automata has lacked the benefit of the insight which symbolic dynamics could have provided.

By now, the basic theorem that cellular automata are the manifestations of continuous mappings of the shift is quite generally known and often cited. Circumstances surrounding the next result, uniform multiplicity of surjective mappings, are fairly murky, even though the result is also widely known. The difficulty arises in explaining how multiple counterimages can be guaranteed and yet only one of them is evident when the automaton is reversible.

The answer that they are indistinguishable in the limit is somewhat misleading, because unless the causes of indistinguishability are evident within a finite and calculably small distance, they will never occur. In other terms, it is all a matter of boundary conditions, and the rate at which their influence diminishes - either almost at once or never.

Still, as Richard Feynman is reputed to have once said, in giving a "physical" reason for numberical instabilities in the solution of differential equations, "You can't hide heat!" For reversible automata, the aphorism seems to be that you can hide multiple counterimages, but you can't ever get rid of them in their entirely, absolutely, completely ... .

This brings us to Welch's indices, defined for surjective mappings, and Nasu's theorem that their subset diagram contains an image of the de Bruijn diagram. It is instructive to work this out, and to observe that although it is an image, it is not always the same.

The subset diagram reveals the Garden of Eden in its entirity, but still it is not nearly as convenient for ascertaining actual counterimages. From its definition, if there is a path in the subset diagram, there are corresponding paths in the base diagram such that every point in the head subset has a path connecting it to some point in the tail subset, but not necessarily in the opposite direction. Consequently unravelling the actual path is just as much work as constructing the diagram in the first place. One solution to the problem is to define a "vector subset diargam;" another is to work with symbolic connectivity matrices, but that is another discussion for another time.

The uniform multiplicity theorem has its consequences for the subset diagram, This shows up in defining the "maximal compatible extensions" which feature in the definitions of Welch's indices, by creating a structure of ideals in the subset diagram (a fancy way of saying that the extensions of maximally compatible subsets are maximal). Paths in the base diagram, which always exist on account of surjectivity, must thread their way through the subsets in the subset diagram; they can coalesce, but never in any way creating too many paths with the same labelling. The result is a series of requirements for the intersections of the ergodic sets that $UB = AU$ with the discrepancy in dimensions being absorbed by zero eigenvalues and associated eigenvectors.

It is also possible that neither matrix is supposed to be diagonal, in which case there could be matrices reducing each to diagonal forms (Lambda) and (Mu), respectively. Then it could be said that $A = UV$, whilst $B = VU$, wherein U and V are variants on the diagonalizers (essentially multiplied by another diagonal matrix). So there is a kind of equivalence relation, which Williams calls strong equivalence, wherein two matrices are equivalent if they arise from the product of two (usually noncomuting) factors, or from a chain of such factorizations. From this it iseasy to get R and S for which $RA = BR$ and $BS = SB$.

On the other hand, a little algebraic manipulation provides matrices U and V and an arbitrary function f for which $Rf(A) = f(A)R$ and $f(B)S = Sf(B)$; often f is simply a power, yielding Williams' weak equivalence. Unravelling its chain of factors shows that strong equivalence implies weak equivalence; it is moreover what turns up in the numerical LR method of diagonalizing matrices, so it is not such a strange concept.

One of Williams' great concerns was whether it worked the other way around, so that once the powers behaved, a chain of factorizations could be discovered. Given that the factorization which defines a matrix and its dual would lead to a nicely intuitive dual tower, it would be nice to characterize equivelent shifts-of-finite-type as those connected by a chain of duality.

Anyway, it is useless to attempt a homomorphism between two shifts if their matrices do not have at least one eigenvalue in common; moreover for reasons having to do with positivity, at least one of the pairs had better be the largest, Perron, eigenvalue.

Working with automata which are known to be reversible, just as with those without Garden of Eden, there is ample opportunity to factor the matrices in the de Bruijn diagram, because of the property in each fragment that there are just as many links as nodes. Of course, the links may be numbered any way that one wishes, leading to an ambiguity according to the permutation group for the links. This is something to be borne in mind for the later discussion.

This review of symbolic dynamics has been undertaken for the bearing it might have for the description of the two new books; it is essentially material which we have just recently studied in detail, in spite of having known about it for several years. Offline correspondence with David Hillman was instrumental in calling our attention to the significance of Williams' series of papers. At the same time, the improved versatility of nxlcau as we learn better how to use its graphics facilities has allowed checking such things as the relation of Welch's indices to the subset and pair diagrams. Mainly we can now save on paper what could only be watched on the screen in the DOS version.

The approach to these questions which we have previously used has already been explained in detail during the course of the review of Wuensche and Lesser's "Atlas" so there does not seem to be much reason to go over it again, except as it may be needed in discussing specific details. That viewpoint worked directly from the trinity of graphs, using tensor powers of the de Bruijn fragments to deduce properties of the automaton.

new books and old articles (7)

Date: Tue, 16 Apr 1996 23:35:55
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The background articles which we have reviewed are classics, but that does not mean

Naturally it is a temptation to study possible relationships between subshifts, especially automorphisms and endomprphisms, just as Hedlund did for the full shift. The results do not seem to be quite as conclusive, there being numerous articles devoted to the subject and its variant. The most comprehensive seems to be

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R. F. Williams,
"Classification of subshifts of finite type,"
Annals of Mathematics 98  120-153 (1973),
with Errata ibid 99 380-381 (1974).
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which depends upon the previous

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R. F. Williams,
"Classification of one-dimensional attractors,"
in: Global Analysis,
Proceedings of Symposia in Pure Mathematics, volume 14
American Mathematical Society, Providence, Rhode Island
pages 341-361 (1970),
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which in turn harks back to the still earlier

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R. F. Williams,
One-dimensional non-wandering sets
Topology 6 473-487 (1967).
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All three articles contain an abundance of category-theory diagrams. Much of that is caused by relating one-sided shifts to two sided shifts, and more is due to situating the derivations in the abstract realm of mappings between topological partitions. Basically, the idea is that equivalences between shifts should correspond to equivalences between their defining matrices. One complication is that the matrices have positive integer elements, possibly just zeroes and ones, and they are rarely symmetric. Thus the Jordan form may be non-trivial, and the equivalences ought to be expressed by the same class of matrix. Even avoiding inverses by making A equivalent to B via the mapping R by demanding AR = RB, there are questions of algebraic propriety.

The principal determinant of the equivalence between subshifts seems to be their entropy, which can be defined topologically, measure theoretically, or algebraically, as the (logarithm of the) largest, or Perron eigenvalue of the matrix from which they derive. Unfortunately, that alone does not suffice; more of the Jordan canonical form being needed, with restrictions caused by the admissible matrix elements.

The final items on the list of Nasu's publications in Part (4) also contain discussions of mappings between Shifts of Finite Type, as well as between Sofic Systems.

For our immediate purposes, the interesting information to be derived from these articles is their reliance on an old property which one learns while studying linear algebra or numerical analysis, but then forgets for lack of an apparent application. The usual definition of equivalence between matrices A and B is that some matrix U satisfies the equation $B = U^{-1}AU$, commonly exhibited when B is a diagonal matrix. Something of the sort still works when A and B are square matrices of different dimensionalities; for rectangular U it is still possible for solving systems of symbolic equations. In particular, Conway's factors describe how sequences can begin or end, this relates to dieals in the regular algebra, and to the ideas of definitiveness and mergibility. These associations need to be explored with greater care. The book is

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J. H. Conway
Regular Algebra and Finite Machines
Chapman and Hall, Ltd. (1971).
ISBN 412 10620 5
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the reference to Backhouse and Carré is

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R. C. Backhouse and B. A. Carr\'e
'Regular Algebra Applied to Path-finding Problems'
Journal of the Institute for Mathematics and its Applications
15 161-186 (1975).
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Finally, we need to comment on the relationship of Shifts of Finite Type to Sofic Systems. Finite Type makes the structure of a subshift depend on the characteristics of one single matrix, whereas either several matrices (one for each label), or a system of labelling, is necessary to discuss Flows in a Labelled Graph. With Finite Type two birds are killed with one stone by insisting that the vertices ARE the labels (or at least wounded, through the byzantine device of using an extended alphabet). Sofic Systems resolve the problem by passing to the dual graph (in effect, putting primes on repeated vertices), but that adds an inconvenient extra endomprphism to the discussion. Whatever the reason, the literature fails to clarify this point.

new books and old articles (6)

Date: Thu, 11 Apr 1996 22:42:51
To: ca@think.com

It is nearly time to take up with the new books, but it may be worth a final look at a few more background articles. The role of Shifts of Finite Type and Sofic Systems in cellular automata theory is still not entirely clear, mainly because they seem to have arisen from other interests.

In the process of topologizing symbol sequences, continuous functions distinguish certain subshifts, namely those which can be the images of the full shift; On the one hand this characterizes cellular automata as having the continuous functions for their rules of evolution, while on the other, subshifts which exclude a countable list of words show up as the resultant images under continuous maps.

There is already a relationship between language theory and cellular automata, wherein languages define the admissible sequences so that their transformation by the evolution of the cellular automaton can be studied; some recent articles are devoted to this exact issue. For whatever reasons, students of differential equation theory, among others, decided to use sequences defined by paths in a certain kind of graph whose connectivity matrix defines the exclusions. Having found that the resulting class was not closed under homomorphism, they decided to close it by introducing sofic systems; essentially beginning with the dual of the matrix from which subshifts of finite type were taken.

Further connectivity properties concern whether or not paths exist between pairs of points, and in which directions. Naturally, when one constructs a derived graph, questions of extensibility and connectivity transfer to the derivative; for example finding a leaf in some subset makes that subset a leaf, but there are others - for example when the subset contains some but not all of the outwardly linked nodes of a woud-be counterimage.

The subset diagram has its own connectivity matrix, whose powers can be used to discover exactly which words generate the Garden of Eden, and what their lengths are. It is a more subtle operation to enumerate the ancestors of those paths which actually have ancestors, which is why the "vector subset diagram" was invented, and why Nasu sometimes ties paths to an origin.

Absent the Garden of Eden, the evolution rule is a candidate for endomprphism, at which point Welch's indices come into play, as well as Nasu's observation that [his portion of] the subset diagram is "well defined" (as an image of the de Bruijn diagram). Subset size (cardinality, to be more formal) stratifies the subset diagram, the indices state which level (yea, each index unto its own handedness); there is also an interesting intersection property relating subsets of the left extensions and those of the right extensions.

There is nothing in the subset diagram which could not have been taken directly from the de Bruijn diagram, but somehow its use seems to clarify the exposition markedly. The same could be said of the pair diagram, which is the third member of the trinity.

By definition, the pair diagram is constructed from the cartesian product of two copies of the vertex set of its base diagram, pairs being linked when both members are linked. If the graph is to be labelled, the label of a pair link is the common label of its coordinates (other labellings are possible, but that is another matter, and gives different graphs). Much the same graph arises when unordered pairs are used instead of ordered peirs, but in either event, the diagonal is (isomorphic to) a copy of the base diagram.

When evolution labels the pair diagram, the existence of paths outside the diagonal implies distinct ancestors. Transients and connectivity in the pair diagram are a consequence of the same qualities in the base diagram, but now a pair can lose linkage when the coordinates behave differently. Note also that the pair diagram is vaguely part of the subset diagram, except that it loses linkages whereas they would move to a different level in the subset diagram. All of Hedlund's discussions concerning (n-1)-separatedness can be read as descriptions of the pair diagram.

Usually the pair diagram is consulted after having used the subset diagram to characterize the Garden of Eden; its emptiness assures the existence of paths in the pair diagram, so the interesting questions concern whether or not a path includes the diagonal.

Paths which contain loops cannot intersect the diagonal without being wholly contained, otherwise counting all possibilities would show a violation of the uniform multiplicity theorem. This is Hedlund's case of total (n-1)-separation. This is also related to mergibility - eventually sequences forget how they began, but only if those which could extend indefinitely in either direction have been excluded [Nasu's theorem 5 (2)].

That leaves transients which have nowhere else to go (or come from) except the diagonal, which [theorem 5 (1)] recognizes as definitivity. Similar distinctions, with variant vocabularies, are to be found in other articles dealing with symbolic dynamics.

Besides the approach from Symbolic Dynamics, much of what is nevessary to understand one dimensional cellular automata can be found in John Conway's book, "Regular Algebra," particularly if it is supplemented by a paper of Backhouse and Carr'e dealing with procedures

The last two portions of this series described the contents of Hedlund's often cited article, which is sometimes taken as a theoretical basis for cellular automata theory, and Nasu's first article whose graph theoretical orientation contrasts with the topological approach of Hedlund. There remains to be discussed Williams articles on the classification of Shifts of Finite Type. Before then, it might be well to describe the graph theoretical basis of the subject.

It has been convenient to call the basic structure a de Bruijn diagram, even though a similar structure was introduced contemporaneously for similar purposes by I. J. Good, both dating from 1946. The ideas apparently had an earlier history, and one cannot help wondering whether the concepts had cryptographic overtones. Shannon's noiseless channel had a similar diagram. Our awareness of the diagrams was a resut of their inclusion in Golomb's book on shift register theory, where they serve to describe the progression of overlapping segments while stepping through a long symbol sequence.

In the realm of one dimensional cellular automata, considerable information can be read directly from properly labelled de Bruijn diagrams, especially when the nodes are sequences just one short of a full neighborhood of cells so that the links join those which overlap, thereby representing complete neighborhoods. Not only neighborhoods, but any of their whole panoply of boolean functions, can be used to label the links, themselves directed arcs because the direction of overlap matters. Labelling the links by the cell into which the neighborhood evolves is especially useful for the reversed attribute of singling out ancestors, which is the implicit relationship to be deduced from Hedlund's article and explicitly spelled out in Nasu's.

By now this relationship is reasonably well known, not the least for having been mentioned regularly in CA-Mail discussions. Two extensions of the diagram are important for developing the theory, namely the subset diagram and the pair diagram. The former is a standard construction in automata theory, used for giving a systematic answer to the question of whether the original diagram does or does not contain a given labelled path. The unit classes for the subsets are the individual nodes of the de Bruijn diagram, subset linkage being defined by linking the subset to the union of the linkages in the base diagram and assigning them the same labels.

The subset diagram serves at least two purposes. First, linkage in the base diagram is not a function because nodes may have multiple out-links, so the subset diagram is a covering space in which linkage is functional, and it might be surmised that it is the least extension having this property. In passing, let it be noted that there are really two subset diagrams, according to whether in-links or out-links are functionalized. Both Hedlund and Nasu enjoy formalizing this symmetry by stating "vector" theorems, but that is a question of style.

Second, the diagram systematizes the search for a path; since it is functional it is only necessary to check paths leading from the full set to the empty set (having exploited the polite mathematical fiction that no-link is a yes-link to the empty set). Its use also avoids having to use the mathematical circumlocution "non-deterministic automaton," but of course there are also drawbacks. One of them is the huge size of the subset diagram, since it grows exponentially with the size of the base diagram. As Voorhees points out and Nasu obtains by not formally mentioning the subset diagram, only a small part of it may be needed in practice.

For any graph, it is customary to discuss a series of connectivity attributes. Nodes laking in-links are leaves (those lacking outlets are rootlets?), which can be extended recursively to those successor nodes having incoming paths of bounded length. That leaves a residue of nodes whose paths can be arbitrarily prolonged in one direction, the other, both, or neither. For a finite graph, unlimited continuation implies loops.

as a "bundle graph." There are actually two graphs, depending on whether union is taken with respect to in links or out links. The important results are that paths for surjective automata are unique between endpoints in the de Bruijn diagram and generally Hedlund's equivalences hold, even in a more general graph theoretical case depending only on uniformity of indegrees and outdegrees.

Welch's indices and maximal compatible extensions are related to the subset diagrams; when the automaton map is an endomorphism, the important results are that all continuations from maximal nodes are maximal, there is a maximal node above each and every unit class, and their level in the subset diagram is given by the L or R indices. TYhe important Theorem 1 asserts that the diagram in the subset graph is well defined (an image of the de Bruijn diagram) just exactly when dealing with an endomprphism.

4. CsubF-surjective local maps
   Finally, the consequences of quiescence at infinity are examined, and summarized in theorem 2. Some trees are required.

5. Mergible local maps

6. Definite local maps
   Even after the subset diagram has been mastered, there seems to be a problem in dealing with the pair diagram. Thus treatments of the shift tend to go into details of which sequences are called closing, resolving, merging, separating, definitie, and so on. Once the pair diagram has been constructed from the de Bruijn diagram, it is necessary to examine its transients, connected components, and the relation of all them to the diagonal; the Welch indices have something to do with the length of transients, just as they relate to levels in the subset diagram. The pair diagram is not used here.

   Bibliography of 20 items

new books and old articles (5)

Date: Tue, 9 Apr 1996 23:24:52
To: ca@think.com

Cellular automata theory has a history beginning with von Neumann's universal constructor and Moore's Gardem of Eden theorem, continuing with attempts at classification, organization, and simplification, and eventually attracting widespread public notoriety with Martin Gardner's publication of Conway's "Game of Life" in Scientific American.

Some years later, Wolfram's simulation experiments with a minicomputer and the construction of the CAM series originated by Toffoli revived the subject by allowing people to visualize the workings of cellular automata on a large scale, and compare the results with models which had grown up in other fields, such as the Zhabotinsky reactions or lattice gasses.

Somewhere along the line, especially for one dimensional automata, realization grew that the subject matter was really the study of sequences in labelled, directed graphs, and that there already existed a venerable tradition amongst those mathematicians concerned with the shift dynamical system. Somehow, the dynamicists seem not to have known about graphs, graph theorists about automata, nor automatists about dynamics. Indeed, the not-knowing-graph strongly resembles K3.

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Masakazu Nasu
"Local Maps Inducing Surjective Global Maps of
 One-Dimensional Tessellation Automata"
 Mathematical Systems Theory, 11 327-351 (1978).

 Masakazu Nasu
 "Indecomposable Local Maps of Tesselation Automata"
 Mathematical Systems Theory, 13 81-93 (1979).

 Masakazu Nasu
 "An interconnection of local maps inducing onto global maps"
 Discrete Applied Mathematics 2 125-150 (1980).

 Masakazu Nasu
 "Uniformly finite-to-one and onto extensions of homomorphisms
 between strongly connected graphs"
 Discrete Mathematics 39 171-197 (1982).

 Masakazu Nasu
 "Constant-to-one and onto global maps of homomorphisms
 between strongly connected graphs"
 Ergodic Theory and Dynamical Systems 3 387-413 (1983).

 Masakazu Nasu
 "An invariant for bounded-to-one factor maps between
 transitive sofic subshifts"
 Ergodic Theory and Dynamical Systems 5 89-105 (1985).

 Masakazu Nasu
 "Topological conjugacy for sofic systems"
 Ergodic Theory and Dynamical Systems 6 265-280 (1986).
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The first of this series is a good place to begin, because that is where the useful definitions and concepts can be found. The article runs 25 pages, which includes six sections and six theorems.

1. Introduction
   the author cites nine references which are traditional cellular automaton theory from the sixties or early seventies, plus Hedlund's article. Promising a graphical and finite-automata-theoretical approach, he then summarizes the remainder of the article.

2. Preliminaries
   a list of definitions and the citation of three theorems from the literature, including a comprehensive one on uniform multiplicity including its ramifications on configurations quiescent at infinity. The section ends with some graph-theoretic definitions.

3. Bundle-Graphs and lambda-Bundle-Graphs
   after calling the de Bruijn diagram a "string graph," the subset diagram is introduced

Bibliography of 28 items

In summary, we have a very long and detailled article, with results of basic importance for cellular automata theory as well as the Shift Dynamical Systems to which it is addressed. Topology is thoroughly interwoven with the presentation, which nevertheless ought to be separable, leaving results derived strictly from automata concepts. The next article to be reviewd, by Nasu, accomplishes this to a great extent.

The article has its small quota of misprints but mercifully, they do not mislead, as such blemishes often do.

new books and old articles (4)

Date: Sun, 7 Apr 1996 18:28
To: ca@think.com

Hedlund's paper on endomprphisms and automorphisms of the Shift dynamical system establishes a role for cellular automata although no mention is made of the fact; the connection seems to have been established only after the study of cellular automata gained independent popularity. In fact, merely continuous mappings received considerably less attention than endomorphisms, characterized by the uniform multiplicity theorem and Welch's index theorem, and the automorphism which were an interesting, albeit complicated, special case.

In spite of the meticulous care evident in the paper's presentation, the result seems cumbersome, and lacks motivation. Why, for example, are (n-1)-blocks and (n-1)-separation so important? From another point of view we know that these are the shift registers to which de Bruijn diagrams apply. Can the author have developed such a complicated theory without knowing that?

Another interesting observation is that it is assumed that neighborhoods have odd length, although that does not affect any results; nevertheless it would have changed a notation which is otherwise spelled out in such exquisite detail.

Other authors have wanted examples of subshifts, settling, for whatever reasons, on the so-called Subshifts of Finite Type. Someone may have had an actual application in mind; there was already an analogy with Markov chains in probability theory.

Alternatively, it may have been due to the temptation arising when the abstract definition of the topology of the Shift was consulted and countable exclusions were encountered, to settle for finite exclusion instead. Choosing a matrix to show exclusions focussed attention on that particular matrix, notwithstanding its constituting a less than ideal representative; a deficiency eventually repaired by the invention of Sofic systems. The whole prolonged episode should confer humility on those who think that Hedlund's article was actually a precursor of cellular automata theory. It may well have been, but if so, the route must have been devious.

The model of Subshifts as Flows in a Labelled Graph had an early proponent in

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Roland Fischer
Sofic Systems and Graphs
Monatshefte fuer Mathematik 80 179-186 (1975)
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but ideas along that line seem to have been most thoroughly elaborated by Masakazu Nasu in a series of more than half a dozen papers dating from the late seventies and early eighties. Some of them are:

9. A Fundamental Property of Inverses

   this section is developed without the use of de Bruijn diagrams and their pair diagrams, so it establishes by topology and the examination of counterimages the requirement that paths outside the diagonal of the pair diagram should remain there.

   (4 pages)

10. Inverse [sic] of Recurrent Points are (n-1)-Separated

    the preceding section takes account of recurrency. (3 pages)

11. Almost all Points Have the Same Number of Inverses

    before stage (1) reaches stage (2), and before limits have been taken, uniform multiplicity prevails. (2 1/2 pages)

12. Invariance of Properties under phi inverse When phi in E(S)

    taking the limit loses (or coalesces) some counterimages, but this varies according to periodicity or recurrence properties (4 pages)

13. Compositions

    definition and properties of composites (1 page)

14. Maximal Compatible Extensions

    The mechanism by which multiplicity may be lost, examined in detail but with the benefit of neither the de bruijn nor the subset diagram. Welch's indices.

    (4 1/2 pages)

15. L(fg) = L(f)L(g) and R(fg) = R(f)R(g)

    the indices are multiplicative by composition (1 page)

16. Cross-Sections of the Mappings f[infinity]

    how to avoid losing multiplicity, in very topological terms. (6 pages)

17. Converse of Theorem 6.7

    keeping the full multiplicity (1 page)

18. Roots of Powers of the Shift

    an application of the index theorem (1 page)

19. Polynomial Mappings

    defining mappings by polynomials in Z/(prime) (1 page)

20. Another Property of the Groups A(S) and A(S)/Sigma(S)

    the author emphasizes the complexity of the automorphism group of the shift (two elements whose product is of infinite order, as well as containing all permutation groups). (1 1/2 pages) reversible. Perhaps these results have not been applied to automata theory as much as they should be.

Naturally the article consists of much more than the three theorems mentioned. Some idea of its organization and orientation can be seen from the sequence of section headings:

Introduction

definitions, antecedents, overview (2 pages)

1. Bisequence Space and the Shift Dynamical System

   basic definitions (1/2 page)

2. Subdynamical Systems of (X(S),sigma) and their Characterization

   more definitions and their topological implications (1 page)

3. A Class of Mappings which Commute with the Shift

   The section in which a construction that can be seen as equivalent to defining a cellular automaton is introduced; in three stages it produces the shift-commiting continuous functions.

   1) the local map

   2) right-extended to one-sided sequences

   3) left translated to encompass two-sided sequences.

   Giving each of these three stages their due creates gives a not insignificant complexity to treatments of the Shift. (2 pages)

4. Properties of F[infinity](S,1)

   Several details concerning permutations, the uncountable multiplicity of counterimages, and restrictions for automata of neighborhood-length 1. (1 page)

5. Multiplicities of the Mappings f[m] and f[infinity]

   various results on the cardinalities of counterimages including the finiteness-and-uniform-multiplicity theorem for surjective mappings (endomorphisms).

   (6 pages)

6. Existence Theorems for the Classes A(S), E(S), and Phi(S)

   existence and strict inclusion running from automorphism through endomprohism to simple continuity is established by examples. (6 pages)

7. Classification of Points of (X(S),sigma)

   a classification, charasteristic of symbolic dynamics, of periodicities and near periodicities (4 pages)

8. Invariance of Properties of (X(S),sigma) under phi in Phi(S) and under phi inverse.

   the behavior of this menangerie with respect to image and counterimage (3 pages) this in mind enables an appreciation of fractal graphs and plaid diagrams, and leads to a clearer style of exposition by clearing away whatever uncertainty there might have been that the configurations of automata are somehow real numbers in disguise.

new books and old articles (3)

Date: Fri, 5 Apr 1996 22:23:27
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The paper

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G.A. Hedlund,
"Endomorphisms and Automorphisms of the Shift Dynamical System,"
Mathematical Systems Theory 4 320-375 (1969).
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is frequently cited in the celular automaton literature in spite of the fact that it is a topological article culminating a long series of investigations arising out of celestial mechanics and differential equation theory. The reason for this is that the central object of discussion - continuous maps of symbolic sequences - turns out to coincide exactly with cellular automata; moreover the discussion of endomprphisms and automorphisms resolves many of the questions which could be asked about reversible automata. Similarly not emphasized in the paper, there is a significant overlap with mathematical communication theory. Indeed the whole article reflects an austerity and style of mathematical exposition characteristic of the author.

Over fifty pages in length, the paper is divided into twenty sections, each one of which could be the subject of a day or two of study in a seminar. At that, a background in topology and real analysis is required; given the origins of the subject it is not surprising that there is a heavy emphasis on the topology of the real numbers, sometimes obscuring differences between that topology and the topology of sequences.

For cellular automata theory, this style of argument has two consequences. The first is that a cleaner separation between topology and the combinatorial arguments, as the author describes them, might have made the presentation more understandable. The second is an overwhelming emphasis on periodicity and varying degrees of near periodicity which are of minor concern for automata therory. This would be a greater annoyance were it not for the fact that later on, if iteration of the automaton's evolution operator replaces or supplements the shift operator, such aspects assume a much greater importance.

For all the article's detail, three of its theorems contain the essence of its applicability to cellular automata theory:

After working up from local maps to the one-sided shift to the two-sided shift, Theorem 3.4 (Curtis, Hedlund, Lyndon) asserts that continuous, shift-commuting maps of the full shift are none other than the evolutionary rules of cellular automata.

The fundamental result is Theorem 5.4, which equates surjectivity with uniform (and finite) multiplicity. The theorem requires careful understanding, to know how the accounting can change while passing from the local map to the global map.

Following some ideas attributed to L. R. Welch, Theorem 14.9 reveals which sheep have strayed from the fold and where they have gone. The indices L, M, and R carry this information, moreover are multiplicative under composition, and have a bearing on which automata are when cuts will suffice, but is necessary if confusion in symbolic dynamics is to be avoided.

In mentioning topologies, or at least distances, it should be mentioned that there is still another which has importance for cellular automata, and that the Hamming distance, which simply counts the number of discrepant terms in two patterns which might otherwise match. Mostly it is used to compare neighborhoods and rule definitions, which are finite in extent, whereas including a divisor which might tend to zero is reserved for such things as configurations, which in principle are infinite.

Turning to the book at hand, Voorhees begins with some sample automata from Wolfram's (2,1) domain (to which the book is primarily devoted), with graphs of their rule of evolution, having previously decided to code configurations as binary numbers. The graphs appear to be perfectly jagged, notwithstanding Hedlund's theorem that these are the continuous shift-commuting functions for symbolic dynamics.

But wait! the discrepancy between the topologies means that the graph has to be much wilder looking to the left than it does looking to the right, and in fact there ought to be a semi-continuity proceeding to the right.

There is a detail which is much harder to see in the graphs of Figures 1.2, which is occasioned by the existence of Garden of Eden configurations, which means that not only are there certain numbers which cannot be values of a given rule of evolution, but they exclude intervals corresponding to numbers where they are an initial segment.

In other places a related diagram can be found, even though it would not be a graph. The plaid diagram, in which Arnol'd's cat map, or Smale's horseshoes can be found, represents the right half of a doubly infinite sequence along the x-axis and the left half, in reverse order, along the y-axis so that the points of the unit square represent configurations. The Shift maps this square in a way which looks anything but continuous until the proper topology is consulted, and clusters of points representing a collection of initial configurations can be followed as the cellular automaton evolves. The plaid arises from observing excluded bands. Unfortunately it would require four dimensions to render the full graph the way Voorhees shows it for semiinfinite configurations.

Naturally one of the reasons for presenting such graphs is to expose their delicate fractal intricacies, something which the figures of the book reveal quite nicely. Graphs have to be carefully drawn and reproduced for all the details to endure close scrutiny; even so, the Garden of Eden, or a lack thereof, does not show up too clearly unless the graph can sustain a rather high degree of magnification.

There is always a question of what to put in a book, and what to leave out. That is why it is nice to have a handy computer program available, to go on beyond the contents of the book. The graphs shown fall in the Wolfram Class III, and IV is missing from (2,1), but ought this to leave Classes I and II without representation? If memory serves, ther have been attempts to Fourier transform, and even Walsh transform, graphs of this nature. But memory does not divulge the results, so maybe someone else knows.

Still, this is just refers to the beginning of the book, still leaveing the main business of this review, namely to compare Hedlund's approach with Wolfram's (to pick two representative names out of a hat) while, at the same time, giving an overview of both the books and the approaches.

To summarize today's discussion — the topology of cellular automata resembles the topology of the real numbers, but it is finer [close(automata] implies close(real) but not conversely] because it distinguishes sequences that would be considered equal, as real numbers. Bearing

new books and old articles (2)

Date: Thu, 4 Apr 1996 09:43
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The books in question are Voorhees Computational Analysis ... and Nasu's Textiles ... ; the articles are Hedlund's Endomorphisms and Automorphisms ... and Nasu's Local Maps ... . To this list ought to be added R. F. Williams' "Classification of subshifts of finite type ," Annals of Mathematics 98 120-153 (1973) with Errata ibid 99 380-381 (1974).

Having remarked that cellular automata theory embodies two rather separate traditions, perhaps some explanation ought to be given for combining them; but that is easy enough to do — the theory of the shift dynamical system long ago gave some pretty definite answers to questions which still seem to be somewhat open in automata theory, while in the meantime automata theory has yielded a wealth of empirical information which never seems to have been suspected, let alone contemplated, by the dynamicists. Reviewing books on the subject probably ought to take this into account.

In referring to Hedlund, a considerable amount of topology is encountered. But it is a strange kind of topology, which brings up some interesting doubts about the history of topology; apparently the "Bourbaki" approach dates back to Poincare. That approach is to be distinguished from the delta-and-epsilon methodology which one has learnt from Hardy's "Pure Mathematics," for example.

One of the criticisms of Bourbaki that we used to hear was that it consisted of incredible generalities of which the only known example was the real number system. Well, that just isn't true, and the Shift Dynamical System provides at least one other. The literature provides numerous examples of confusion between the two, both in differential equation theory and in cellular automata theory. That is entirely aside from the fact that the topologists have thought up all kinds of strange limits, periodicity conditions, and pathologies.

To make the point in concrete terms: the real line, even confined to the interval 0-1, depends on real number topology, derived from a metric defined by differences. The topology of sequences depends on the degree to which the sequences agree — from the beginning for one sided sequences, or in the middle for two-sided sequences. One way to get this is to take the reciprocal of the length of the largest common stretch of agreement as the distance for a metric topology. Sometimes the symbols are taken as integers modulo k, the sequences are treated as k-nary (not decimal) fractions, and the distance is the size of the last significant figure at which agreement occurs. That makes it easy to develop a nice graph.

The formal approach is to give the symbol set the discrete topology and then give sequences the cartesian product topology taken from the symbol sets. This is NOT the real number topology, even though school children are led to think that real numbers are just infinitely long decimals. Salvaging that view is sort of like explaining the easter bunny; in both cases reality is found to be somewhat more complicated than expected.

For real analysis and advanced calculus, mathematicians turn to Dedekind cuts and define real numbers as equivalence classes of finite decimals (rationals), but to practice symbolic dynamics, it is better to keep the sequences and understand that the mapping from sequences (arithmetic without carry) to real numbers (arithmetic with carry) is not as continuous as one would like, and to make the most of it. Thus, two long decimals which are close as sequences are close as real numbers, but some real numbers are close even though their decimal expansions (in the dyadic topology) are not. Exploring this correspondence is usually not undertaken

Totally distinct from this were some of the ideas which Henri Poincare put forward to make some sense of the n-body problem, celestial mechanics, and the solution of differential equations in general. One of their consequences was the growth of the subject called Symbolic Dynamics, one of whose major proponents in the early part of the century was G.D. Birkhoff and some of his associates. Later on, G.A Hedlund devoted many, perhaps most, of his years to formalizing the topic, including the Americal Mathematical Society publication with that title.

As time went on, his approach became even more abstract, culminating in the document which is frequently cited in the cellular automaton literature, "Endomorphisms and Automorphisms of the Shift Dynamical System" published in Mathematical Systems Theory 4 320-375 (1969). Ten years in preparation according to the text, it is not an easy paper for non-specialists to read, placing it somethat on a par with von Neumann's "Foundations of Quantum Mechanics " in that regard.

Hedlund's manner of approach has much in common with communication theory in the tradition of Shannon, so its effects are rather noticeable in areas which have to do with cryptography and signalling, as well as the more traditional area of its origins, namely differential equation theory. Stephen Smale has been pretty much at the center of the latter, whereas the former apparently had its influence in such places as the Signal Corps laboratories at Fort Monmouth, some of whose employees seem to have eventually returned to Japan.

It might be an interesting project to try to untangle all the influences and cross influences, but for present purposes suffice it to say that cellular automata and symbolic dynamics were evolved by workers who were substantially unaware of each other. The discomfiture of Lewis Carrol's intrepid band of snark hunters seems to have been matched in recent times by whomever has tried to come to grips with Curtis, Hedlund, and Lyndon's Theorem 3.4.

Whereas Hedlund seems to have been content to topologize sequence space in order to describe its continuous functions and their ramifications, others seem to have wanted concrete examples, leading to the concept of Shifts of Finite type, a line pursued by William Parry from the University of Warwick, and its elaboration into Sofic Systems, by Benjamin Weiss and others. Most of this work is an inextricable mixture of topology, measure theory, and just plain combinatorics. Yet those who work with cellular automata theory see it as little more than the theory of "all paths through a maze" based on hardly more than the theory of regular expressions, if you will.

Here, a central reference is Masakazu Nasu's "Local Maps Inducing Surjective Global Maps of One-Dimensional Tessellation Automata" which also appeared in Mathematical Systems Theory, 11 327-351 (1978). While referring constantly to Hedlund, the exposition is nevertheless based on graphs; concretely, the de Bruijn diagram, the subset diagram, and the "vector subset diagram" which seems to bear on the subset diagram in much the same way that sofic systems relate to shifts of finite type.

Although our intention here is to review these two new books, the foregoing commentary should emphasize our conviction that creating a sufficiently detailled analysis practically requires going back and reconstructing automata theory from the ground up, bearing in mind that whatever merit is to be found in the topological and measure theoretic versions, the situation is analogous to the plight of a little girl in a family I once knew where English, German and Italian were spoken with equal fluency: "Dear, Please DON'T mix your languages!"

For next time, the topic will be a survey of Hedlund's article, the one referenced above.

new books and old articles (1)

Date: Mon, 1 Apr 1996 22:13:09
To: ca@think.com

From time to time the question comes up, what is a good book to read on cellular automata? There aren't too many of them - Wolfram's World Scientific book (now reprinted), the proceedings of the 1989 CA conference, the manual for the CAM's, and maybe one or two others. Various suggestions are listed in the CA-Mail FAQ's.

Six weeks or so ago Burton Voorhees ¡burt@cs.athabascau.ca¿ announced:

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> Subject: New CA Book From World Scientific
>
> Computational Analysis of One-Dimensional Cellular Automata
 > By Burton H. Voorhees
> Recently published by World Scientific.  ...
> [that's ISBN 981-02-2221-1]
> [... extract from the preface followed by the table of contents ...]
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Those whose memory reaches a bit further back will recall that the book

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Textile Systems for Endomprphisms and Automorphisms of the Shift
By Masakazu Nasu
Memoirs of the Americal Nathematical Society # 546
ISBN 0-8218-2606-9
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was also suggested, on the level of "to be published" by an earlier correspondent. Both of these items are now available, and we are in the process of reading them, finally having bought copies. Of course, the two books are rather different, the latter being much more specialized.

Voorhees has summarized a considerable part of his own work during past years, placing emphasis on linear rules of evolution and the classification of one dimensional automata. But so also has Nasu, who has carried one- dimensional automata into two-dimensions, considering both the spatial shift operator and the temporal evolution by an automaton rule. Previous treatments of the Shift dynamical system have tended to concentrate on the shift operator alone.

Nevertheless, the reading of a book on cellular automata is not an activity to be undertaken lightly, the reason for this being that the subject is far more complicated than appears at first sight. These two books exemplify the confluence of two widely separated traditions; those currents have not at all developed in isolation from one another, but it would appear that the practitioners of the two specialties tend to have rather different skills and interests.

Cellular Automata seem to be an outgrowth of the theory of ordinary automata with roots in the work of von Neumann and his universal constructor, but drawing on the ideas of McCulloch and Pitts, the lore of language theory and especially regular expressions, and the emerging role of iteratitive arrays as electronic circuits became smaller and cheaper. If the invention of the Garden of Eden was not intended to refute universal construction, that was nevertheless the philosophical interpretation that arose for a time. Just a few words hardly do justice to a whole line of investigation, but it does not hurt to think of formal language theory, automated pattern recognition, the theory of relays, electrical circuits, and neurophysiology as topics which grew into a theory of automata.

New books and old articles

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January 29, 1998

Abstract

The collection of commentaries on the book:Computational Analysis of One-Dimensional Cellular Automata, By Burton H. Voorhees, published by World Scientific. 1996 (ISBN 981-02-2221-1) and old articles:Endomorphisms and Automorphisms of the Shift Dynamical System" published in Mathematical Systems Theory 4 320-375 (1969) and Textile Systems for Endomprphisms and Automorphisms of the Shift, by Masakazu Nasu, Memoirs of the Americal Nathematical Society No. 546, ISBN 0-8218-2606-9 which were posted on CA-MAIL during April and June, 1996, is reproduced with the correction of misspellings and adaptation to TeX format. Citations to some of the references mentioned have been included.