Pentagonal Flexagons

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Abstract

Maps and cutouts for a variety of flexagons are presented, emphasizing those which can be cut out, mostly from single sheets of paper. Since TeX may not align front and back images, and in any event if cutting up the booklet is not desired, the .eps files can be printed directly to get sheets suitable for cutting. In the same spirit, only those sheets which are going to be used right away need be printed.

Contents

1 Introduction

Flexagons can become fairly complicated. The ones based on triangles are most conveniently made from long strips of paper; a roll of adding machine or calculator tape is ideal for this purpose given its convenient width. Crooked strips can be gotten by gluing faces together, or just cutting out segments and then joining them together. Leaving one extra triangle in each segment for overlapping and later gluing leads to efficient constructions.

Coloring the triangles is another problem, which can be done with crayons or markers once it is known which colors ought to be used. Aside from copying an already existent design, this is best done by drawing the Tukey triangles and then lettering or numbering the triangles in the strip. That information is sufficient to fold up the strip, since pairs of consecutive numbers are to be hidden by folding them together. Painting can be done before folding by following a color code for the numbers, or after the folding is done, when the faces can be painted wholesale, or even embellished with designs.

Other flexagons, even the ones folded from "straight" strips, require a higher degree of preparation, although it is relatively easy to assemble a collection of primitive components which later can be glued together according to the necessities of the individual flexagon.

2 First Level Pentagonal Flexagon

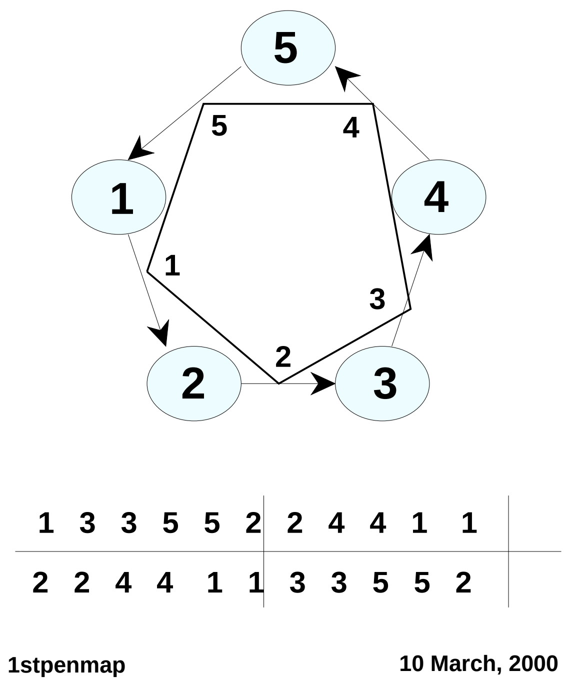


Figure 1: The full first level pentagonal flexagon has 5 vertices.

Flexagons constructed from regular pentagons are instructive in many respects. Unlike flexagons constructed from squares or triangles, the figure does not lie flat in the plane. Notwithstanding their nonplanarity, single cycle flexagons whose central angle sum exceeds 360° can always be run through their cycle even when they can't be laid out flat, so that the principle which bases a flexagon on a stack of polygons can always be confirmed.

Pentagon flexagons are good for observing that any number of leaves may be taken out of one pat and placed in the other. Single leaves give a figure anchored on the vertices of the pentagon, but taking two leaves gives a result which looks like tubulation, since it is anchored on the prolongations of edges surrounding the one which was skipped. With squares, that makes an exact tubulation.

Skipping three leaves forward looks like skipping two backward which is akin to folding the flexagon backwards. However, for polygons with a large number of sides, small numbers of leaves may be grasped simultaneously to advance rapidly around the cycle of faces.

The inductive part of the construction, which allows the substitution of a single polygon for an inverted stack spanning the same angle is most readily confirmed with the binary flexagon, wherein one single polygon has undergone this replacement. A stack of four pentagons is sufficiently thick as to be noticeable as an entity, and requires sufficient exertion to run through either one of the two cycles that it probably illustrates the principles of flexagons better than some other choice. A stack of two triangles is rather inconspicuous, while three squares have too much in common with coordinate axes and smooth folding to be entirely convincing. Pentagons do nicely, avoiding the ever thicker stack which results when the number of sides of the polygon is increased. Still, binary flesagons of all orders illuminate the principles of flexagon construction.

Much of the excitement of exploring flexagons resulted from the multitude of was in which triangle flexagons as well as square flexagons could be compounded by adding new cycles to the Tuckerman traverse. Of course, all the other polygons offer the same possibilities, each time with a greatly increased scope of alternatives. A more systematic approach would be to replace *all* the polygons in the basic cycle with inverted stacks, arriving at what one might call the second level of flexagon. Working along similar lines would lead to third level flexagns, fourth level, and so on. All can be prepared from winding up previously prepared polygon strips, but the thickness of a paper implementation rapidly makes physical realization difficult and then impossible.

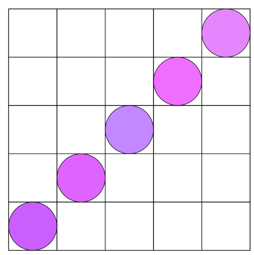


Figure 2: Permutation of the pentagons along the strip for a first level Pentagonal Flexagon.

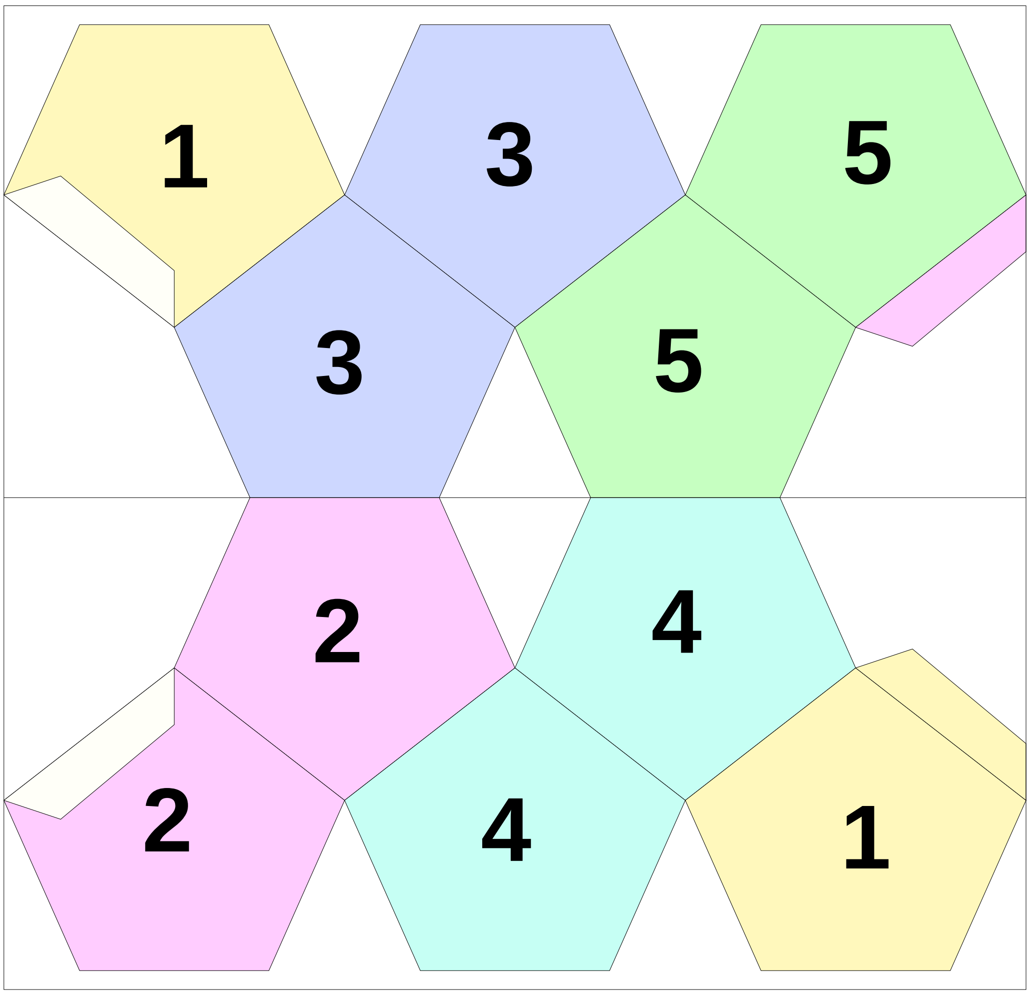


Figure 3: Top side of a Pentagonal Flexagon consisting of a single cycle. The figure should be cut horizontally through the middle, and one tab pasted to match colors. Once the figure has been folded up, the other tab should be pasted where space for it has been provided. The flexagon will not lie flat, but the two sectors will divide easily into twp pats each, within which leaves can be separated and moved from one pat to the other by flexing.

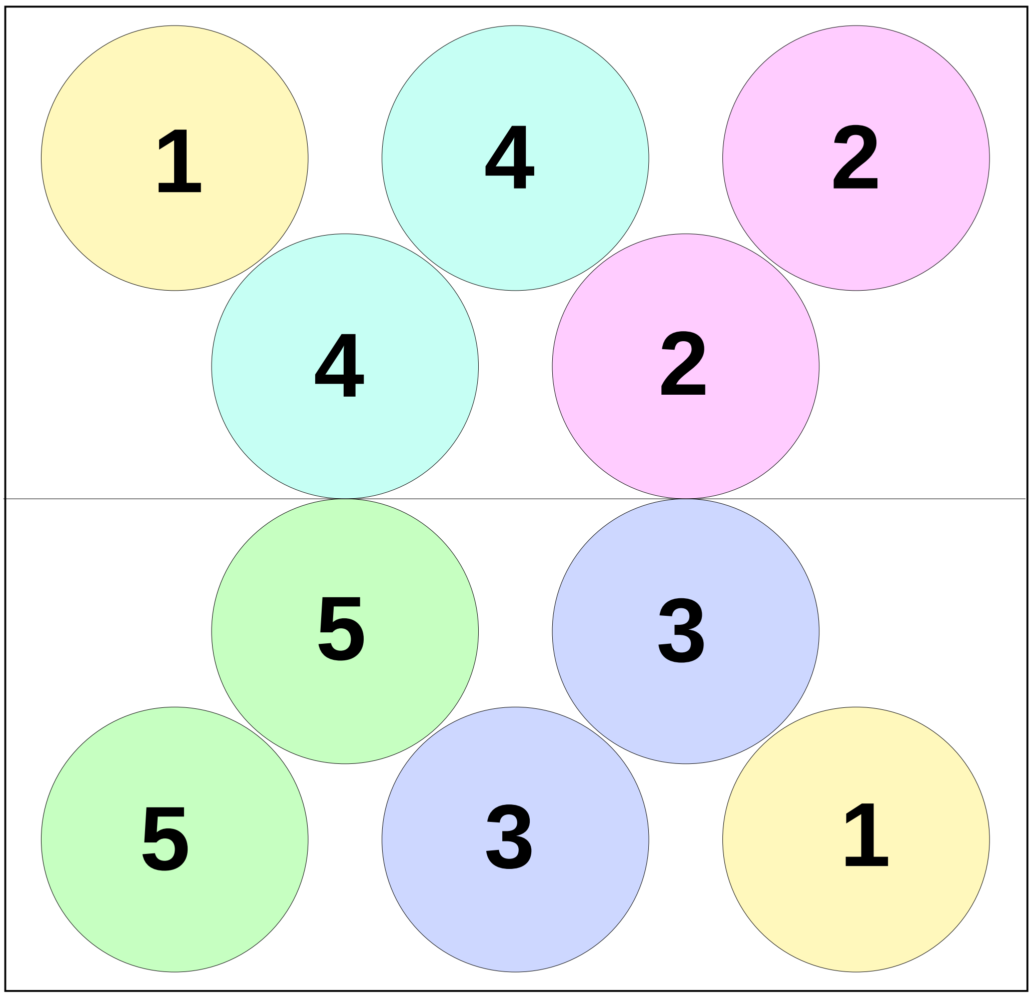


Figure 4: Bottom side of a Pentagonal Flexagon consisting of a single cycle.

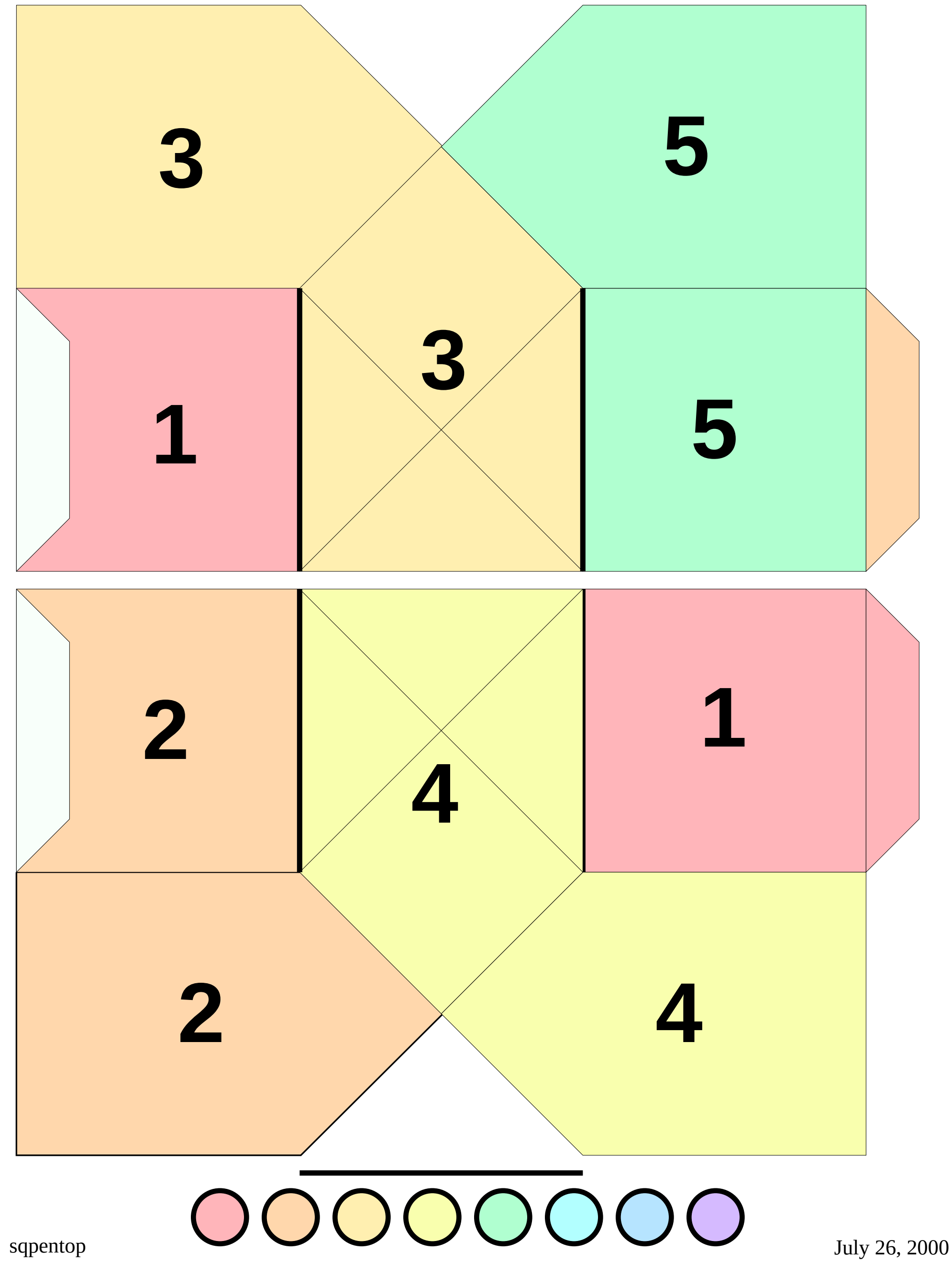


Figure 5: Top side of a squared up pentagonal flexagon. In this figure there are three 90° angles and two 45° angles, but the figure resembles a house with a peaked roof because two of the right angles raise the walls off the floor, while the third sits up in the gable.

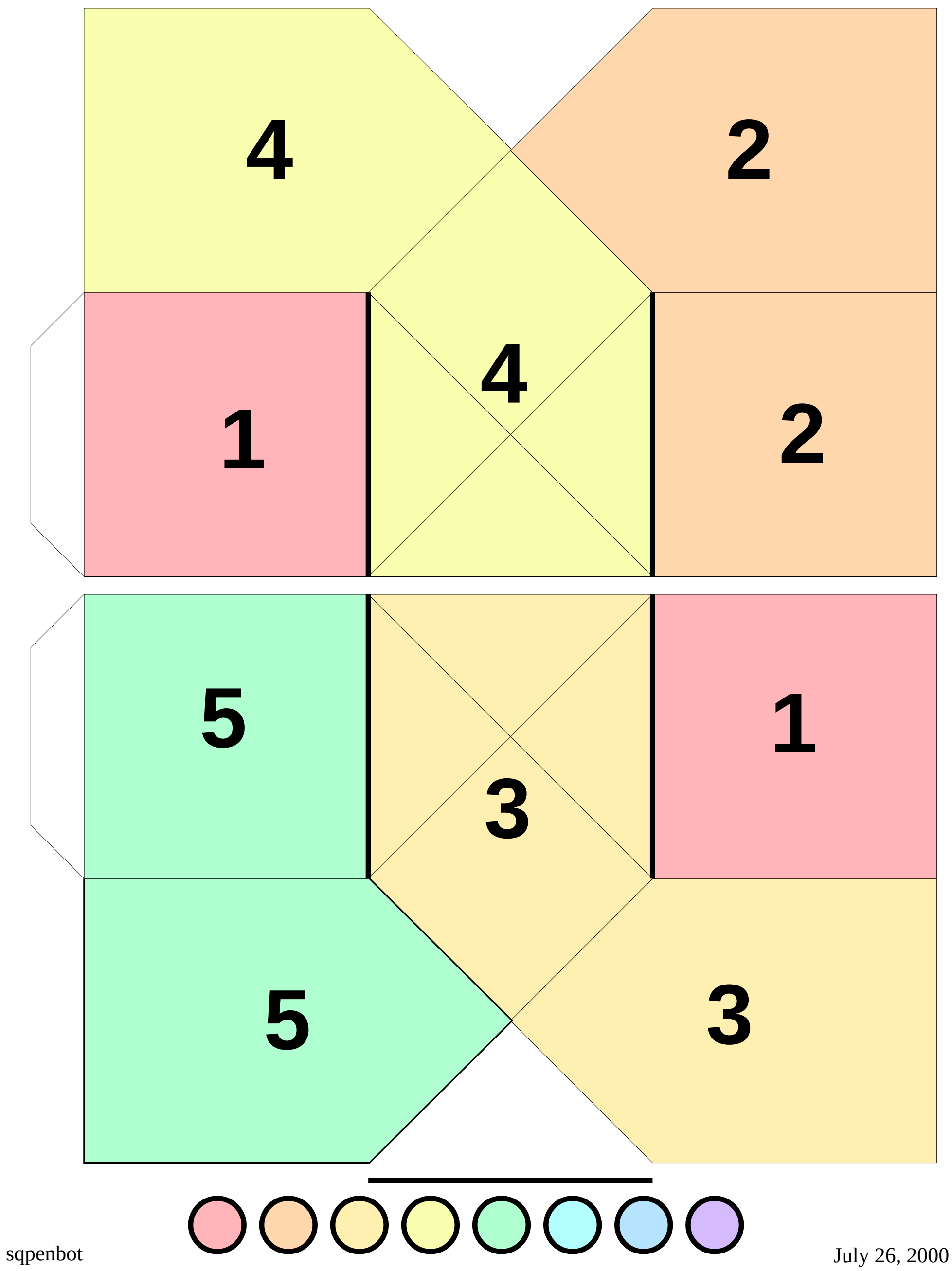


Figure 6: Bottom side of a squared up pentagonal flexagon.

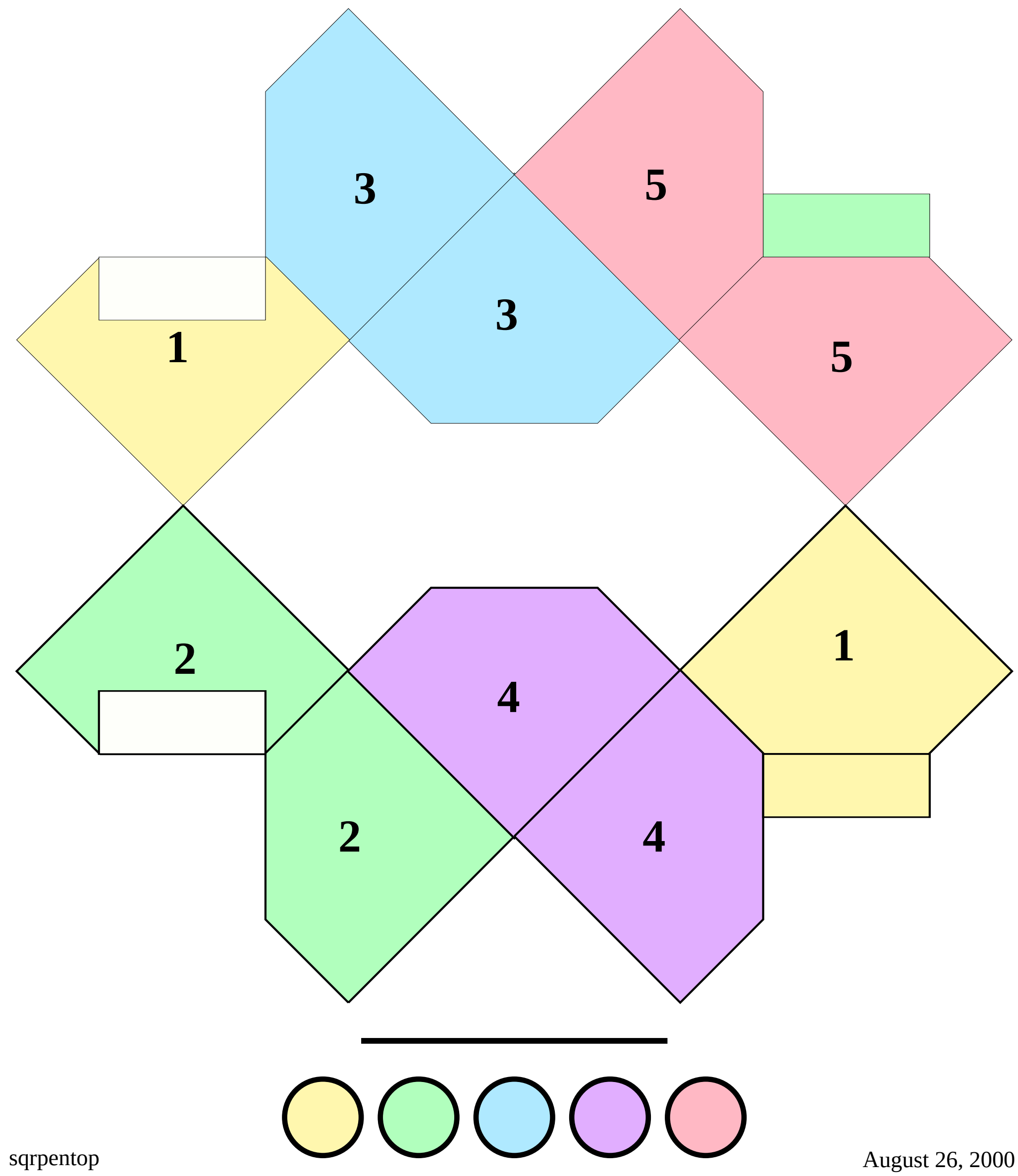


Figure 7: Top side of a lopped off square pentagonal flexagon. In this figure there are three 90° angles and two 45° angles, but the figure resembles a lopped off square because the three right angles run in sequence.

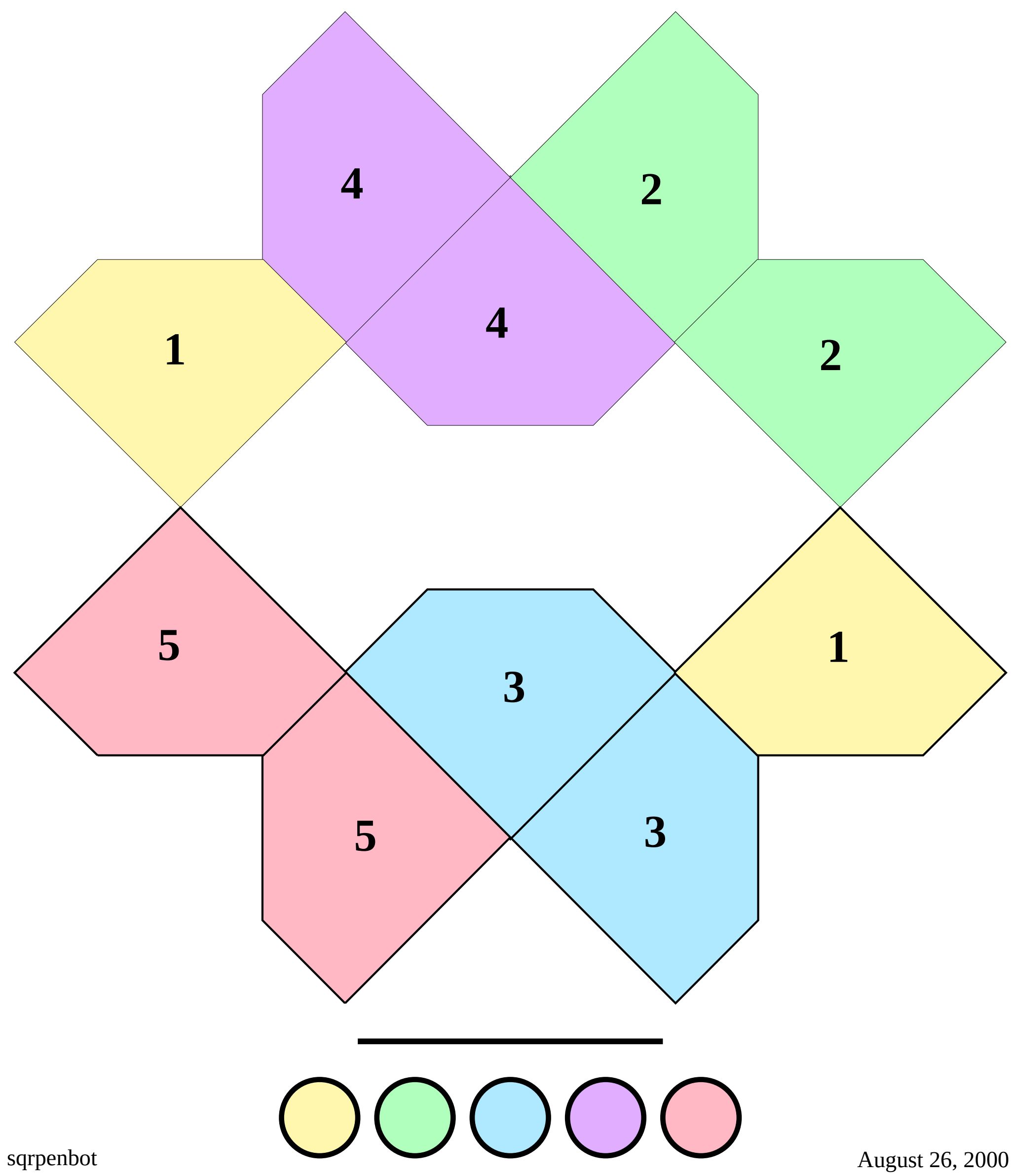


Figure 8: Bottom side of a lopped off square pentagonal flexagon.

3 First Level Pentacle Flexagon

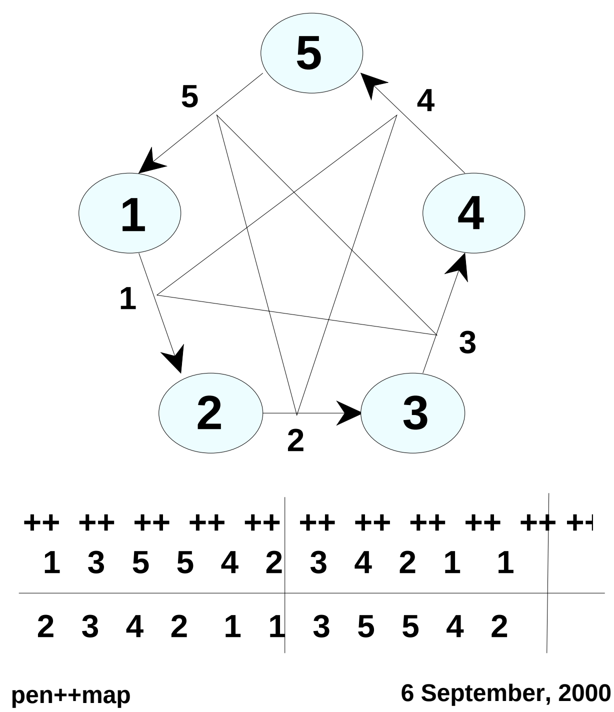


Figure 9: The full first level Pentacle Flexagon has 5 vertices.

The pentacle flexagon works out well because of the cancellation of two conflicting tendencies. The angle between two consecutive pentagons joined by skipping an edge is too large to accomodate the mountain fold - valley fold transition which is sometimes necessary, but by taking two leaves together to make the flex, the fourfold angle reduces modulo 360° to the normal angle for a pentagonal flexagon, so flexing may procede without incident.

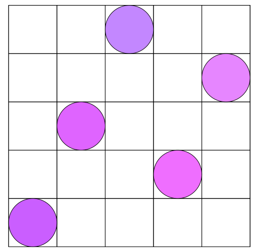


Figure 10: Permutation of the pentagon labels along the strip for a first level Pentacle Flexagon.

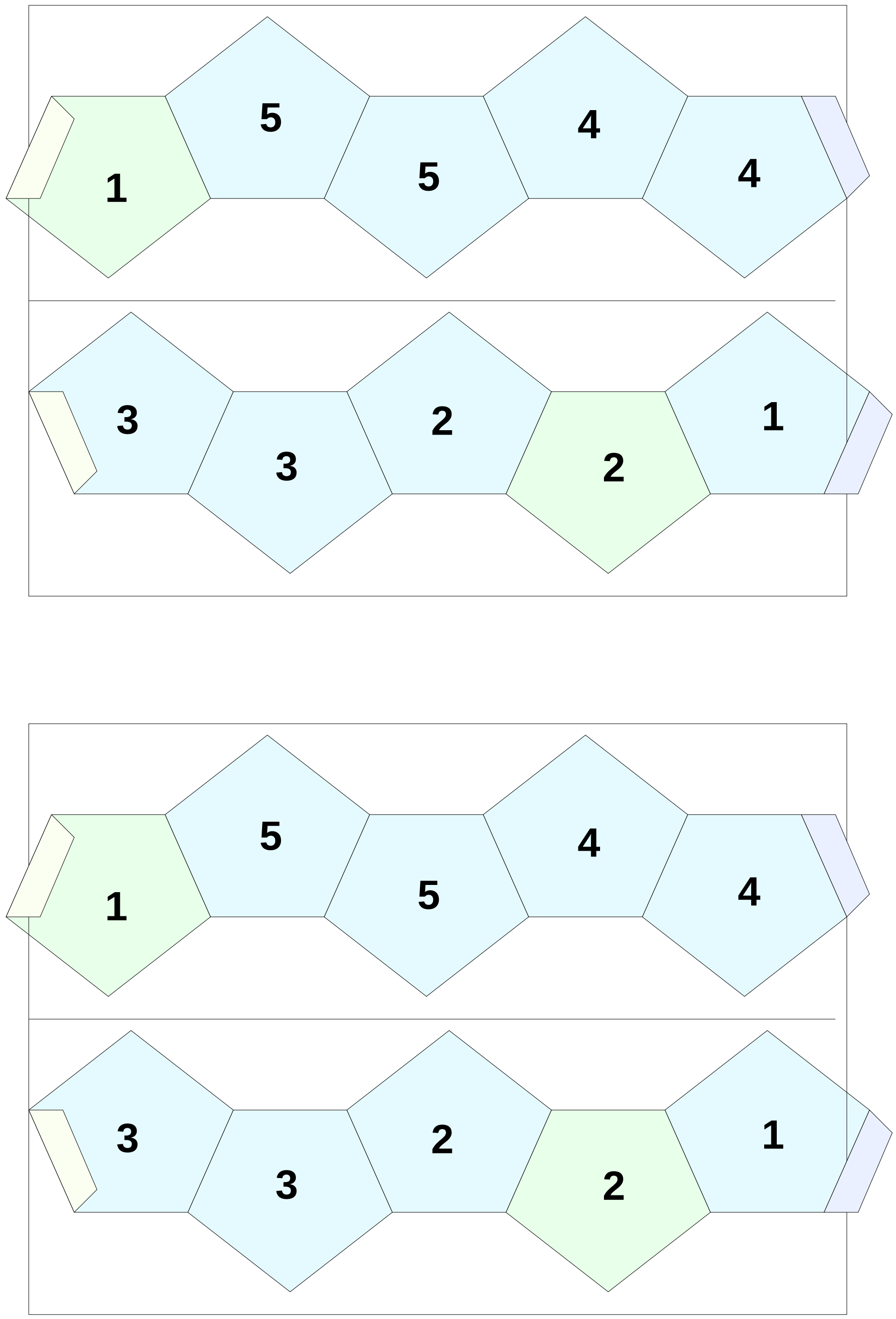


Figure 11: Top side of a Pentacle Flexagon. The pentagons are joined, not by consecutive edges, but by skipping an edge. Five pentagons comprise one sector, a minimum of two of which are needed for a flexagon. Thus sufficient material is shown for two minimal flexagions.

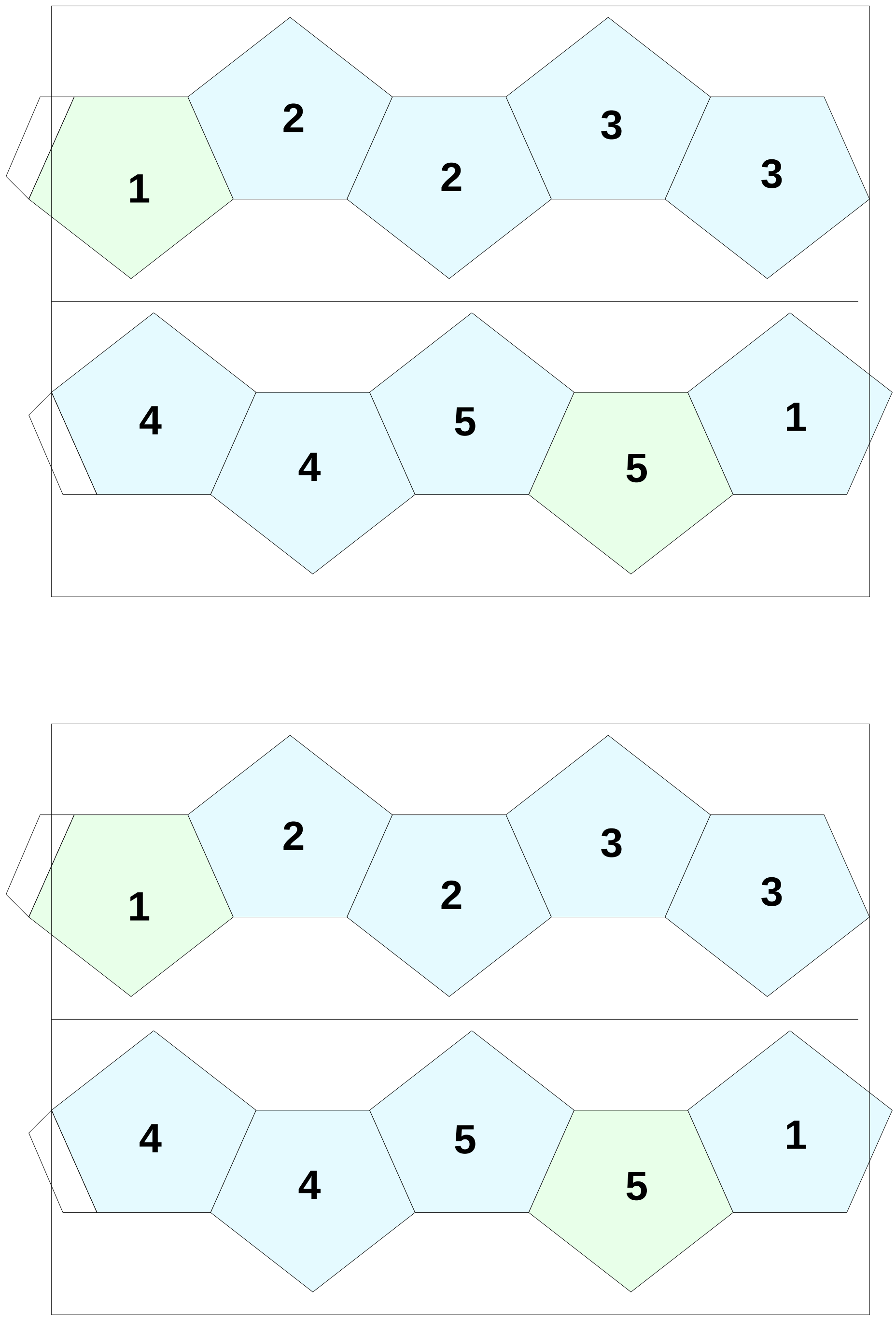


Figure 12: Bottom side of a pentacle flexagon.

4 Binary Pentagonal Flexagon

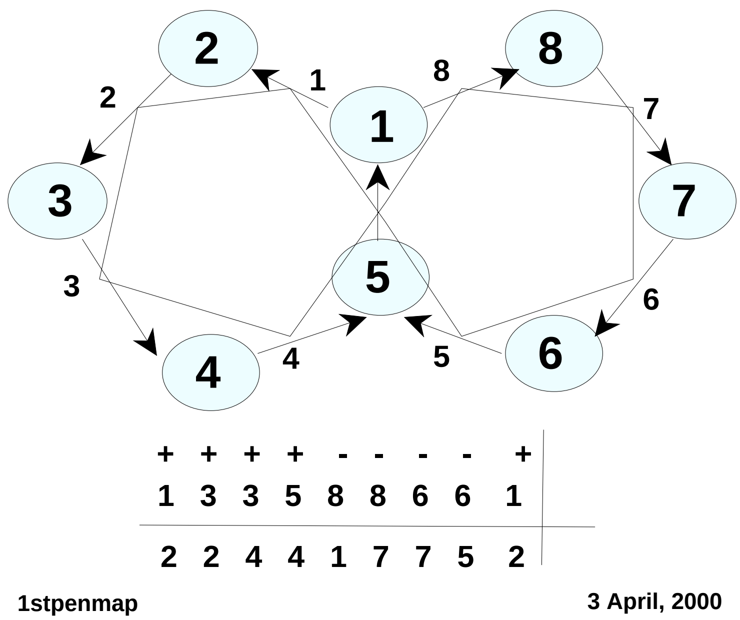


Figure 13: The binary pentagonal flexagon has two cycles, each of which has two vertices in common with the other one, for a total of eight.

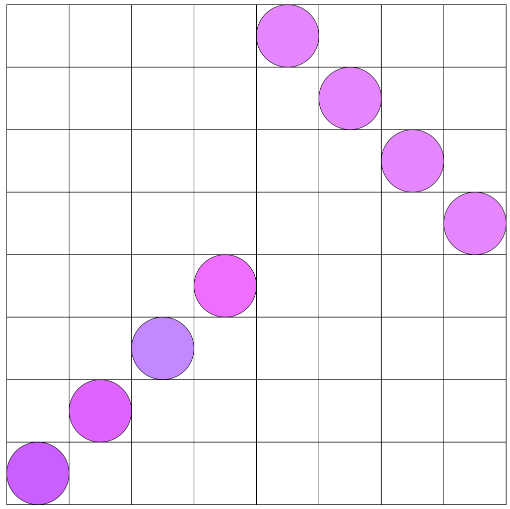


Figure 14: Permutation of the pentagons along the strip for a binary pentagonal flexagon.

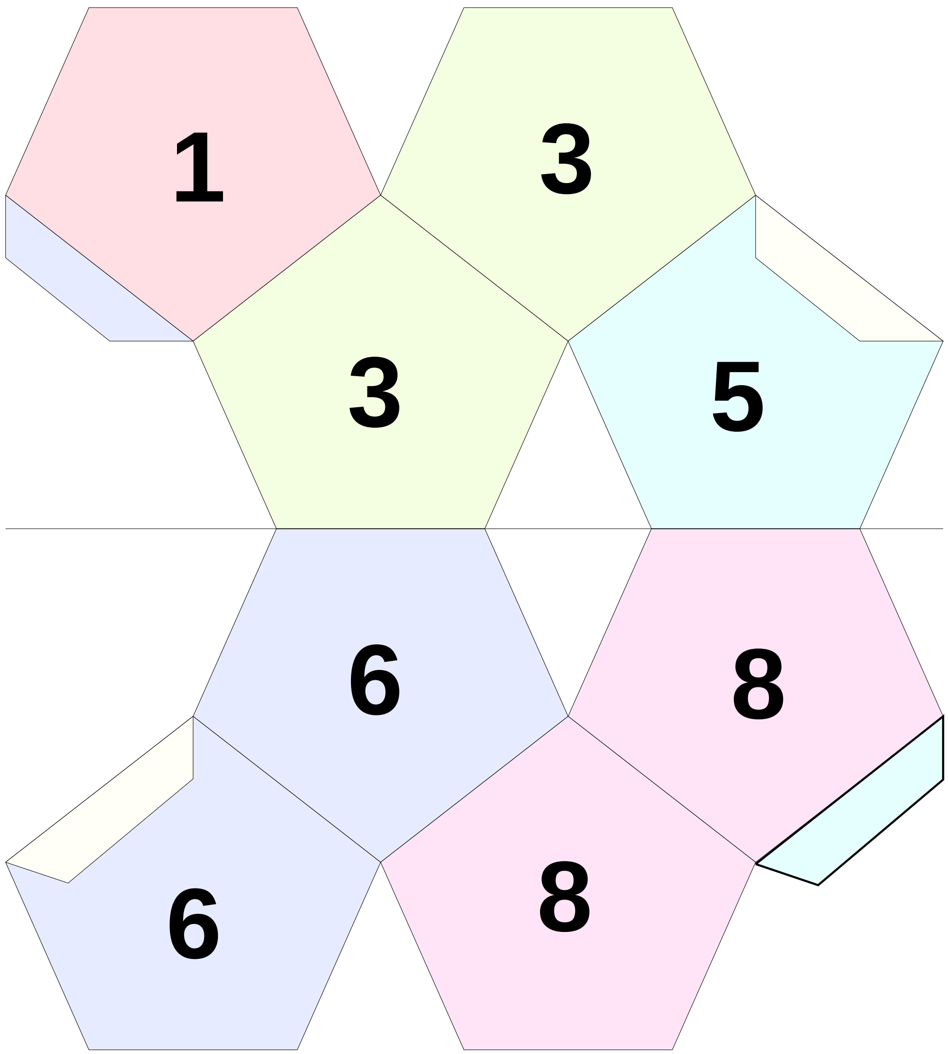


Figure 15: Top side of the binary pentagonal flexagon cutout. The flexagon has ten faces, so this cutout provides material for just one of the sectors needed for the flexagon.

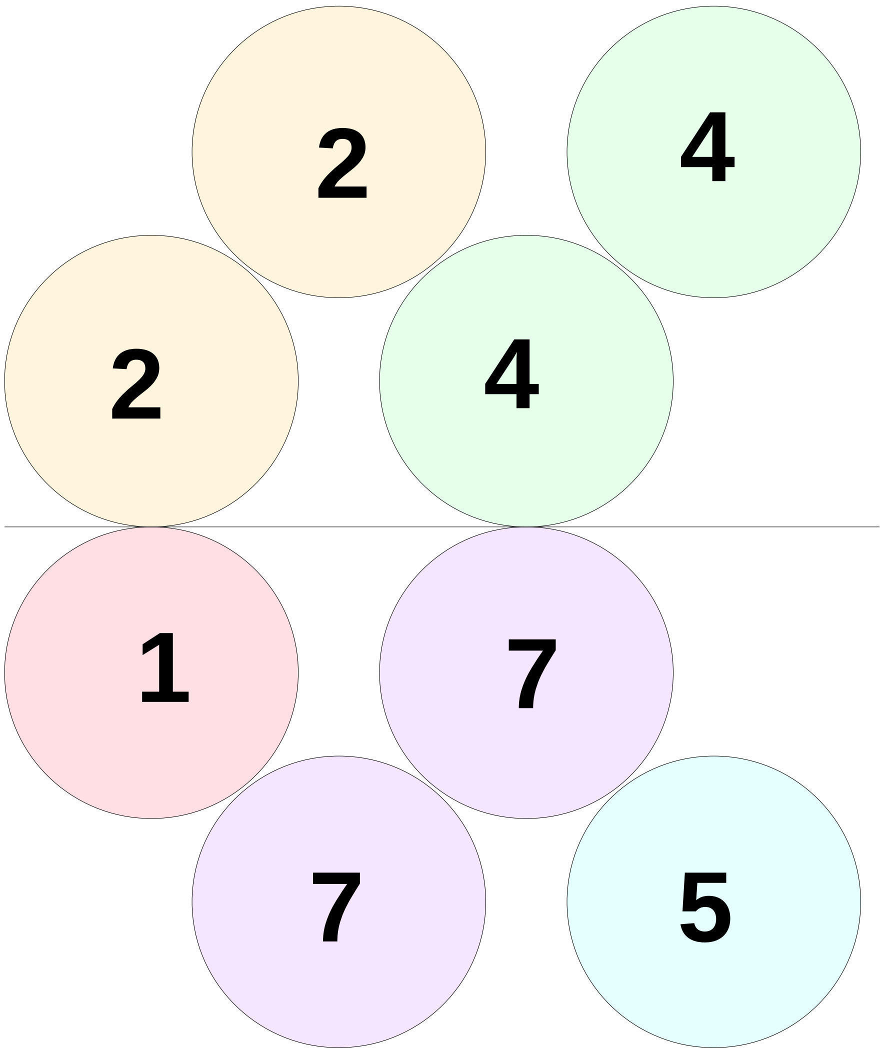


Figure 16: Bottom side of the first level binary pentagonal flexagon cutout.

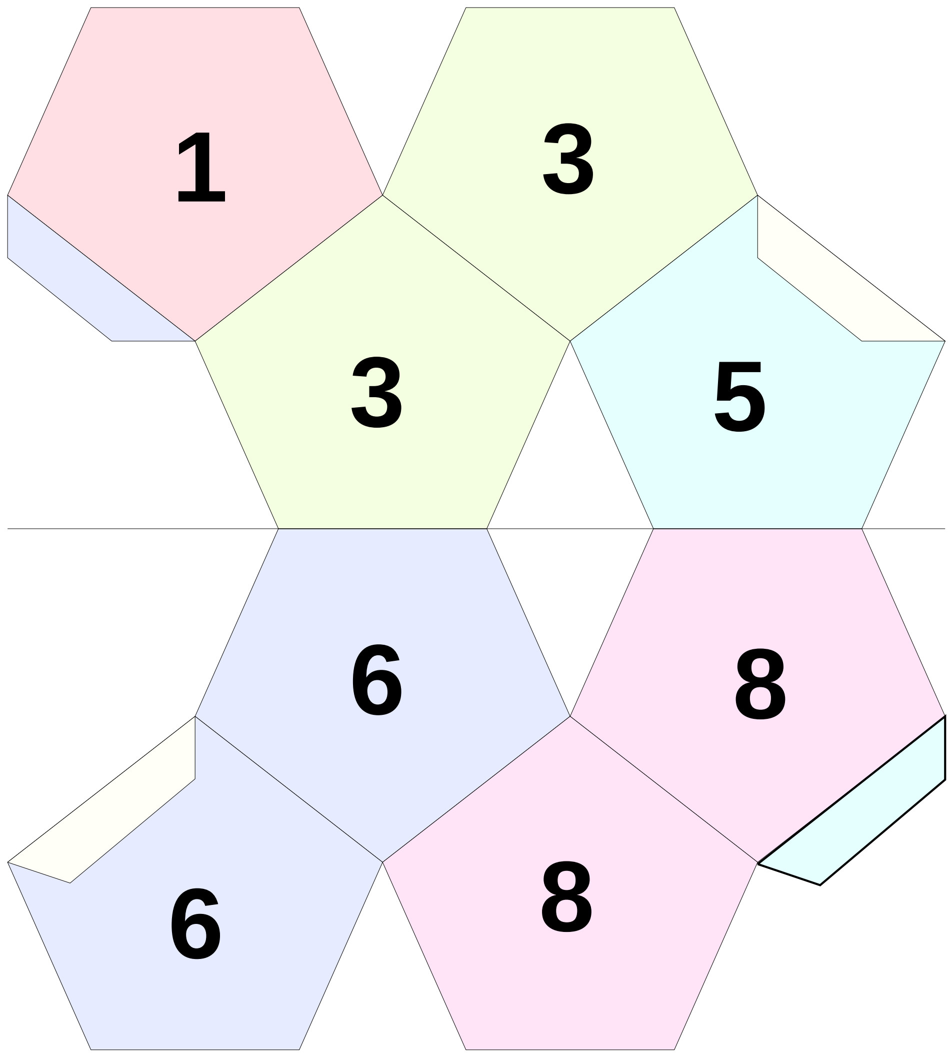


Figure 17: Copy of the top side of the first sector of the binary Pentagonal Flexagon, for making the second sector.

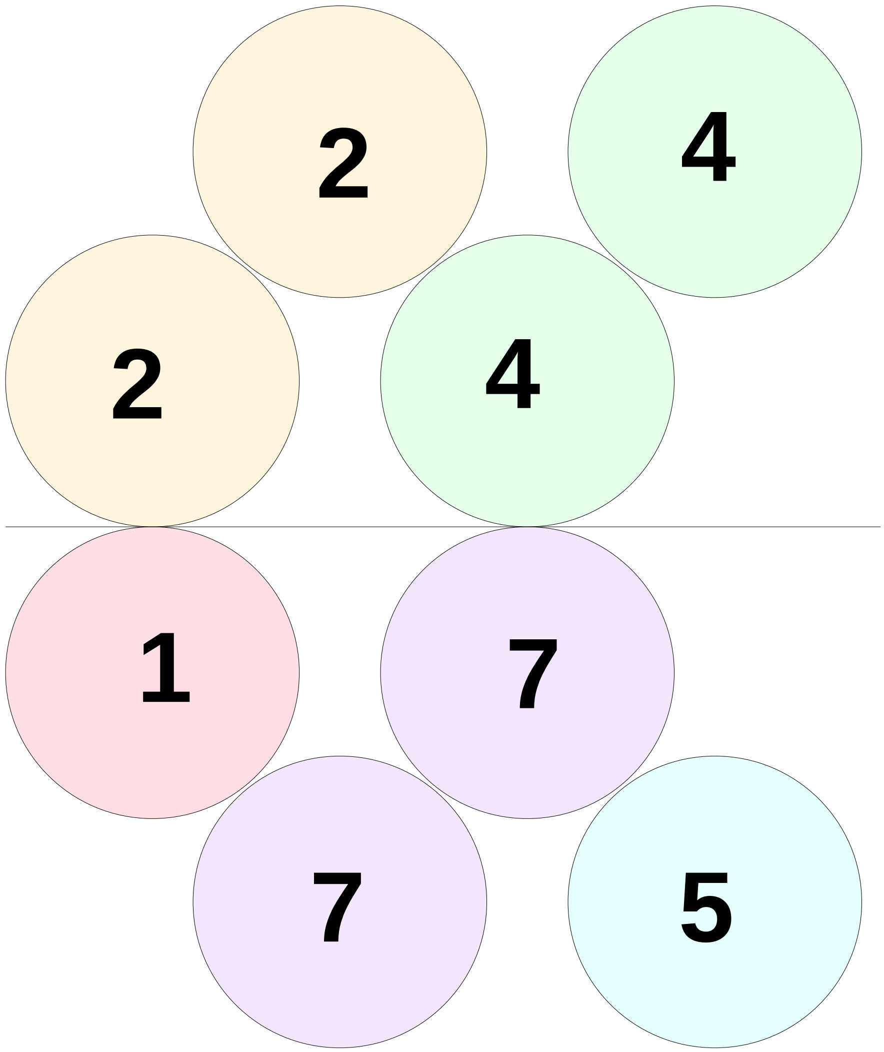
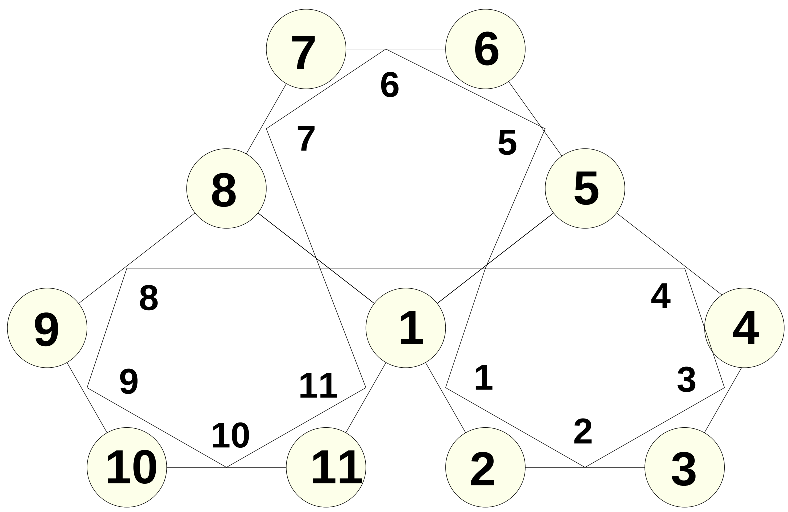


Figure 18: Copy of the bottom side of the first sector of the binary Pentagonal Flexagon, for making the second sector.

5 Ternary Pentagonal Flexagon

Ternary flexagons are those whose map has three cycles. For symmetry reasons, using regular pentagons, here are only two ternary flexagons, which can be called cis-ternary and trans-ternary, from the viewpoint of chemical nomenclature. In the cis- form, the two secondary rings are adjacent; in the trans- form, they are separated.

5.1 Cis-ternary pentagonal flexagon



+	+	+	+	+	+	+	+	-	-	-	+
1	3	3	5	8	10	10	1	7	7	5	2
2	2	4	4	9	9	11	11	8	6	6	1

Figure 19: Map, with Tukey pentagons, of the cis-ternary pentagonal flexagon.

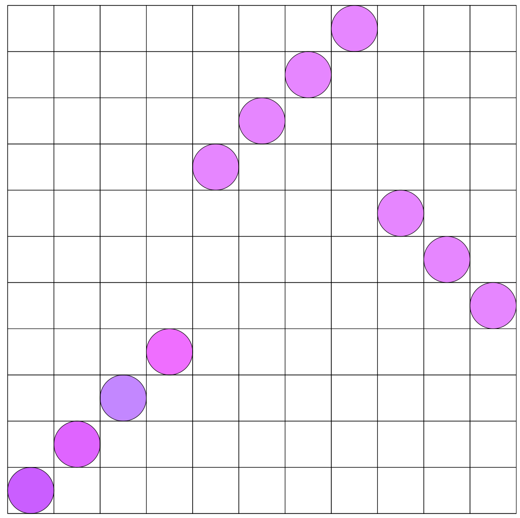


Figure 20: Permutation of the pentagons along the strip for a binary pentagonal flexagon. Three cycles are manifected by the three blocks in the permutation. That two of them result from recursive replacement manifests itself in the size and cyclic structure of two of the blocks.

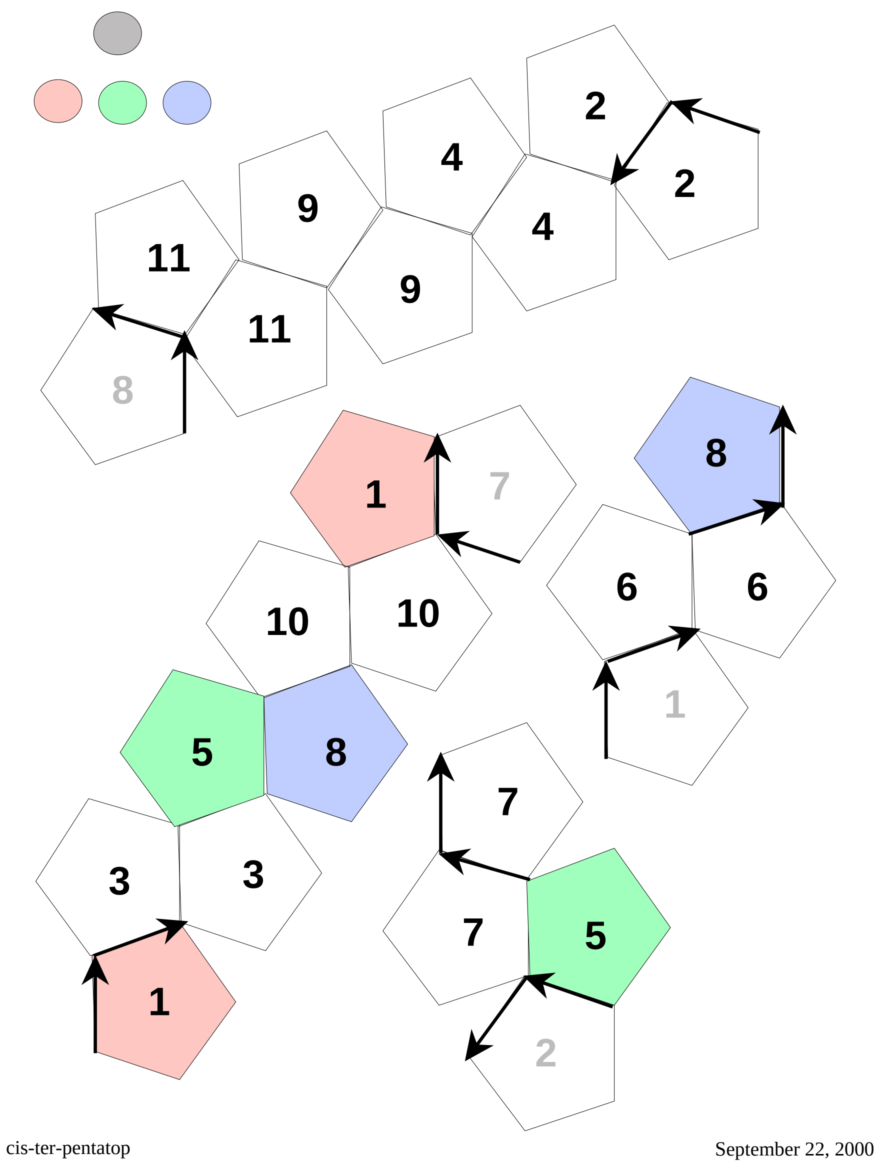


Figure 21: Top side of the cis-ternary pentagonal flexagon cutout. The flexagon has eleven faces. Because the number is odd, one of the sectors has to be turned over to get the other with a smooth match. The figure will make one flexagon.

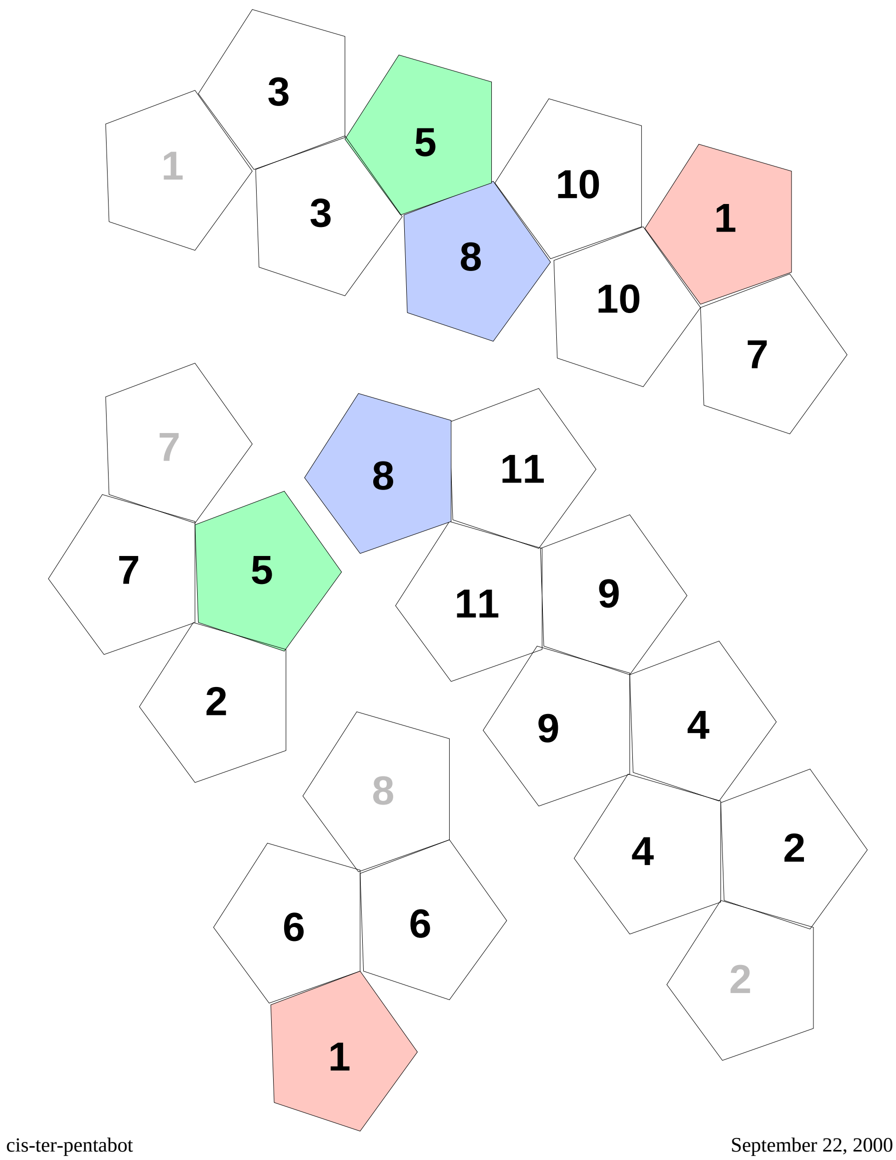
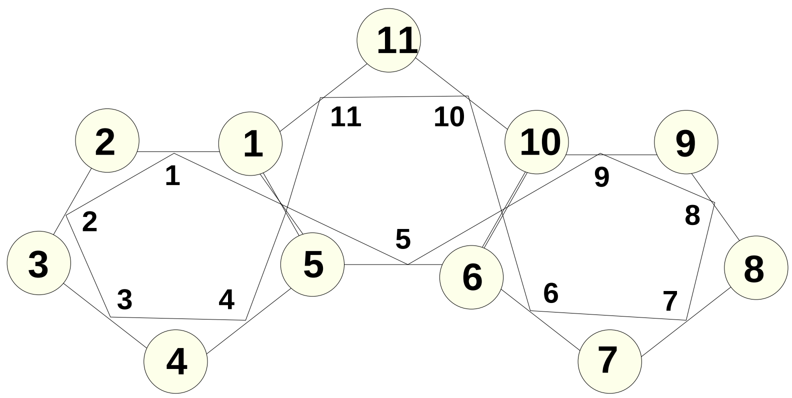


Figure 22: Bottom side of the cis-ternary pentagonal flexagon cutout.

5.2 Trans-ternary pentagonal flexagon



+	+	+	+	-	-	+	+	+	+	-	+
1	3	3	5	11	11	6	8	8	10	5	2
2	2	4	4	1	10	7	7	9	9	6	1

Figure 23: Map, with Tukey pentagons, of the trans-ternary pentagonal flexagon.

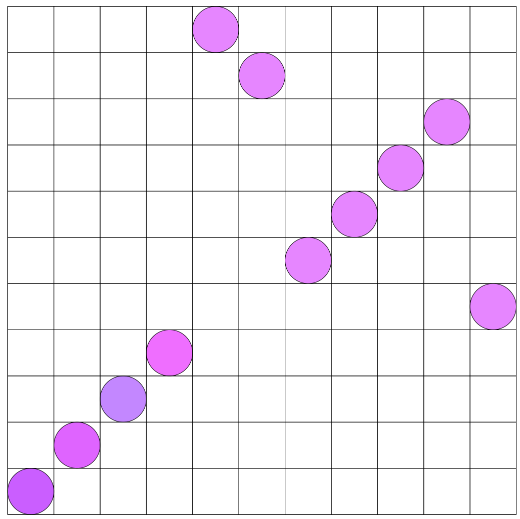


Figure 24: Permutation of the pentagons along the strip for a binary pentagonal flexagon. Three cycles are manifected by the three blocks in the permutation. That two of them result from recursive replacement manifests itself in the size and cyclic structure of two of the blocks.

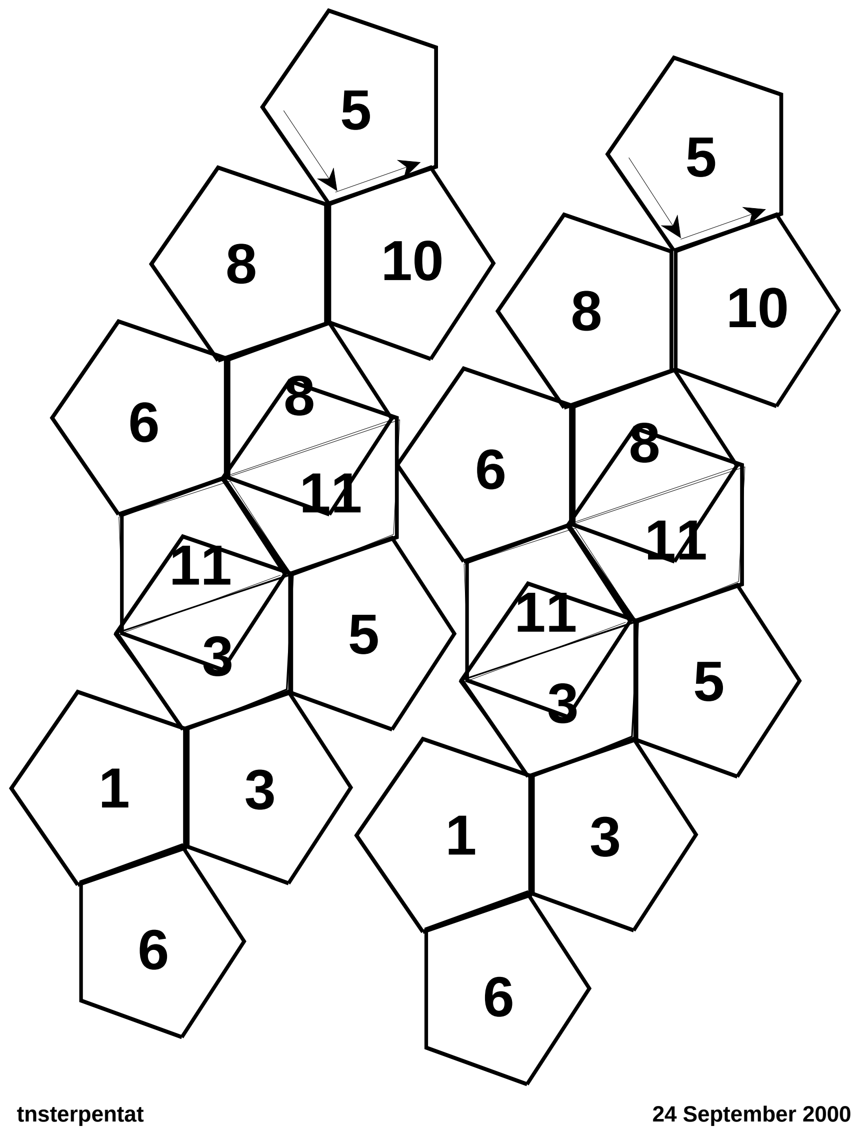


Figure 25: Top side of the trans-ternary pentagonal flexagon cutout. The flexagon has eleven faces. Because the number is odd, one of the sectors has to be turned over to get the other with a smooth match. The figure will make one flexagon.

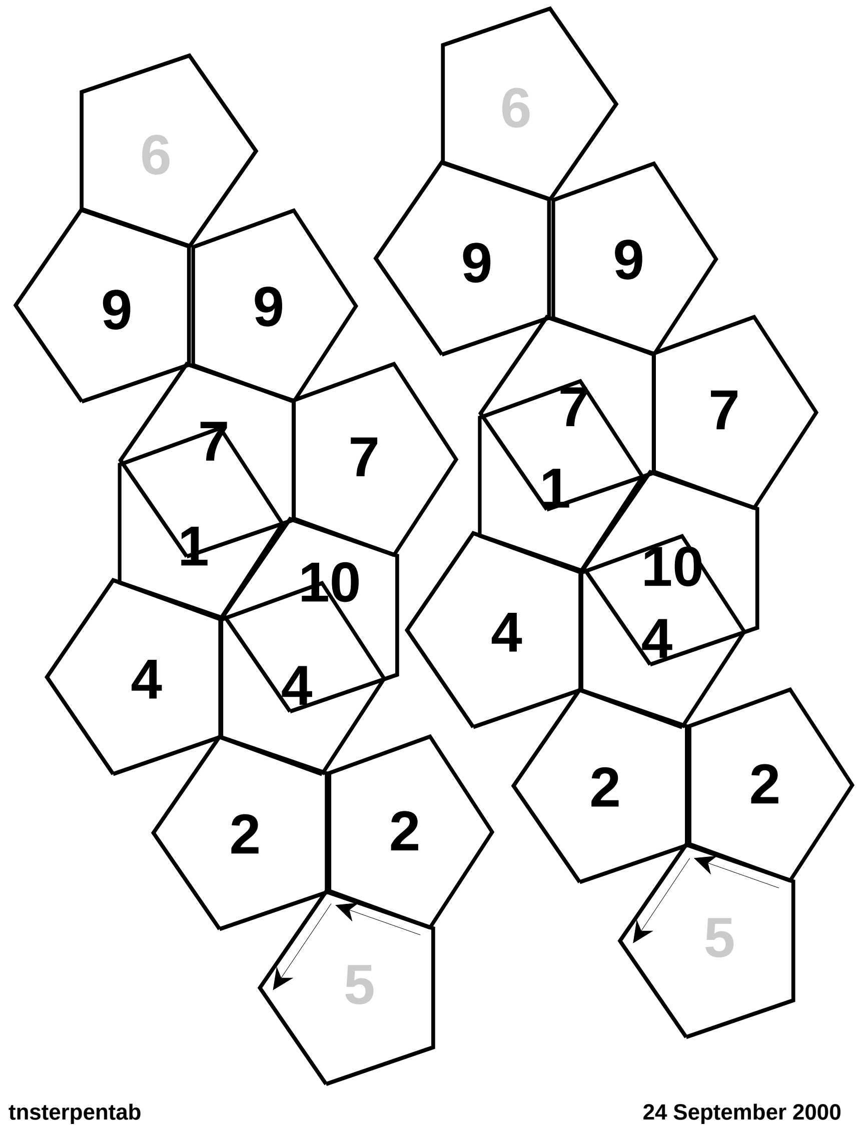


Figure 26: Bottom side of the transs-ternary pentagonal flexagon cutout.

6 Second Level Pentagonal Flexagon

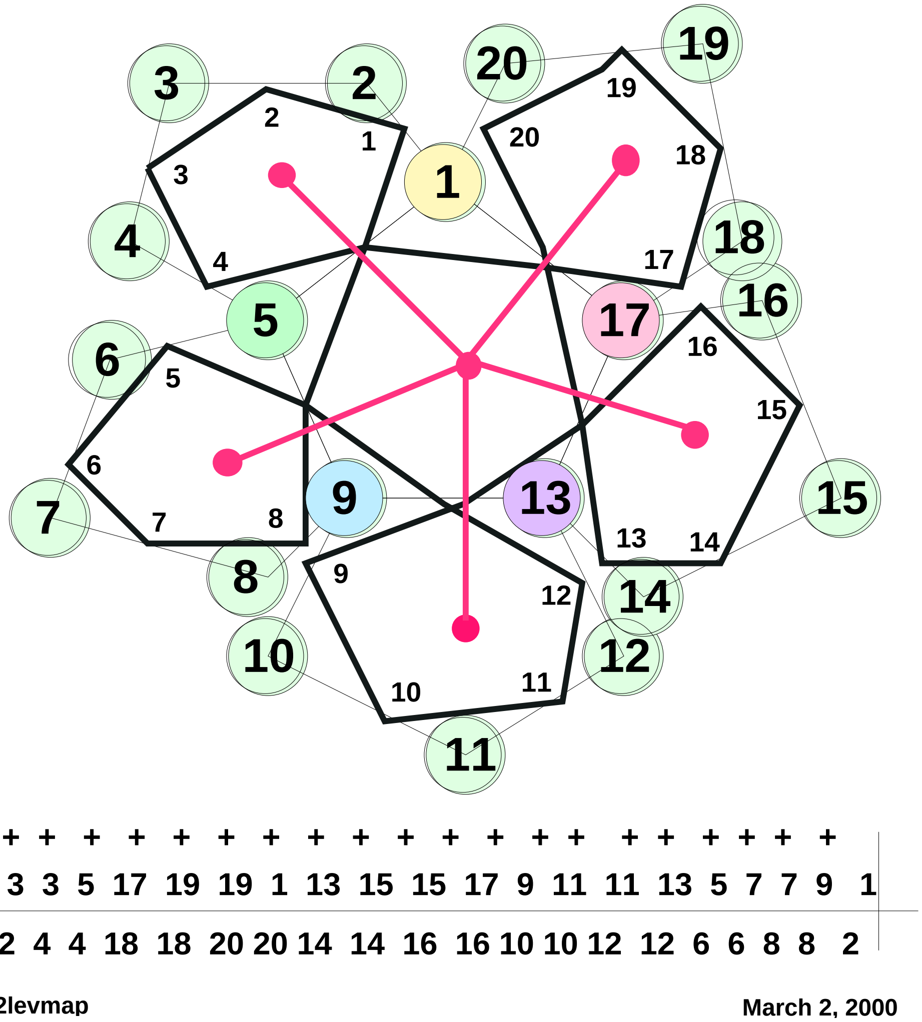


Figure 27: Since each edge of the first level Pentagonal Flexagon spawns three new vertices, the full second level Pentagonal Flexagon has 20 vertices.

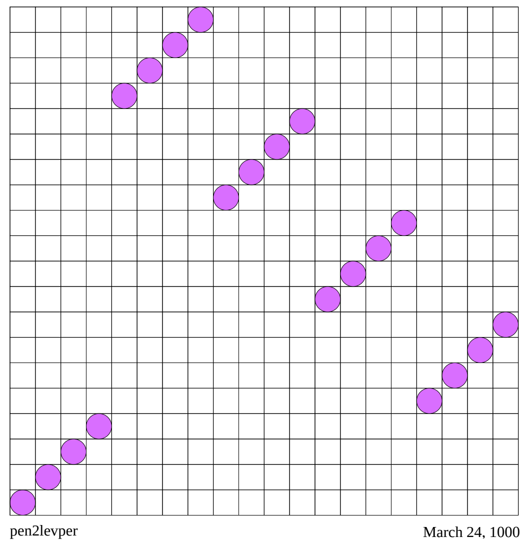


Figure 28: Permutation of the pentagons along the strip for a second level Pentagonal Flexagon.

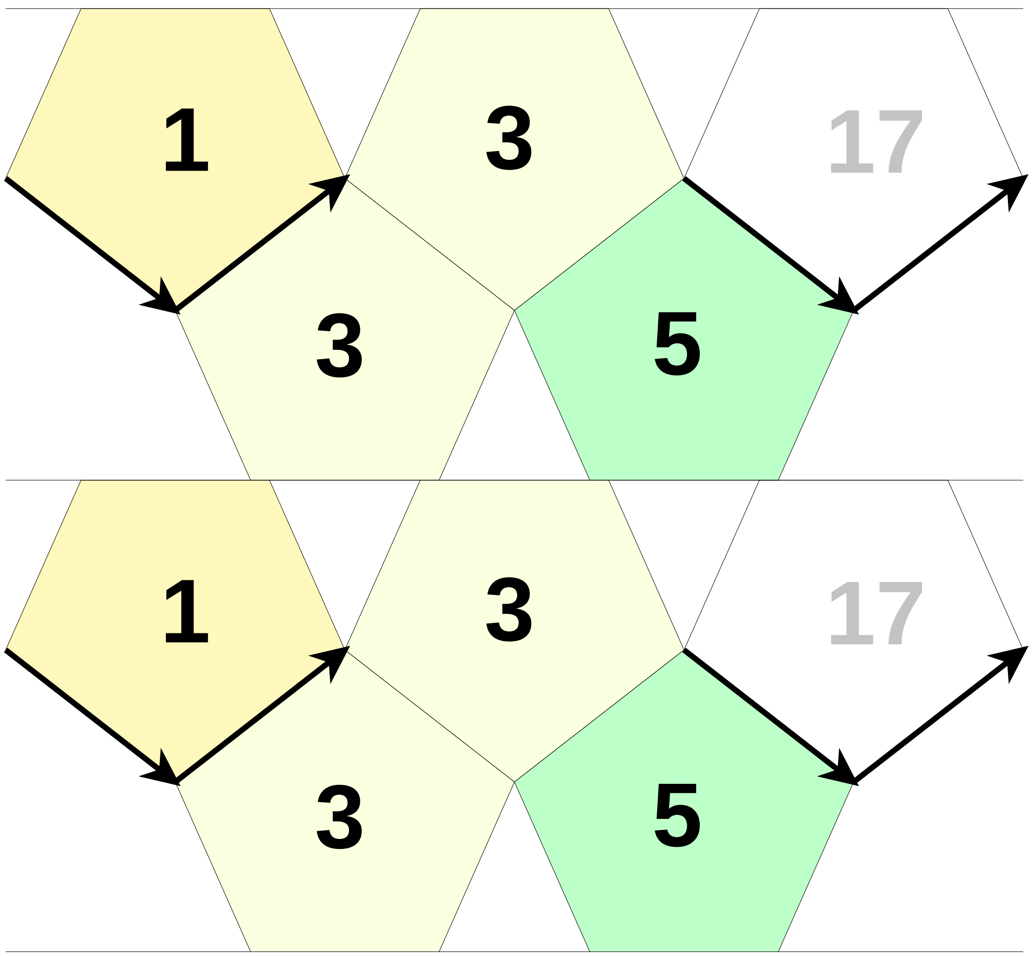


Figure 29: Top side of a segment which can be cut out to make a second level Pentagonal Flexagon. This is the first of five segments.

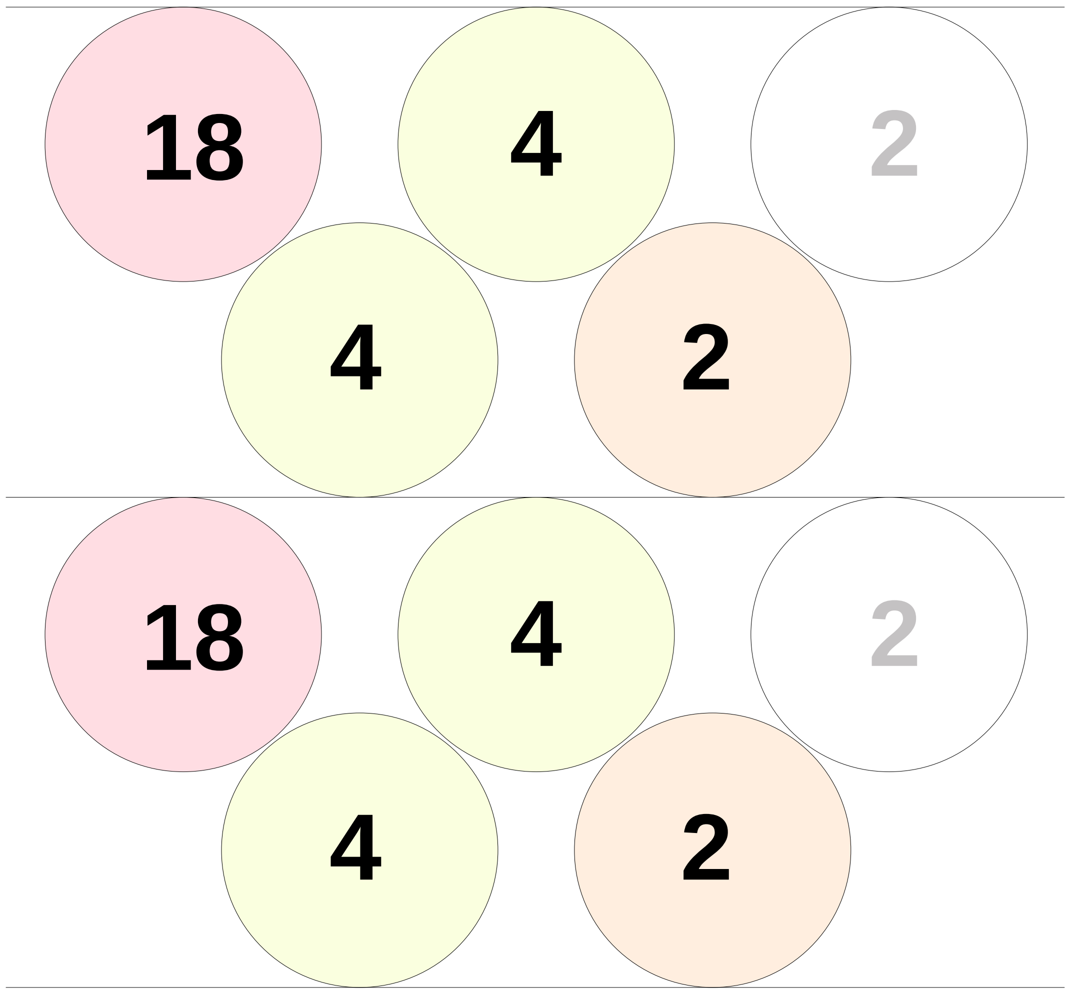


Figure 30: Bottom side of a second order Pentagonal Flexagon, first of five segments.

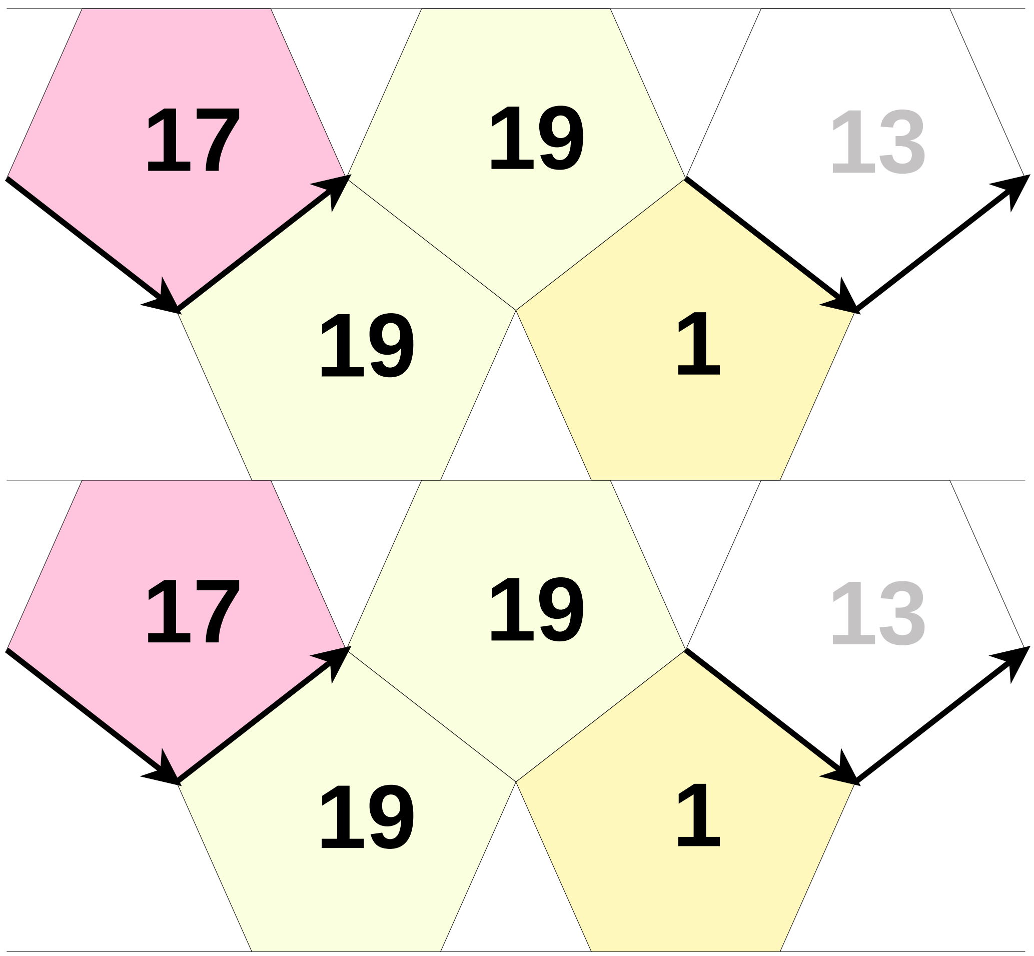


Figure 31: Top side of a segment which can be cut out to make a second level Pentagonal Flexagon. This is the second of five segments.

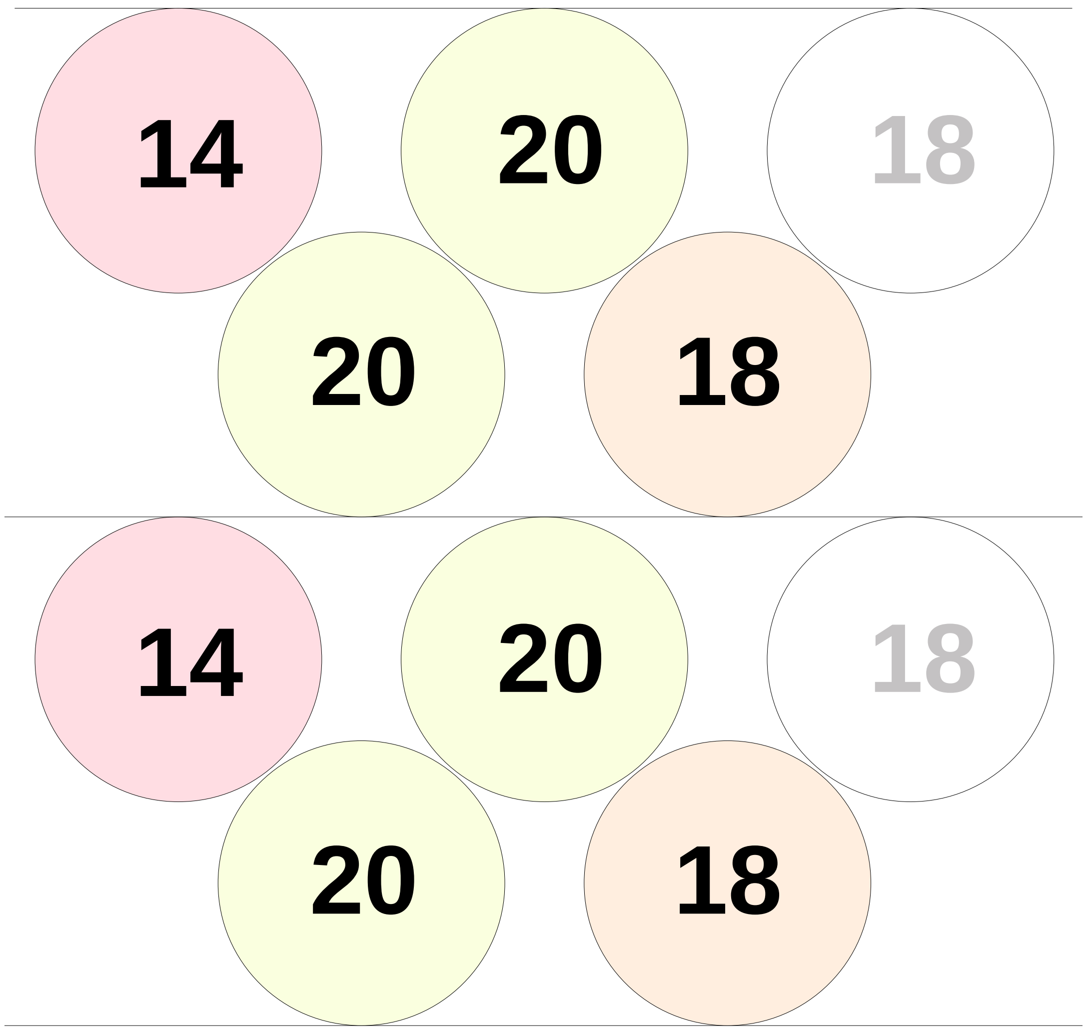


Figure 32: Bottom side of a second order Pentagonal Flexagon, second of five segments.

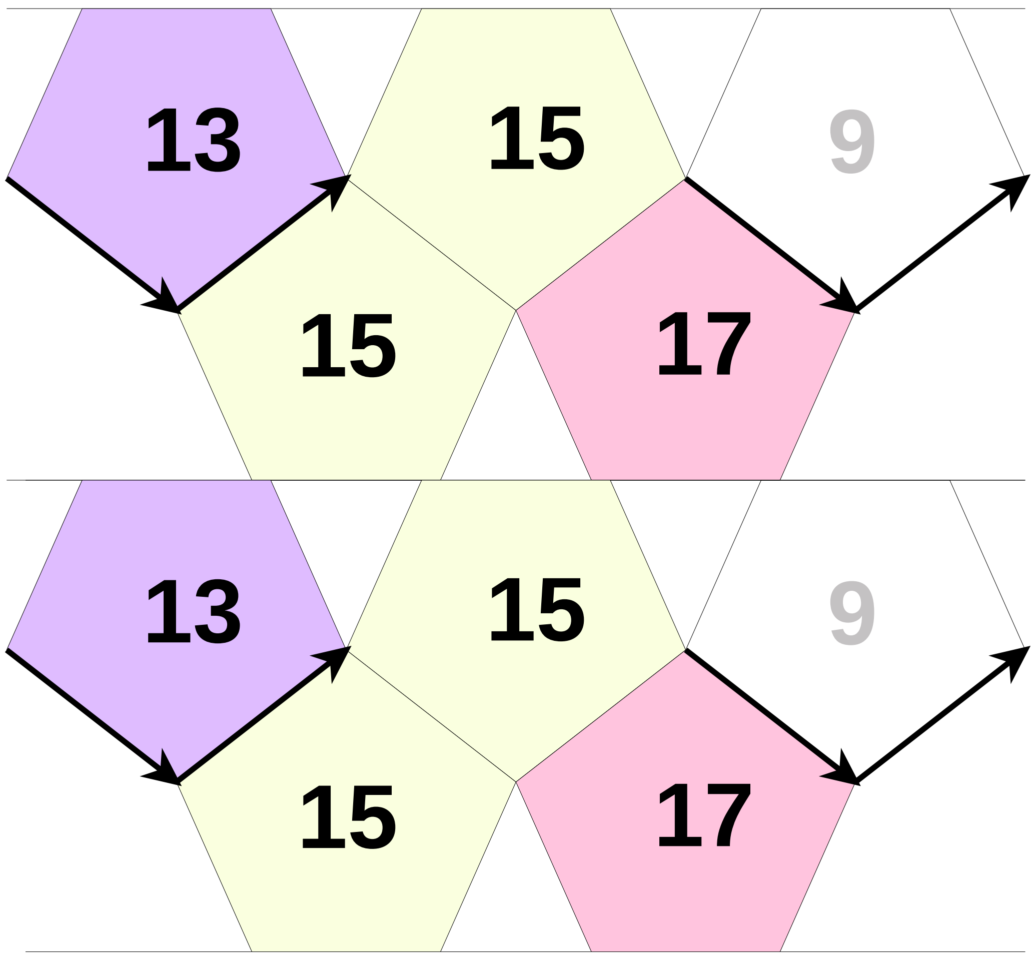


Figure 33: Top side of a segment which can be cut out to make a second level Pentagonal Flexagon. This is the third of five segments.

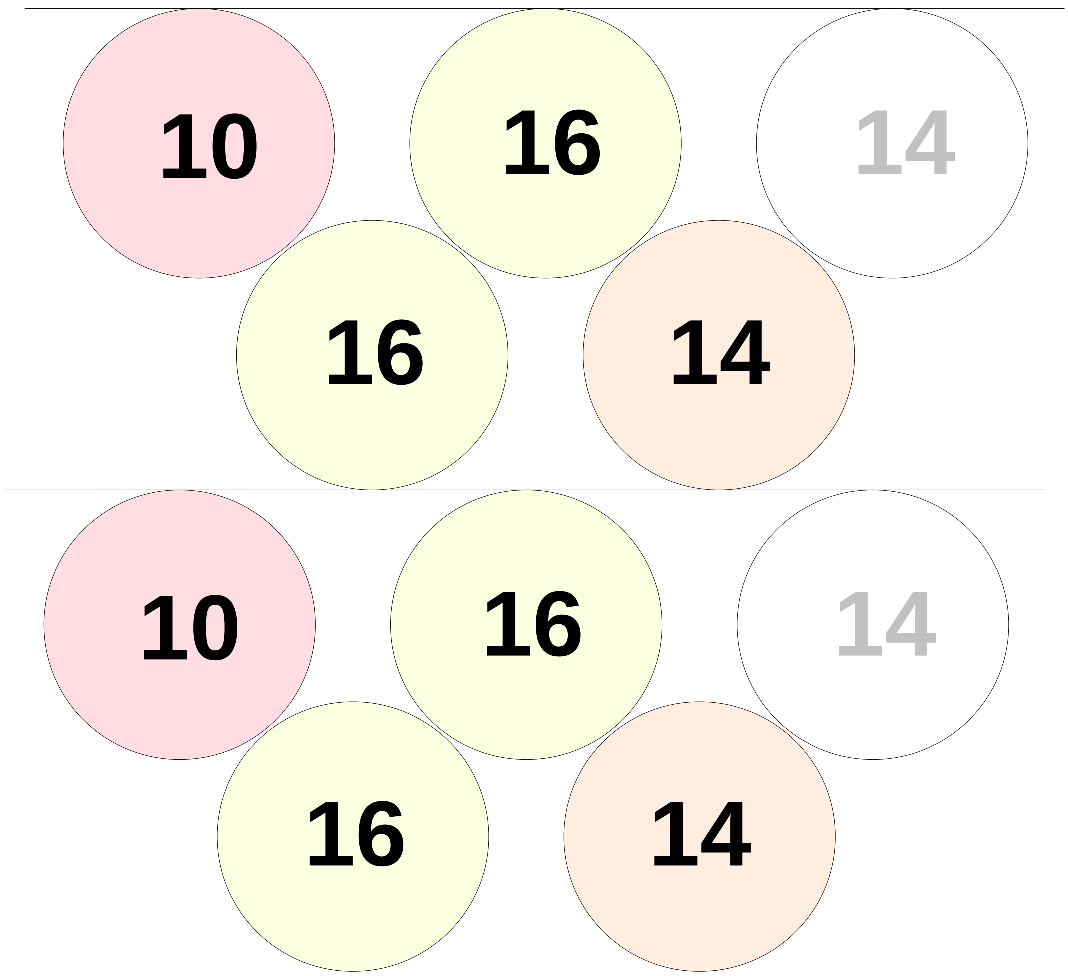


Figure 34: Bottom side of a second order Pentagonal Flexagon, third of five segments.

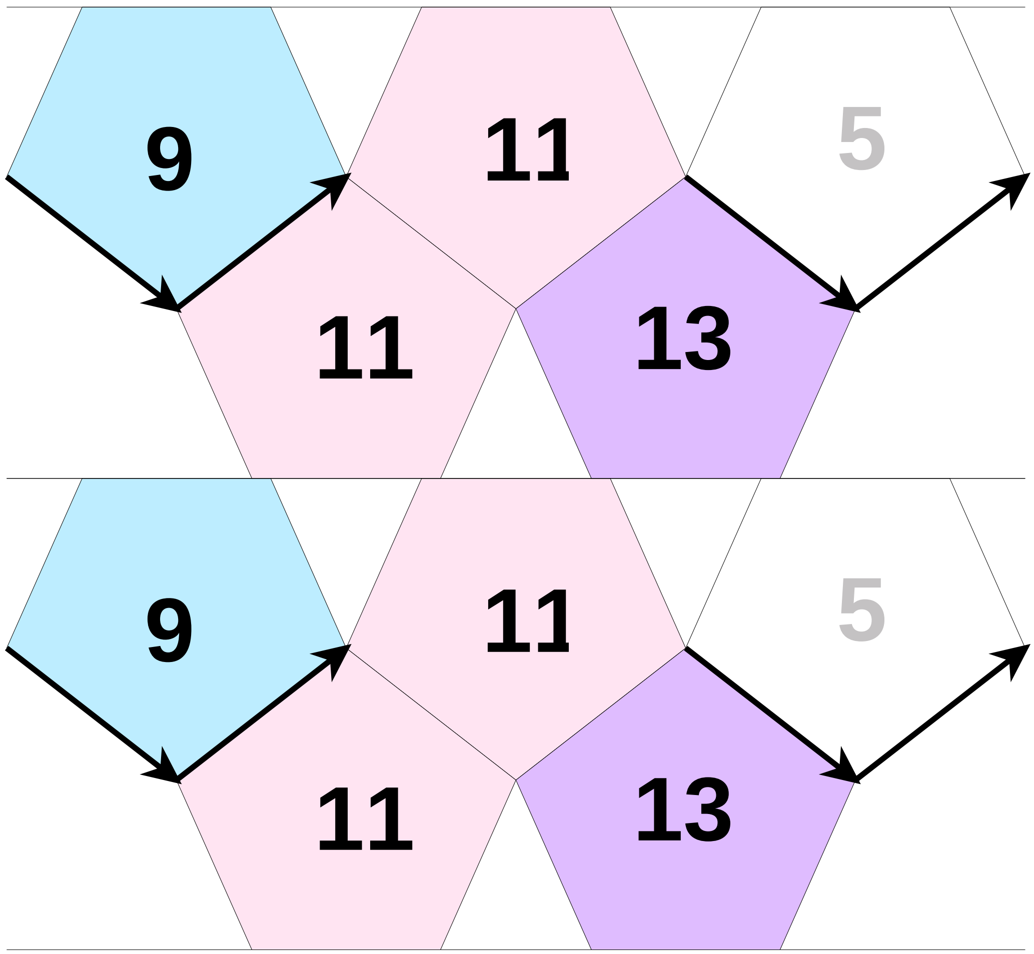


Figure 35: Top side of a segment which can be cut out to make a second level Pentagonal Flexagon. This is the fourth of five segments.

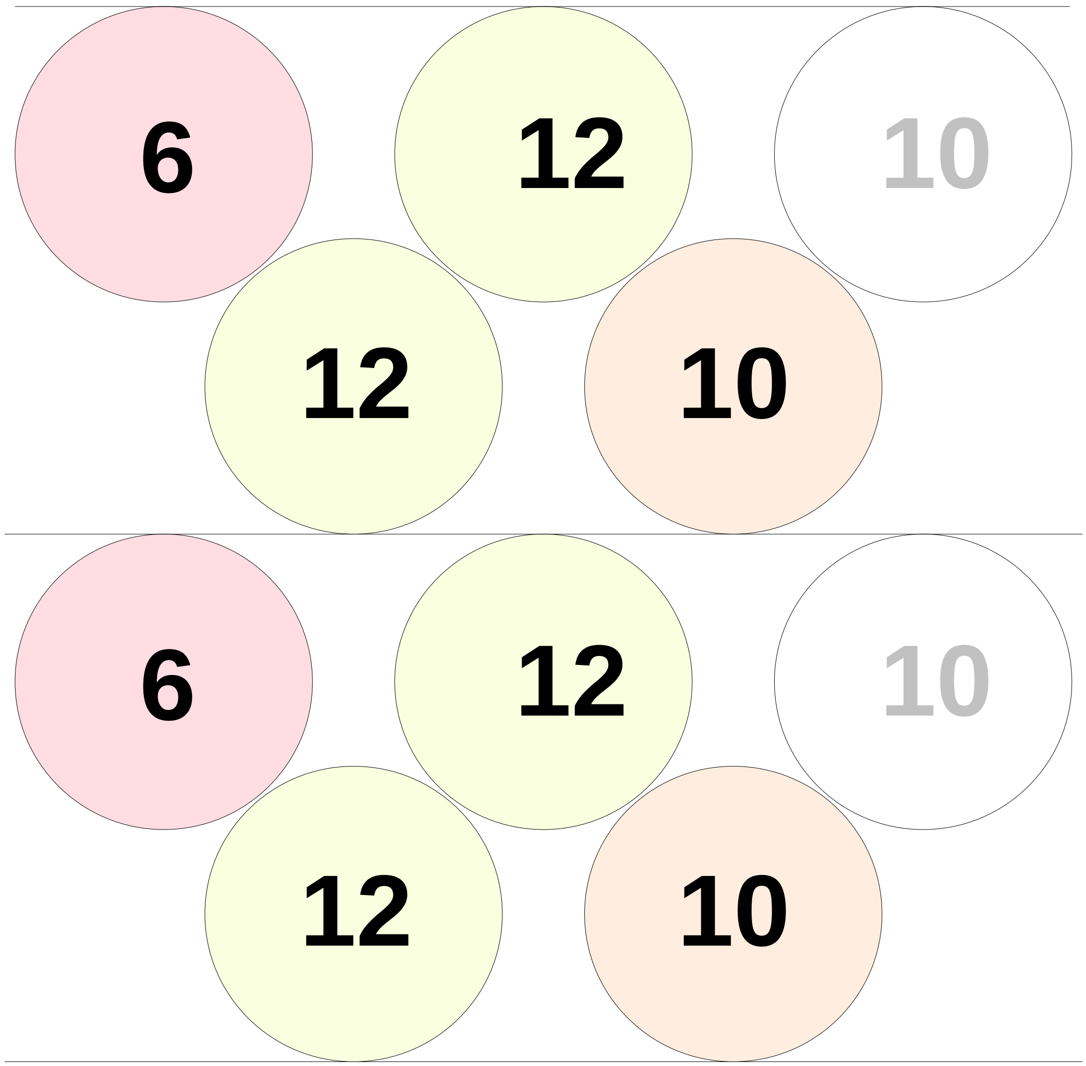


Figure 36: Bottom side of a second order Pentagonal Flexagon, fourth of five segments.

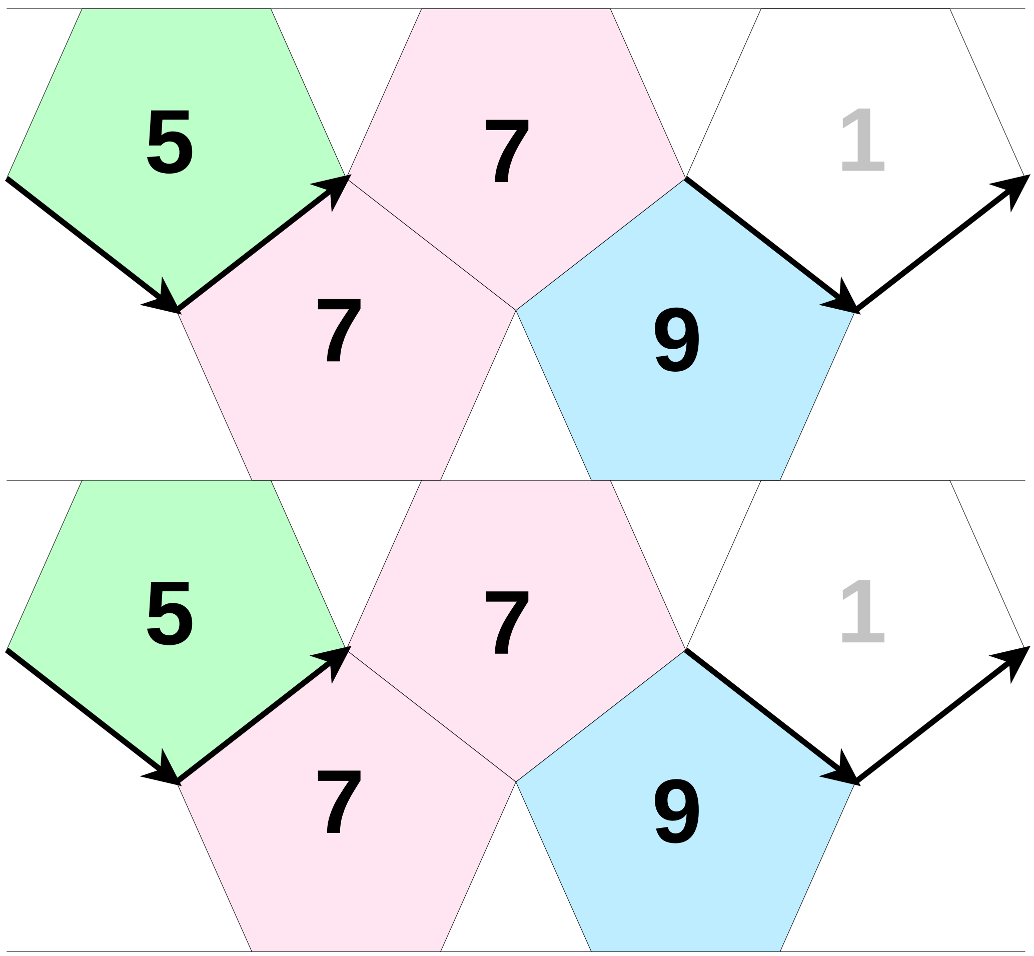


Figure 37: Top side of a segment which can be cut out to make a second level Pentagonal Flexagon. This is the fifth and last of five segments.

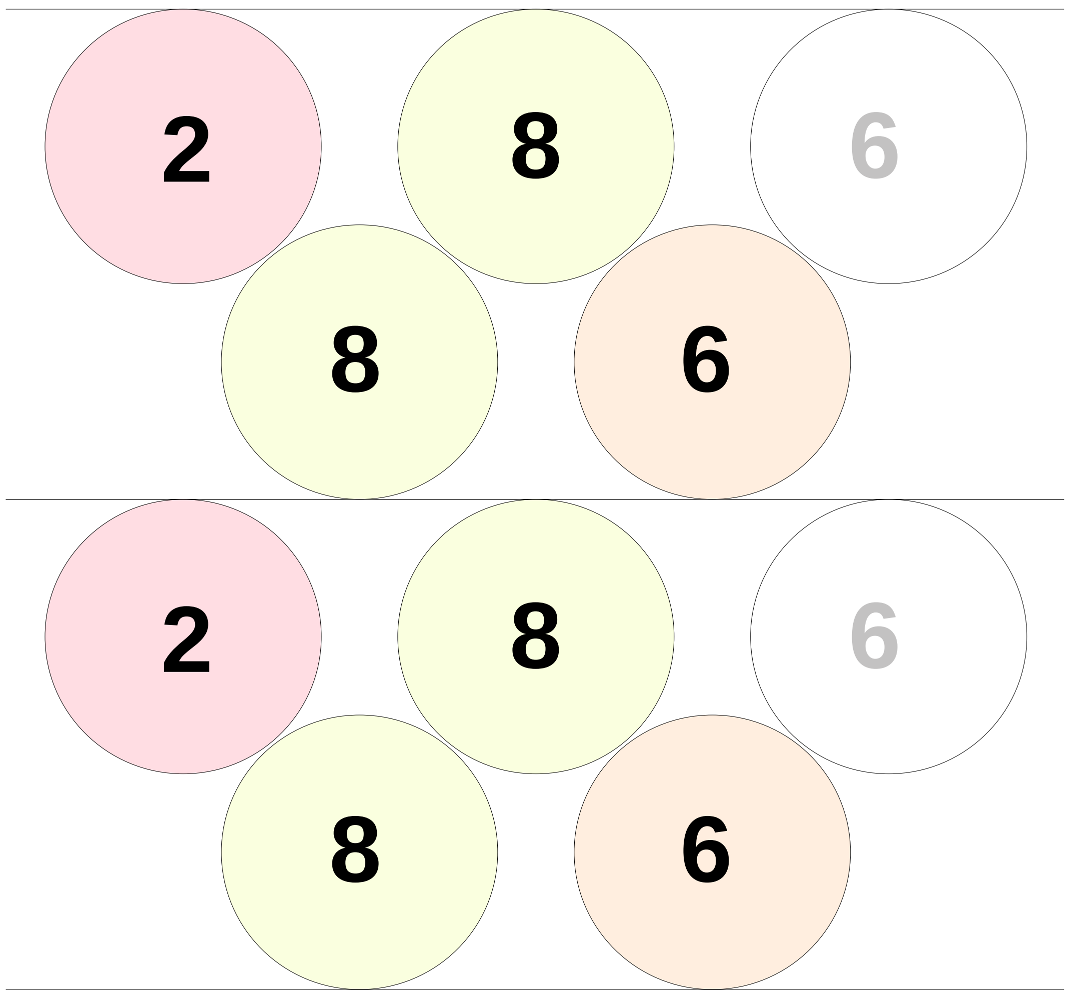


Figure 38: Bottom side of a second order Pentagonal Flexagon, fifth and last of five segments.