Tetragonal Flexagons

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March 11, 2000
revised June 10, 2000

Abstract

Maps and cutouts for a variety of flexagons are presented, emphasizing those which can be cut out, mostly from single sheets of paper. Since TeX may not align front and back images, and in any event if cutting up the booklet is not desired, the .eps files can be printed directly to get sheets suitable for cutting. In the same spirit, only those sheets which are going to be used right away need be printed.

Contents

1 Introduction

Flexagons can become fairly complicated. The ones based on triangles are most conveniently made from long strips of paper; a roll of adding machine or calculator tape is ideal for this purpose given its convenient width. Crooked strips can be gotten by gluing faces together, or just cutting out segments and then joining them together. Leaving one extra triangle in each segment for overlapping and later gluing leads to efficient constructions.

Coloring the triangles is another problem, which can be done with crayons or markers once it is known which colors ought to be used. Aside from copying an already existent design, this is best done by drawing the Tukey triangles and then lettering or numbering the triangles in the strip. That information is sufficient to fold up the strip, since pairs of consecutive numbers are to be hidden by folding them together. Painting can be done before folding by following a color code for the numbers, or after the folding is done, when the faces can be painted wholesale, or even embellished with designs.

Other flexagons, even the ones folded from "straight" strips, require a higher degree of preparation, although it is relatively easy to assemble a collection of primitive components which later can be glued together according to the necessities of the individual flexagon.

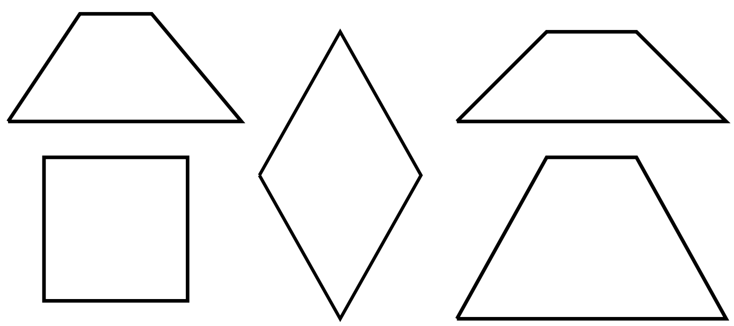


Figure 1: Some regular and semi-regular tetragons. In designing a flexagon, the more angles which are small divisors of 360°, the more readily will the flexagon lie flat in the plane.

Figure 1 shows seveal candidates for tetragonal flexagons. The square, being a regular 4-gon, wiull lead to mormal flexagons which will lie flat with only two sectors.. However, the rhombus, with 60° and 120° angles will lie flat with three sectors, and still produce a very attractive flexagon. Trapezoids also give good flexagons, especially when made with a choice of these same angles.

All of them are susceptible to recursive elaboration, although the presence of different angles will lead to somewhat different shapes while they are being flexed if the recursion is not applied uniformly throughout.

2 First Level Tetraflexagon

2.1 Tukey triangles

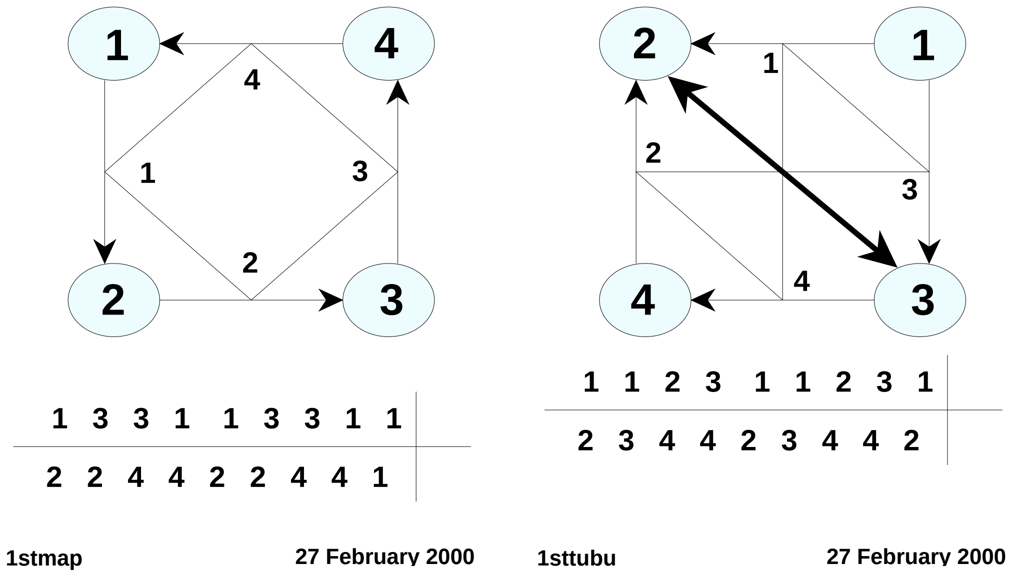


Figure 2: Left: Tukey quadrangle map for the normal first level tetraflexagon, which consists of a single cycle of four colors. Right: Triangle map for the tubulating first level tetraflexagon, likewise displaying four colors, but which contains a single tubulation.

2.2 Flexagon permutations

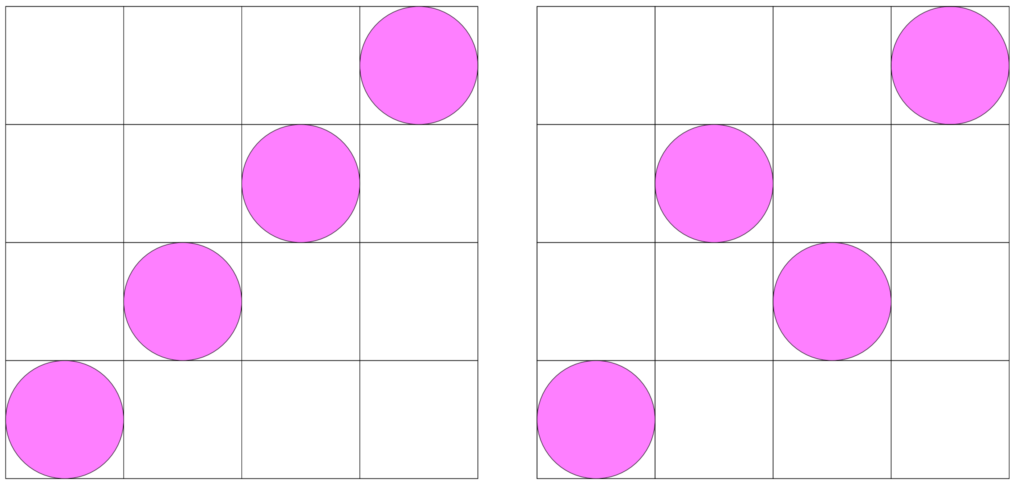


Figure 3: Left: Permutation of the squares along the strip for a normal tetraflexagon. They run in order, subject to being flipped over, so the permutation is the identity. Right: Permutation for the tubulating flexagon. The normal order is not preserved, the permutation is not the identity.

2.3 First level normal tetraflexagon cutout

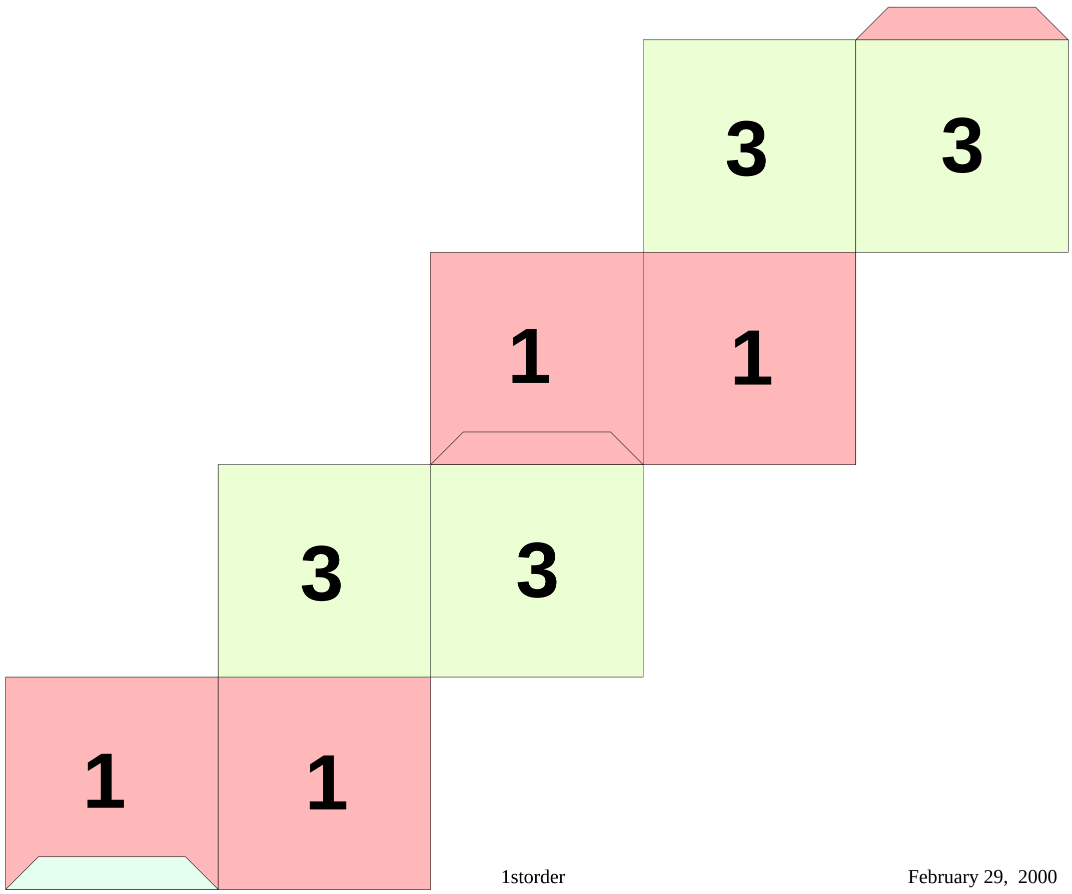


Figure 4: Top side of the first level normal tetraflexagon cutout. Together with its backside, the figure contains one single tetraflexagon.

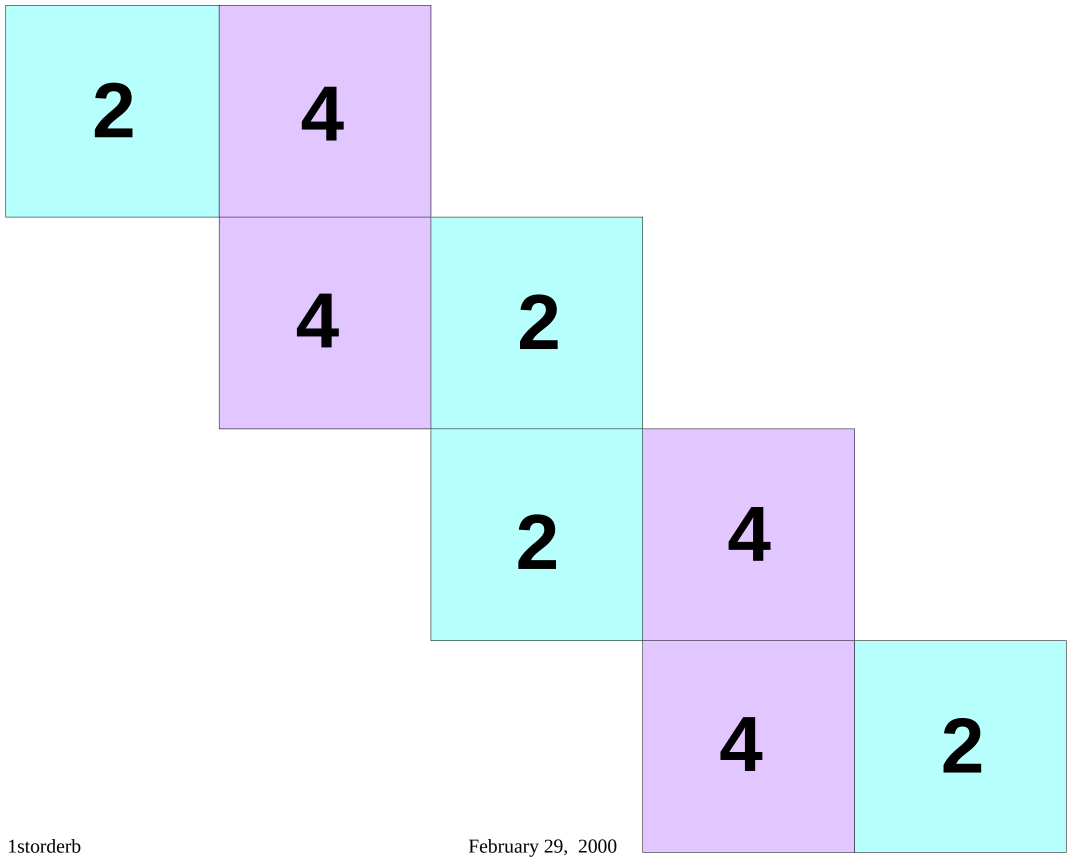


Figure 5: Bottom side of the first level normal tetraflexagon cutout.

2.4 First level tubulating tetraflexagon cutout

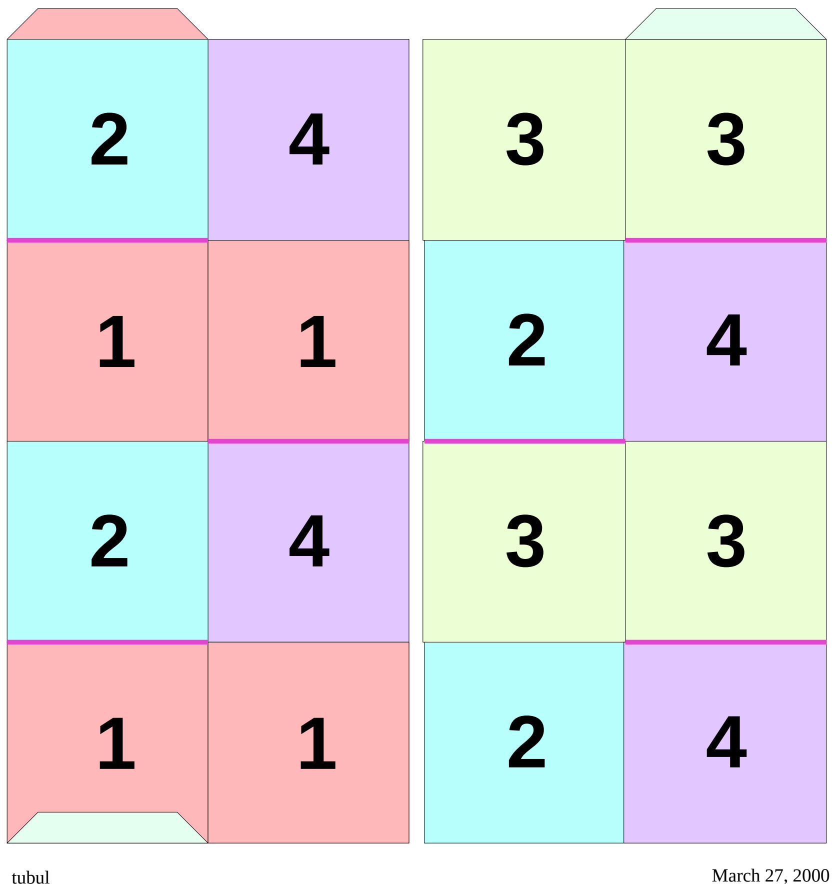


Figure 6: Top side of the first level tubulating tetraflexagon cutout, characterized by the sign sequence (+ + - -). The cutout is small enough that both top and bottom for two sectors are shown, enough to make two separate flexagons.

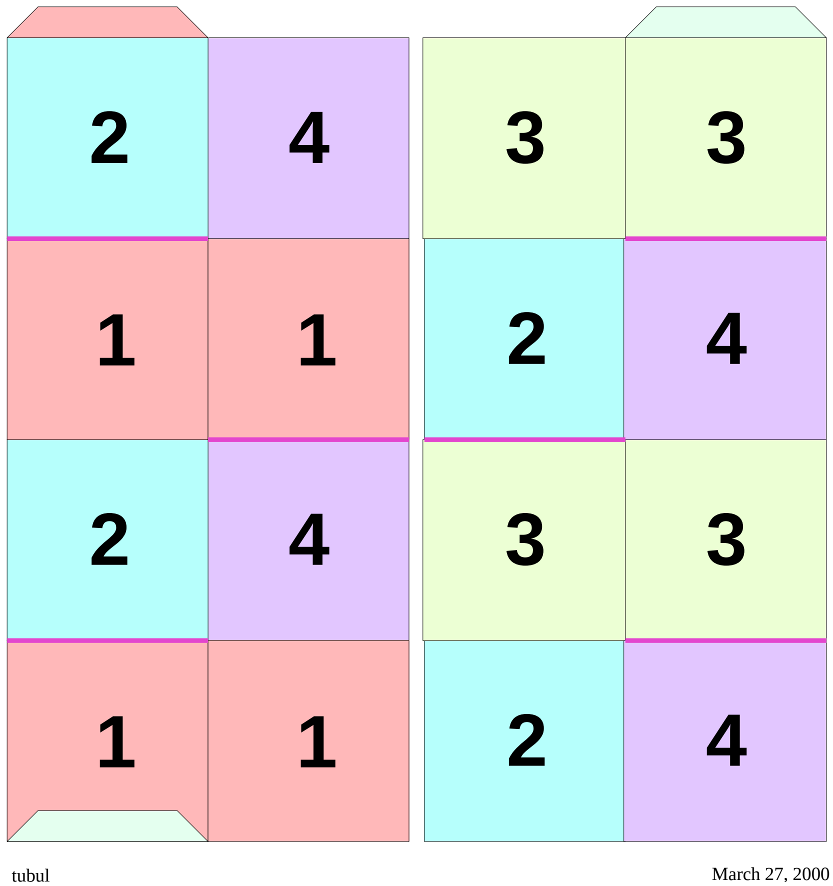


Figure 7: reverse side of the first level tubulating tetraflexagon cutout.

2.5 Alternative first level tubulating tetraflexagon cutout

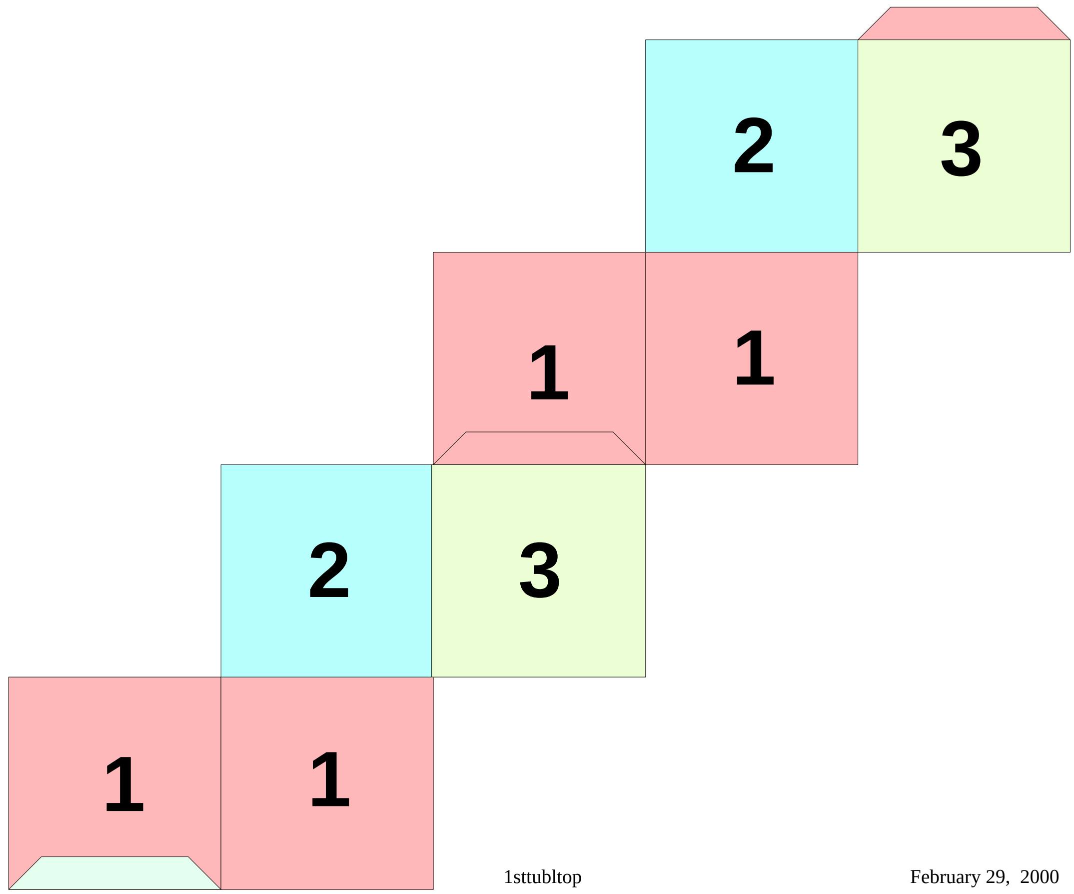


Figure 8: Top side of the alternative first level tubulating tetraflexagon cutout, characterized by the sign sequence $(++++)$. The figure, taken with its backside, makes one single flexagon consisting of two sectors.

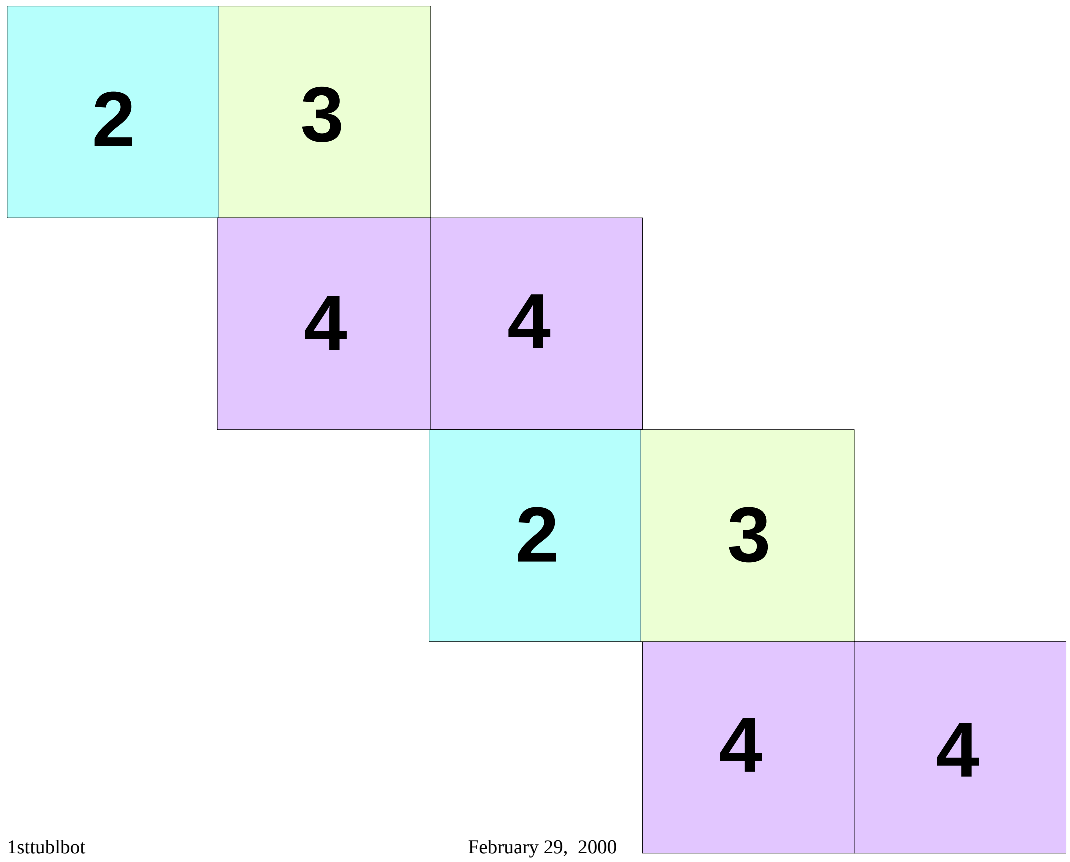


Figure 9: Bottom side of the alternative first level tubulating tetraflexagon cutout.

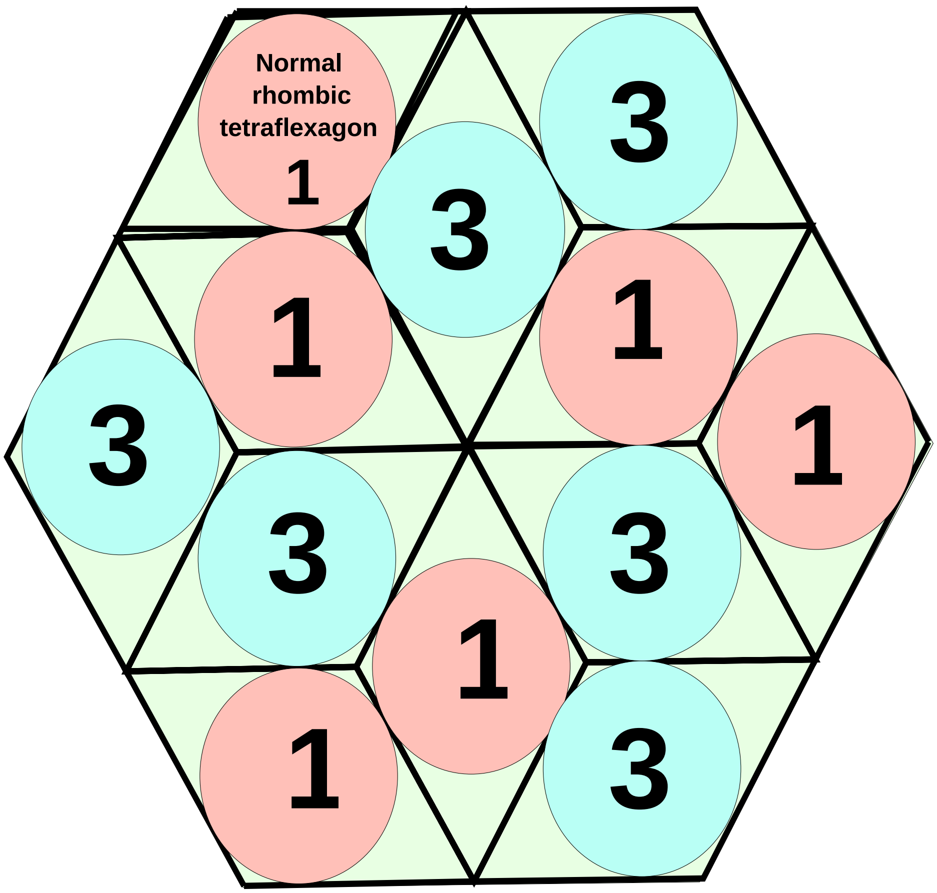


Figure 10: Top side of the first level normal rhomboidal tetraflexagon cutout. Together with its backside, the figure contains one single tetraflexagon. Three sectors are required to make some of the faces lie flat on account of the 60° angles.

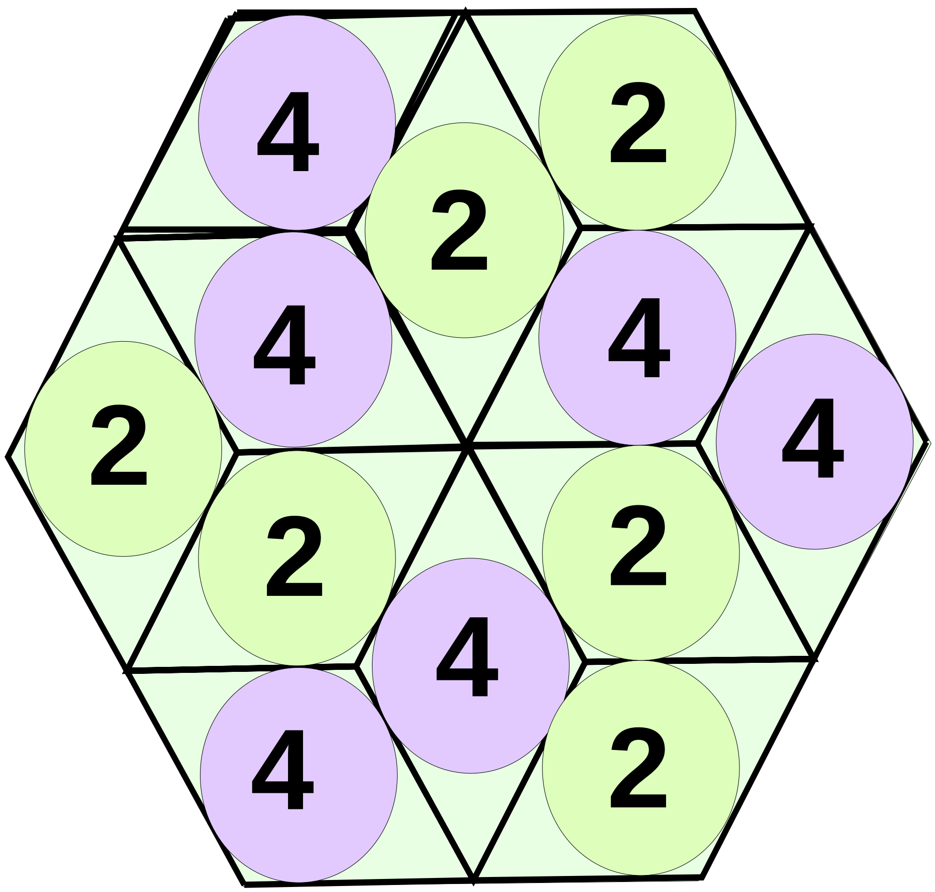
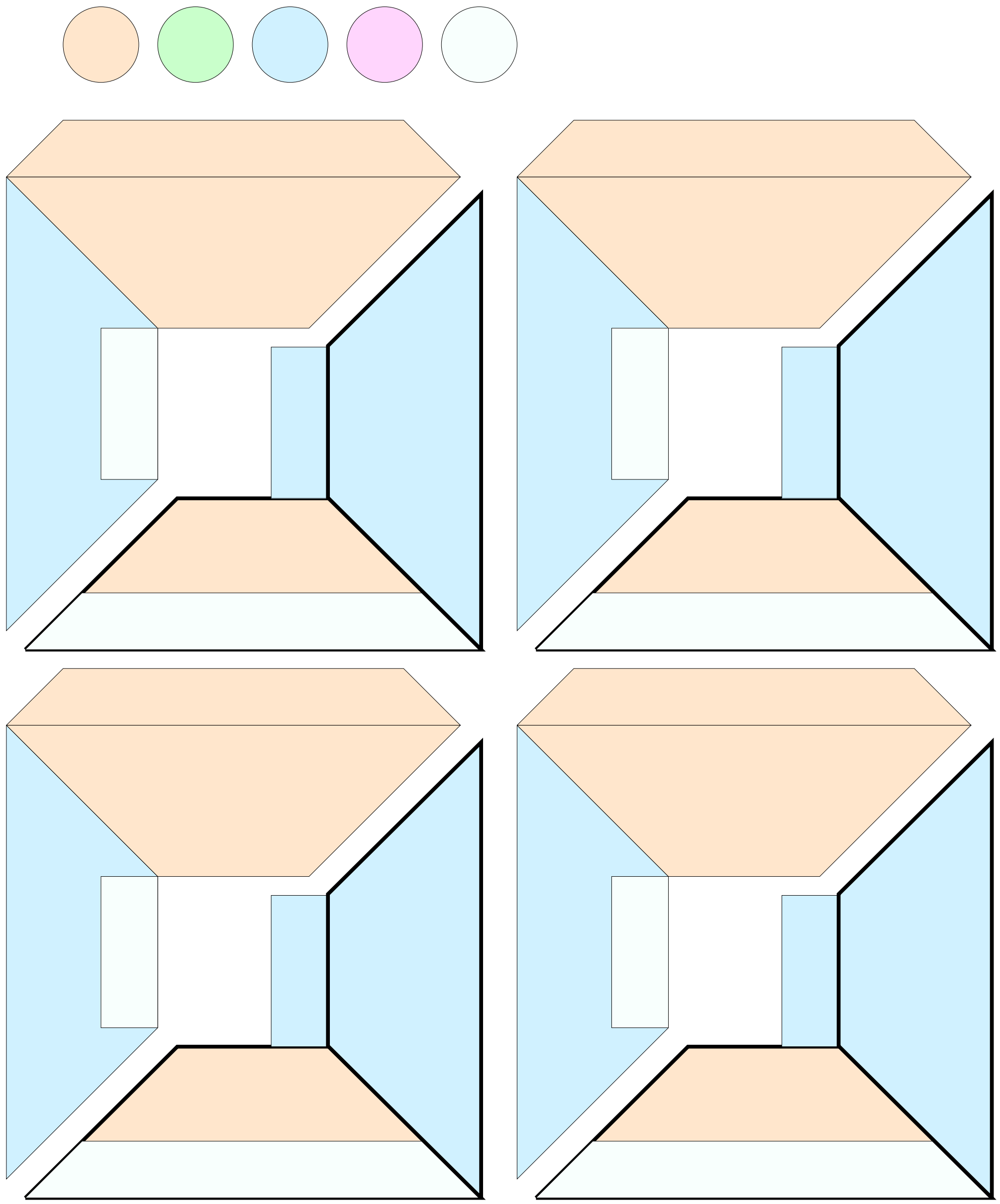


Figure 11: Bottom side of the first level normal rhomboidal tetraflexagon cutout.



45trapplant September 27, 2000

Figure 12: Top side of the 45° normal trapezoidal cutout. Together with its backside, the figure contains one single tetraflexagon. Four sectors are required to make at least one of the faces lie flat on account of the 45° angles.

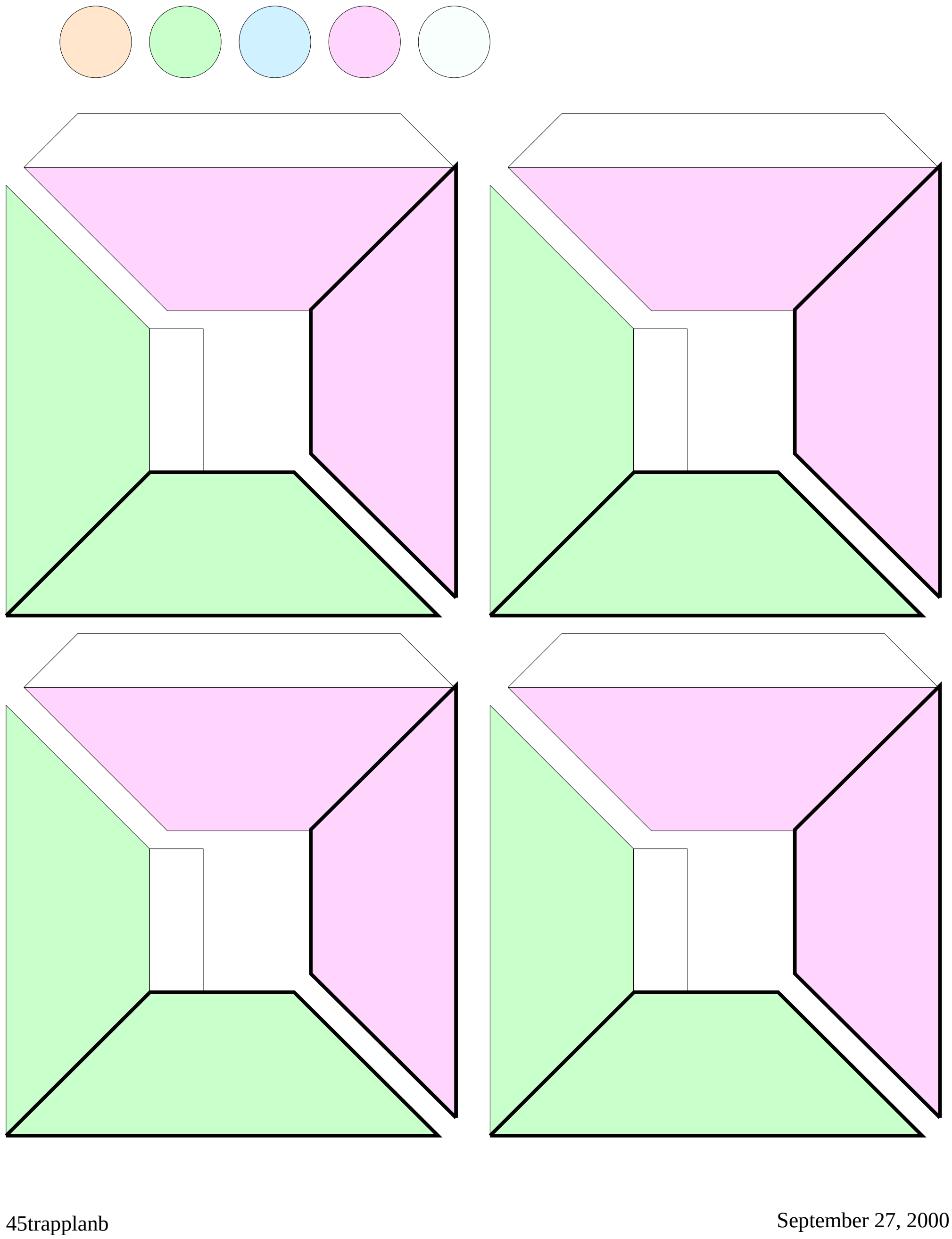


Figure 13: Bottom side of the 45° normal trapezoidal tetraflexagon cutout.

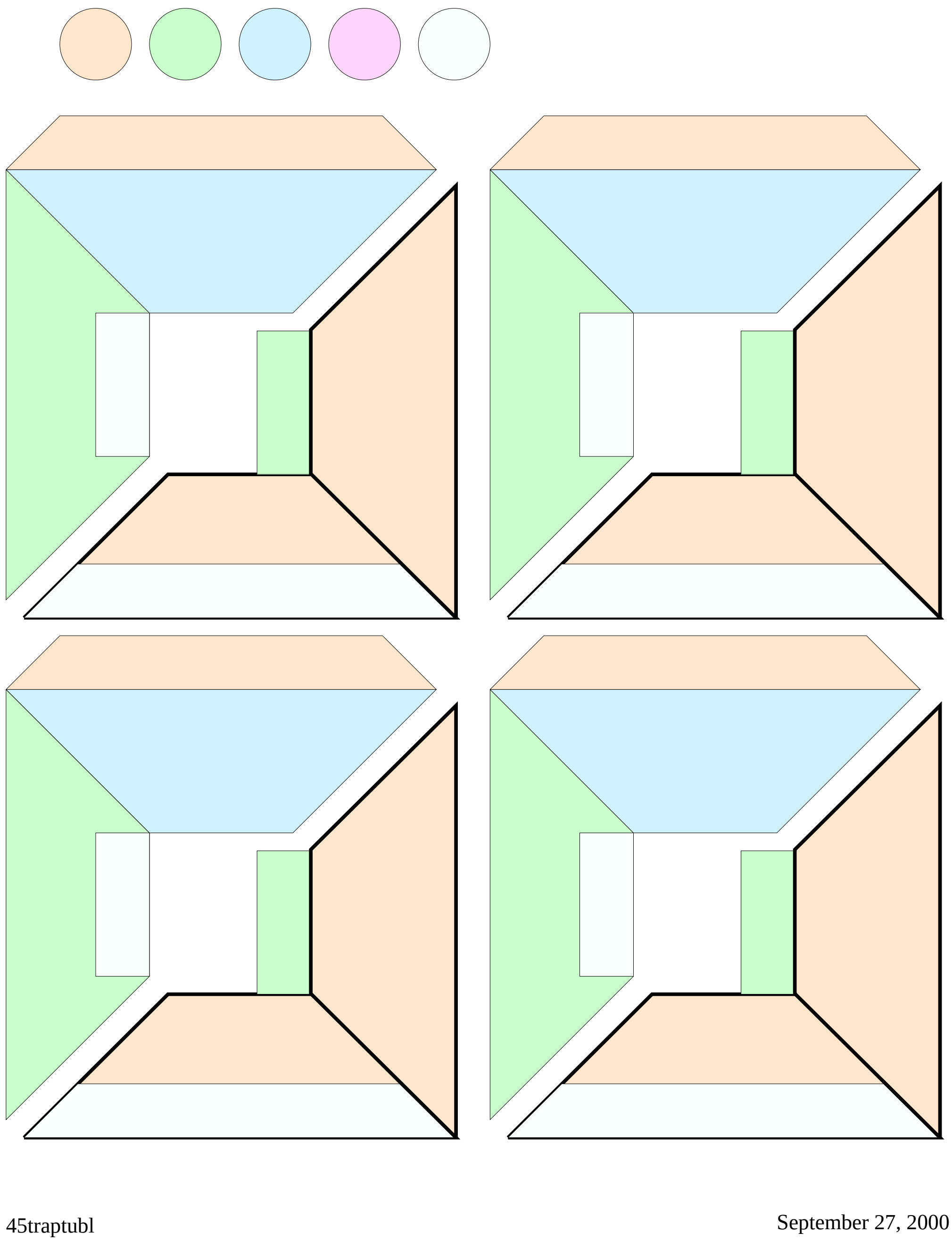


Figure 14: Top side of the 45° tubulating trapezoidal cutout. Together with its backside, the figure contains one single tetraflexagon. Four sectors are required to make at least one of the faces lie flat on account of the 45° angles.

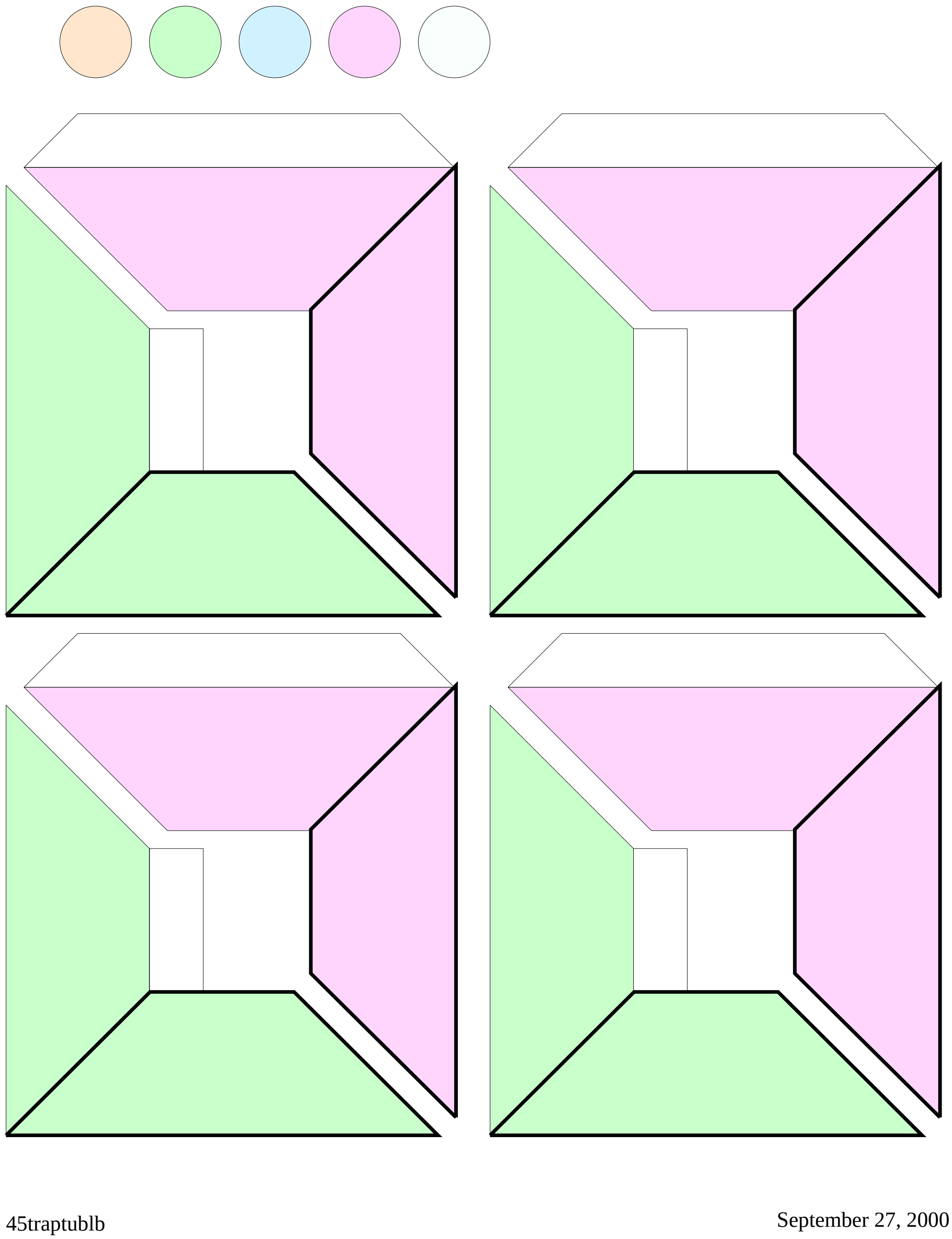


Figure 15: Bottom side of the 45° normal trapezoidal tetraflexagon cutout.

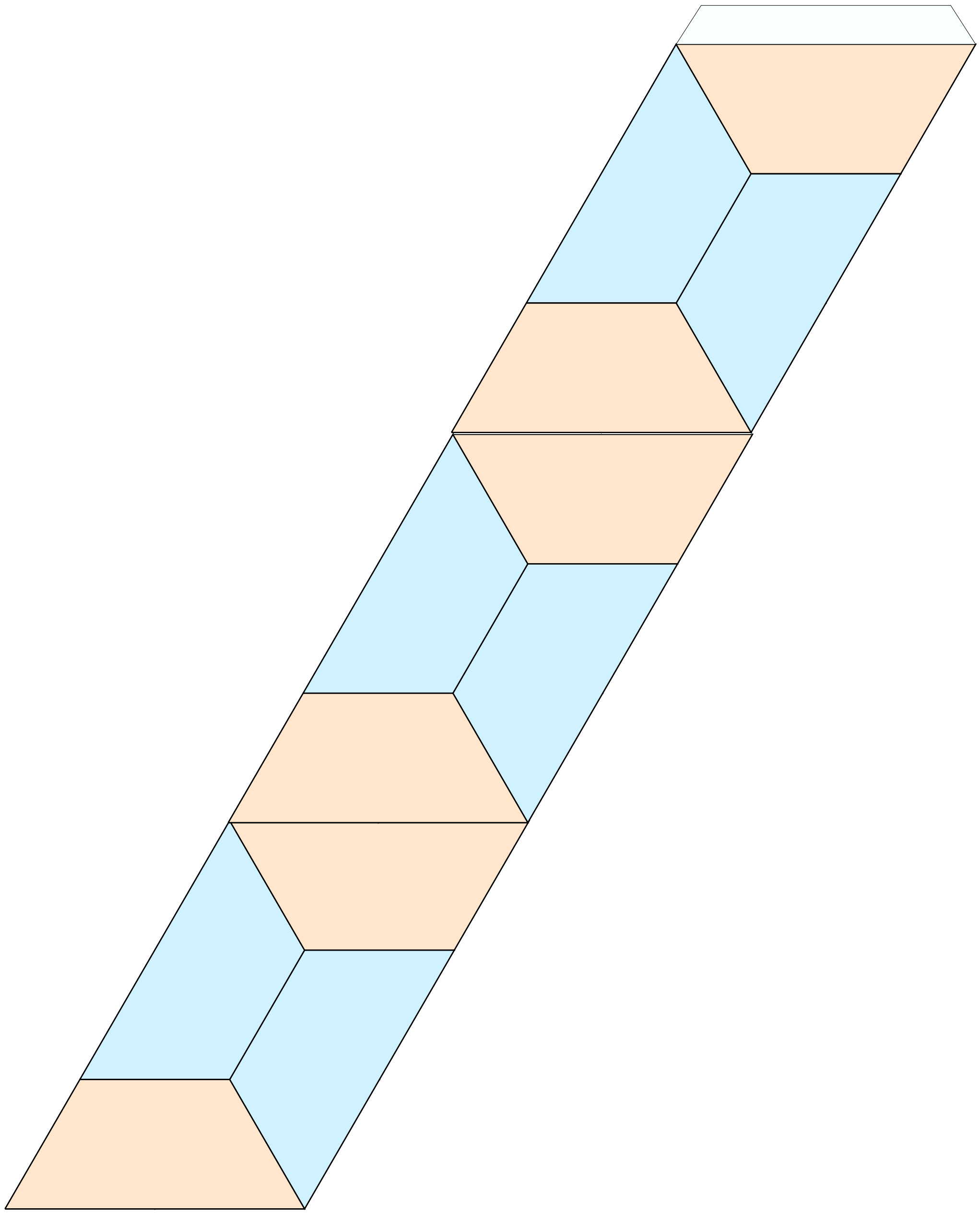


Figure 16: Top side of the 60° normal trapezoidal cutout. Together with its backside, the figure contains one single tetraflexagon. Three sectors are required to make at least one of the faces lie flat on account of the 60° angles.

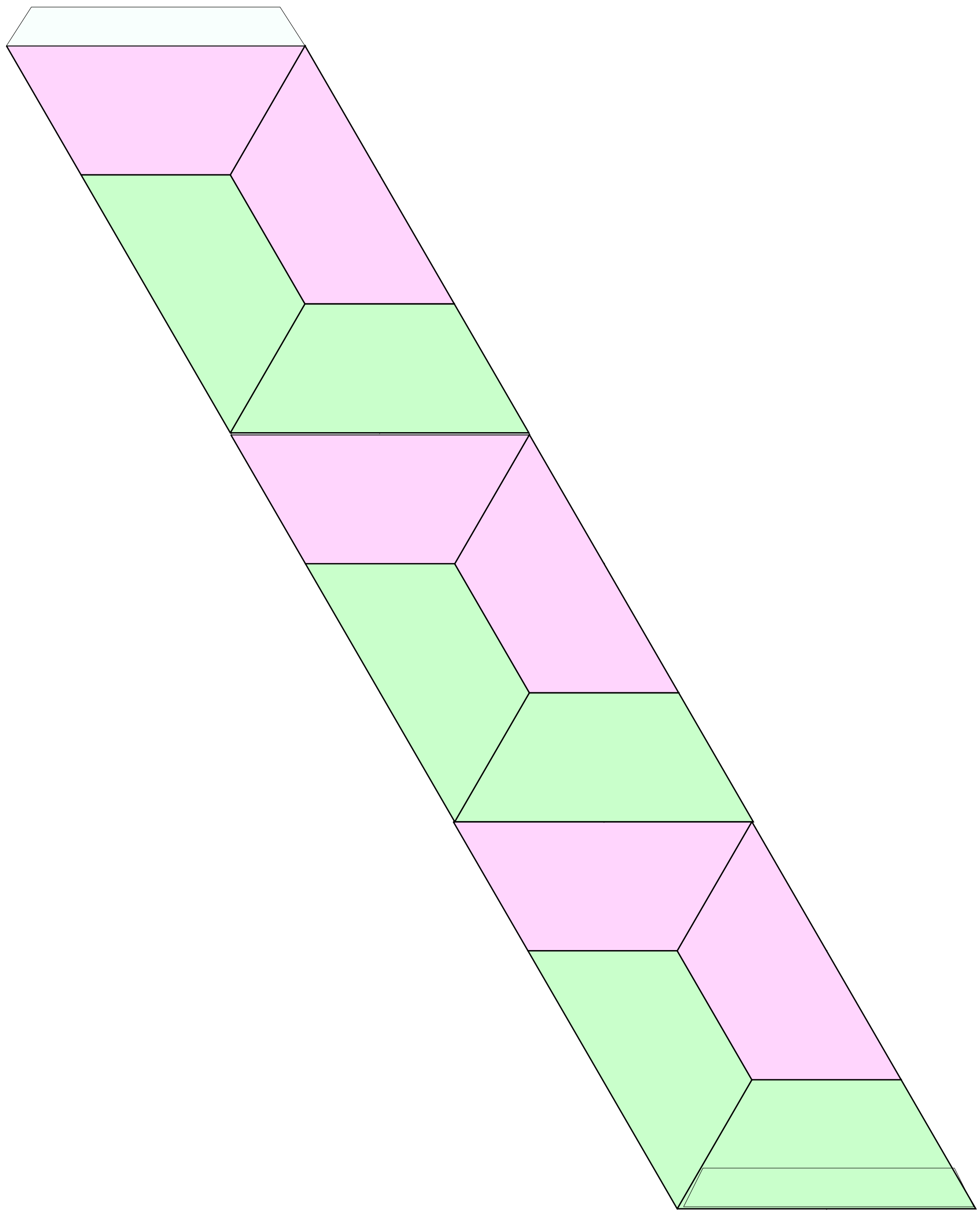


Figure 17: Bottom side of the 60° normal trapezoidal tetraflexagon cutout.

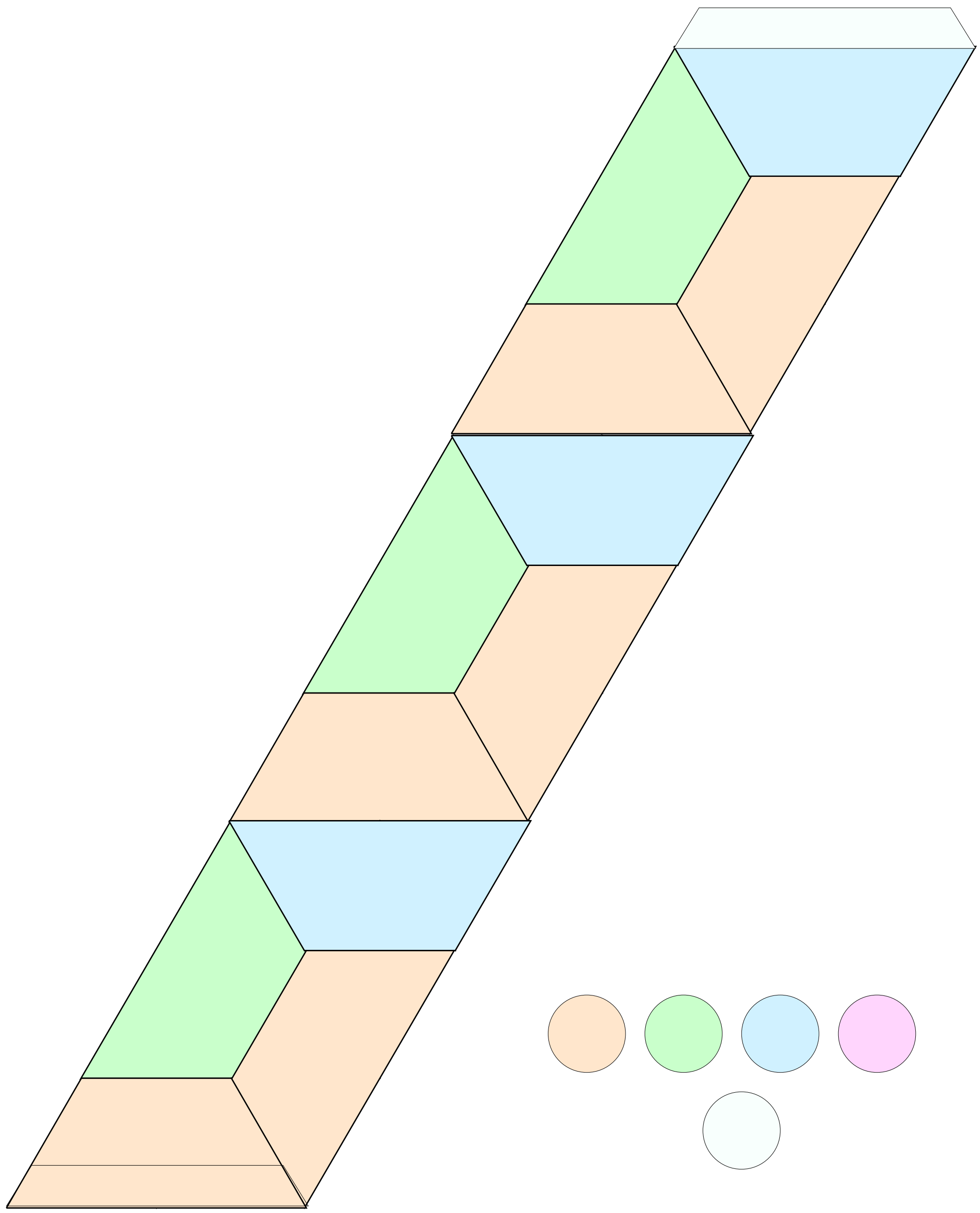


Figure 18: Top side of the 60° tubulating trapezoidal cutout. Together with its backside, the figure contains one single tetraflexagon. Three sectors are required to make at least one of the faces lie flat on account of the 60° angles.

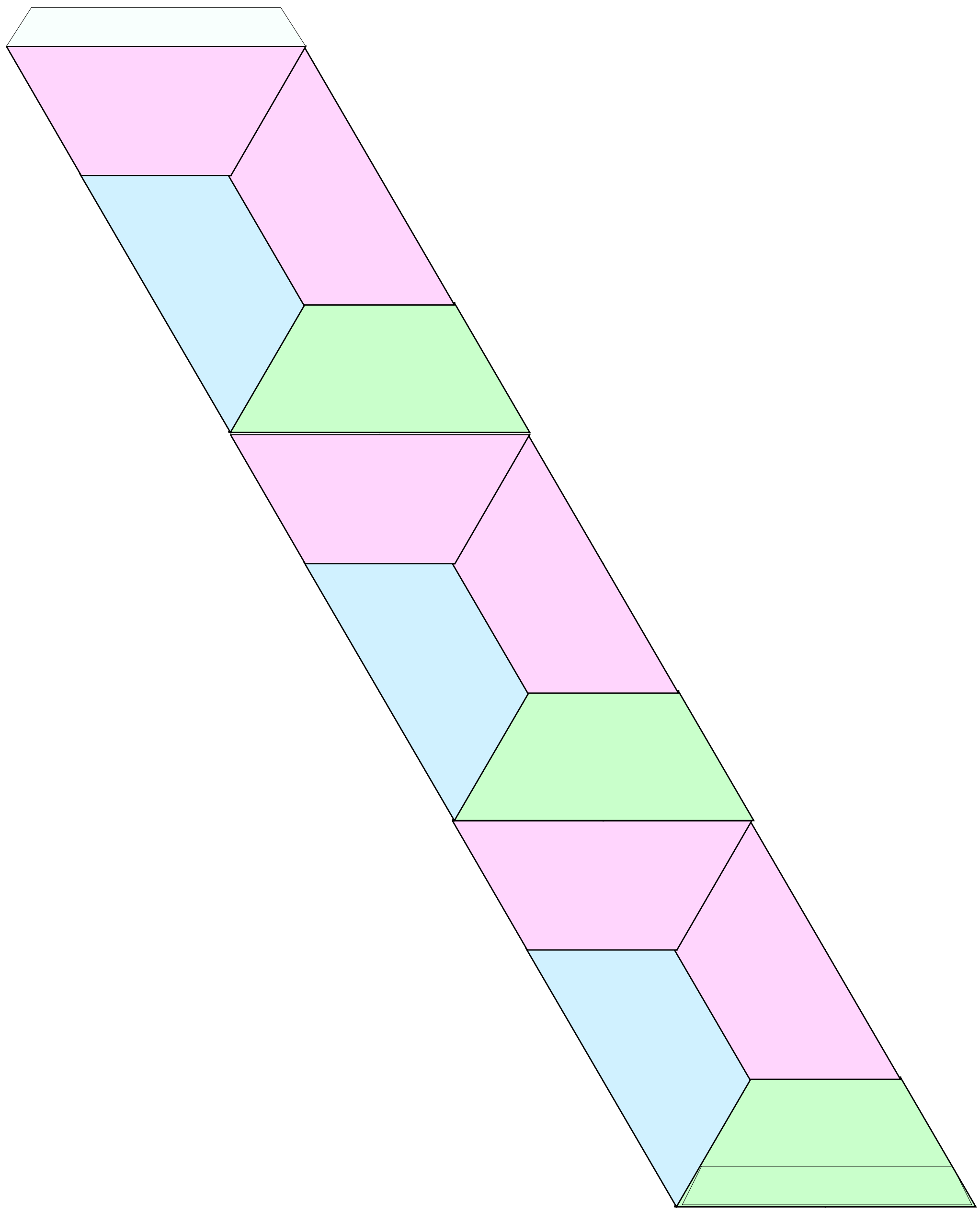


Figure 19: Bottom side of the 60° tubulating trapezoidal tetraflexagon cutout.

3 Binary Tetraflexagon

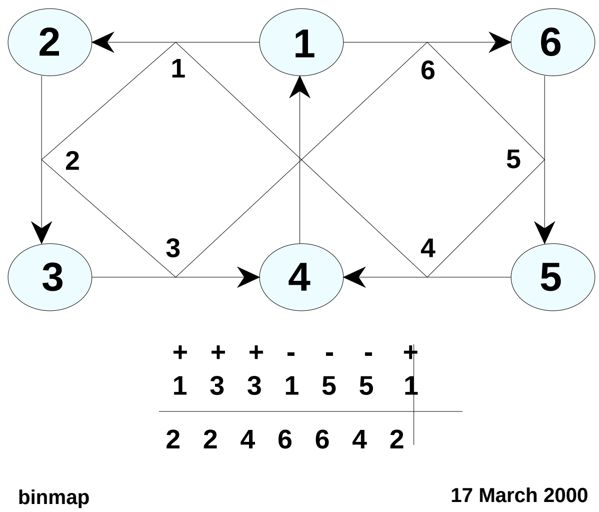


Figure 20: The binary tetraflexagon has two cycles, each of which has two vertices in common with the other.

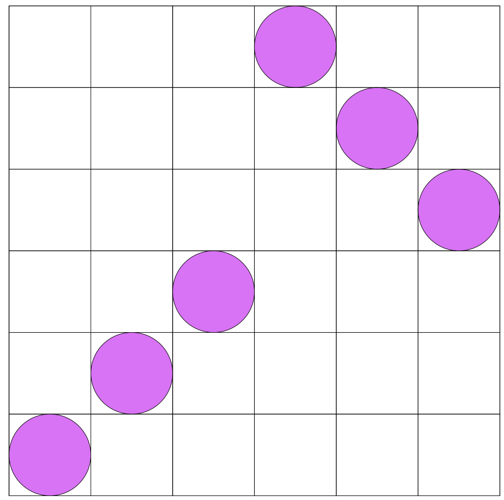


Figure 21: Permutation of the squares along the strip for a binary tetraflexagon.

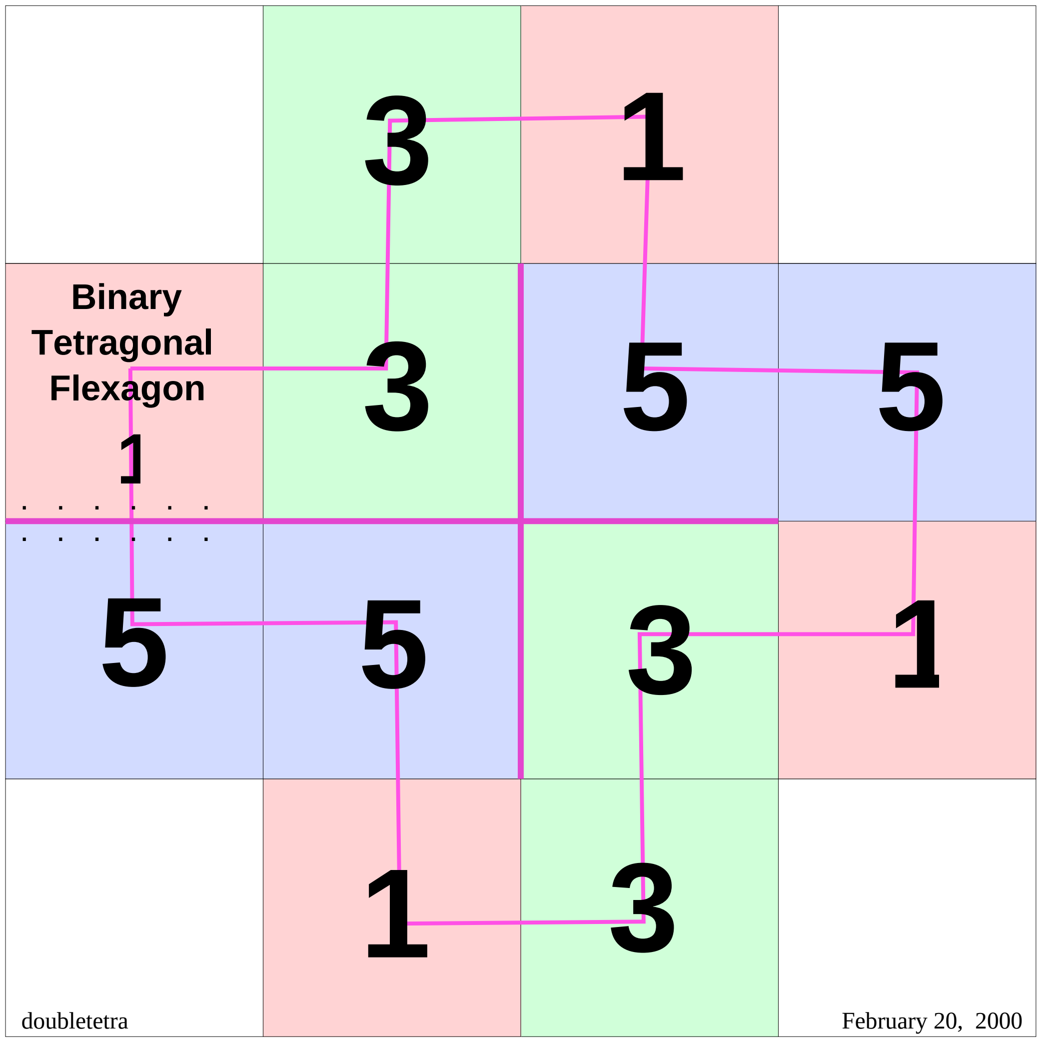


Figure 22: Top side of the binary tetraflexagon cutout. The flexagon has six faces, so this cutout provides material for both of the sectors needed for the flexagon. Cuts should be made along the heavy lines, edges joined (using an extra little strip of paper taken from an unused spot) along the edges marked by dots. Together with the backside, the figure makes one individual flexagon.

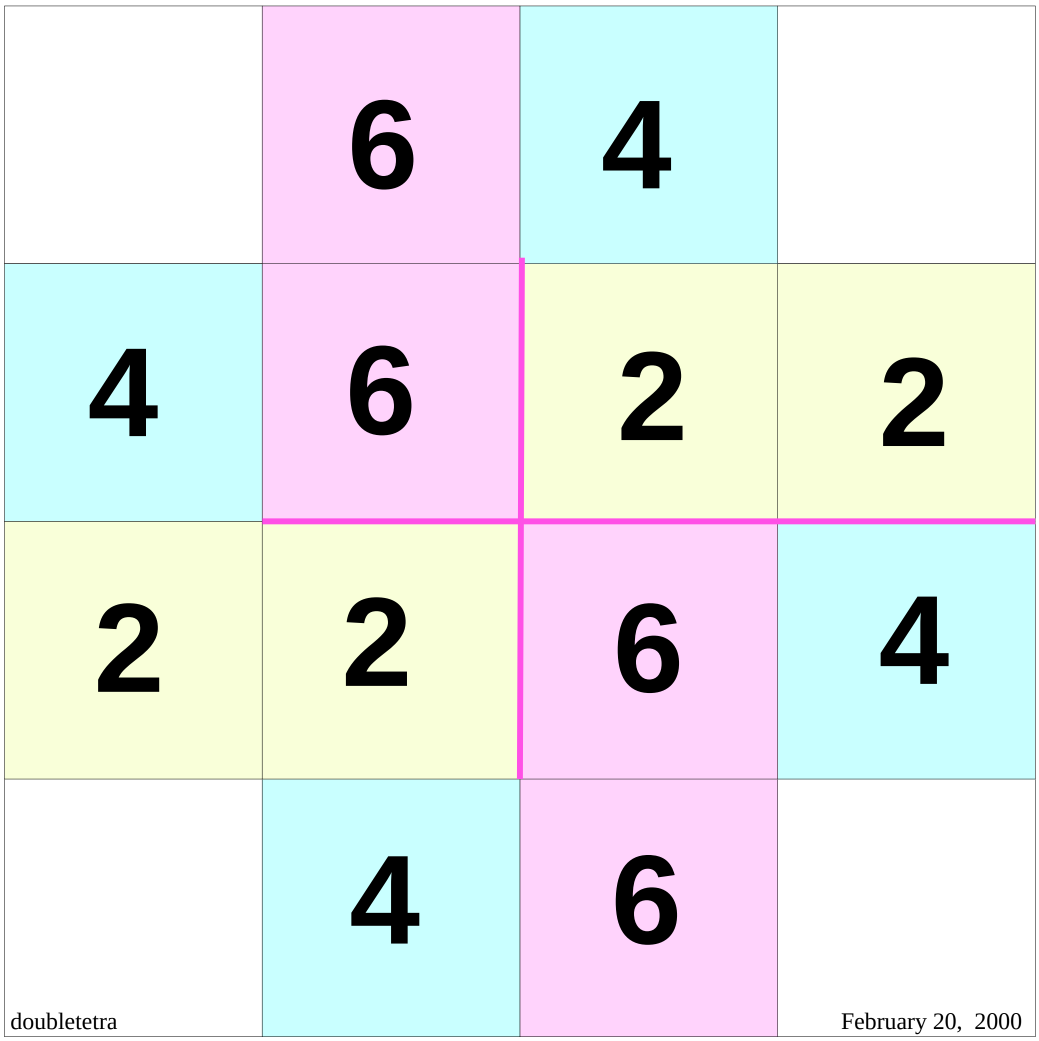


Figure 23: Bottom side of the first level binary tetraflexagon cutout.

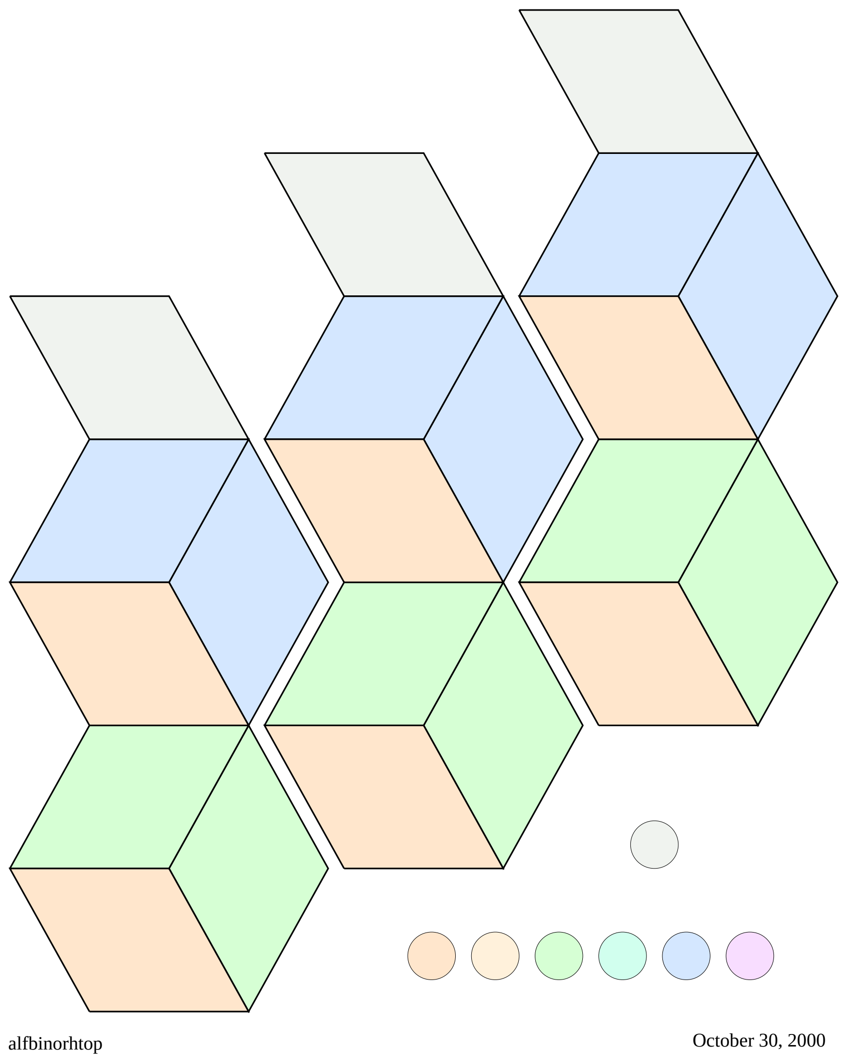


Figure 24: Top side of the first level alpha binary rhombic tetraflexagon cutout. Together with the backside, the figure makes one individual flexagon.

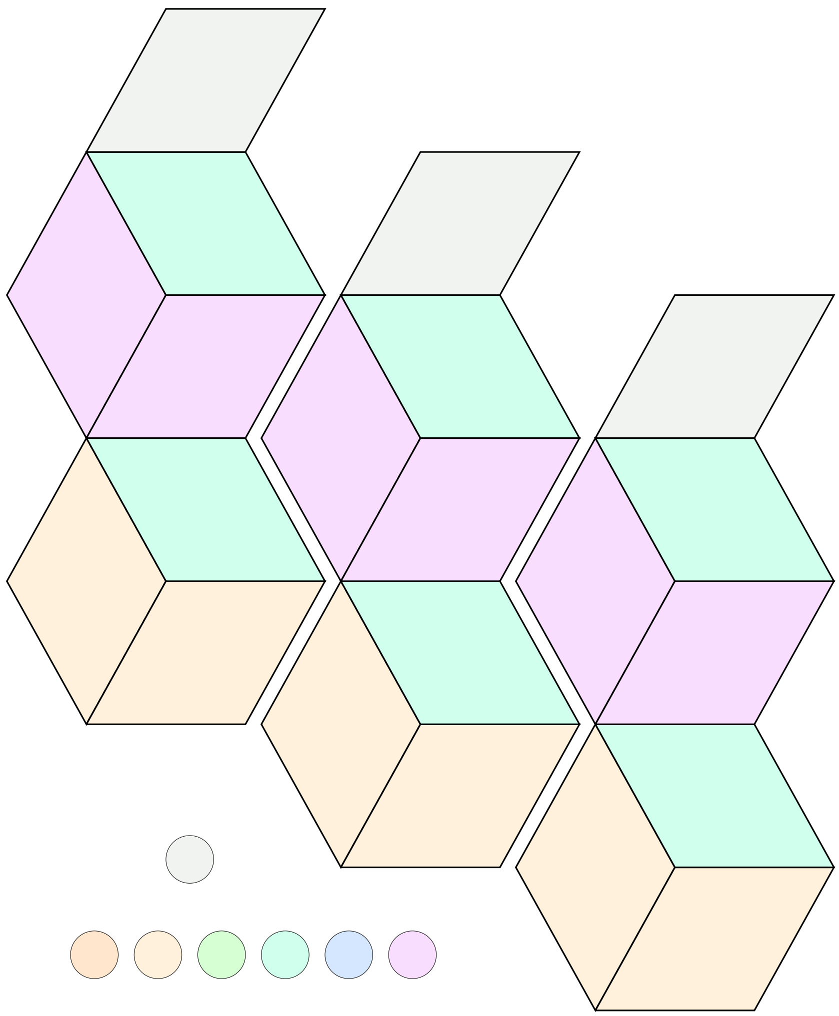


Figure 25: Bottom side of the first level alpha binary rhombic tetraflexagon cutout.

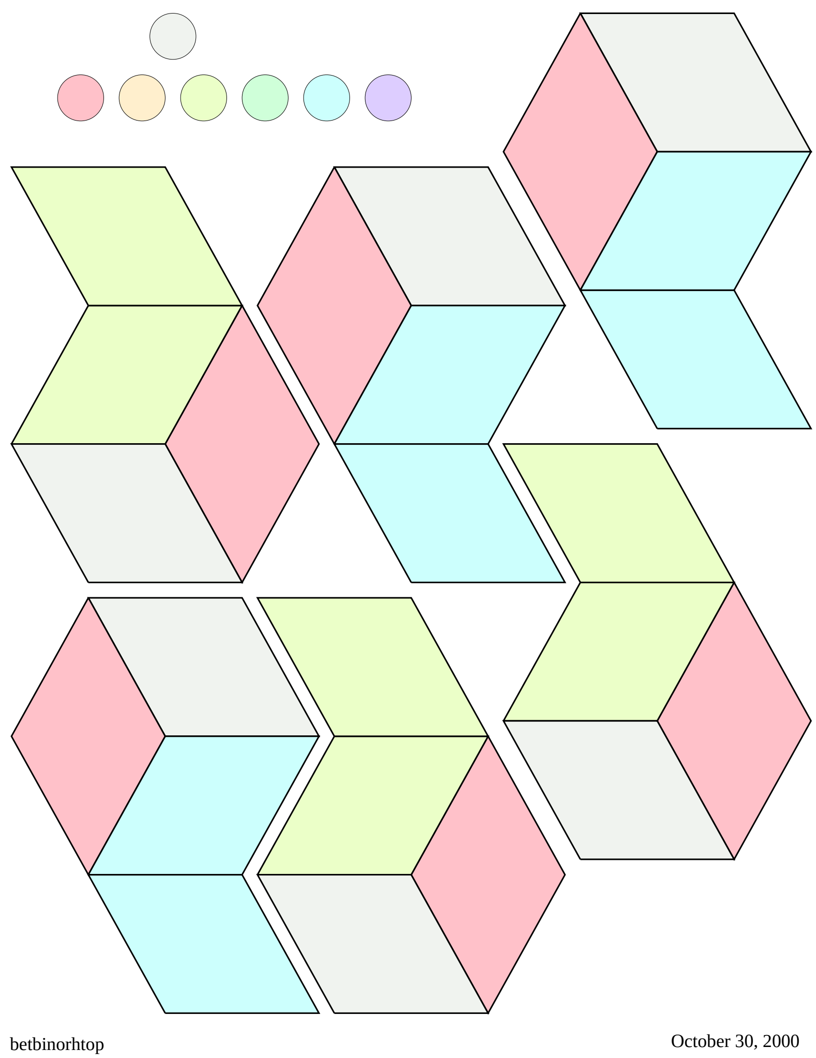


Figure 26: Top side of the first level beta binary rhombic tetraflexagon cutout. Together with the backside, the figure makes one individual flexagon.

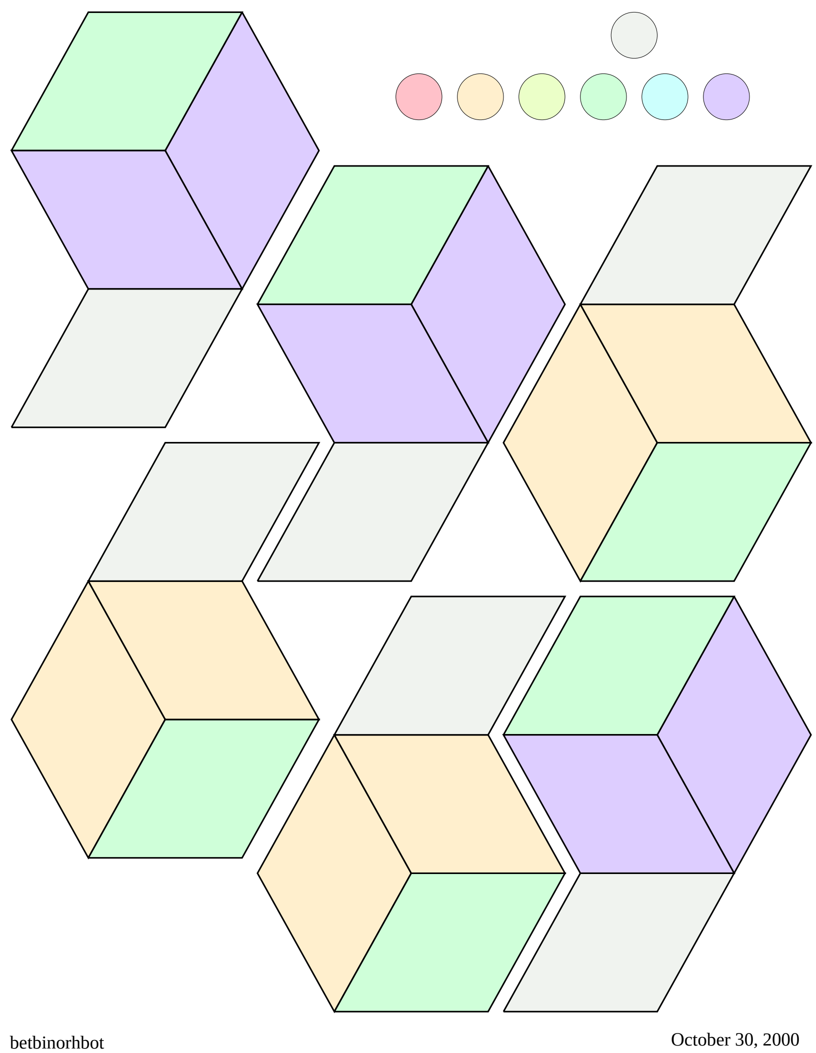


Figure 27: Bottom side of the first level beta binary rhombic tetraflexagon cutout.

4 Second Level Tetraflexagon

Starting from the generic second level tetraflexagon map, which consists of a node linked to four surrounding nodes, any of the linked nodes can be dropped for a total of sixteen possibilities, not all of which are symmetrically independent. Thus one cound drop one node, an opposite or an adjacent pait, or three of the four. That makes six combinations, plus the full second order flexagon or just the full first order flexagon.

Amongst all of these, various possibilities for tubulations exist.

4.1 Second level normal tetraflexagon

The generic second level tetraflexagon has twelve faces, and is still quite easy to fold up from a previously prepared strip of paper.

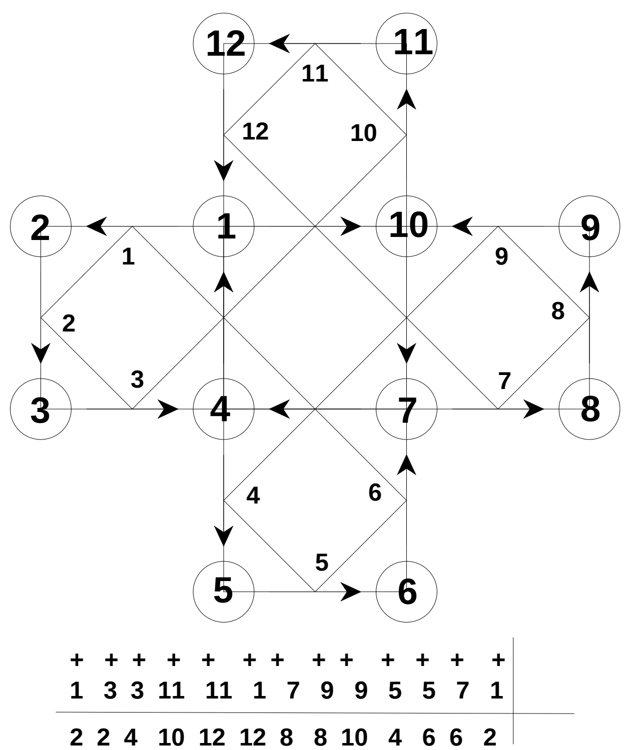


Figure 28: Since each edge of the first level tetraflexagon spawns two new vertices, the full second level tetraflexagon has 12 vertices.

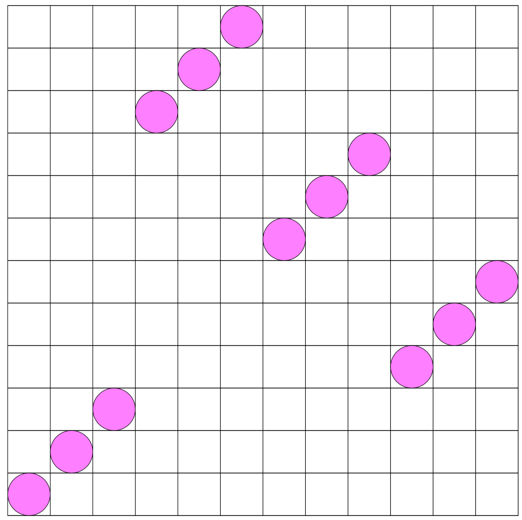


Figure 29: Permutation of the squares along the strip for a normal second level tetraflexagon. They run in order, subject to packages being flipped over.

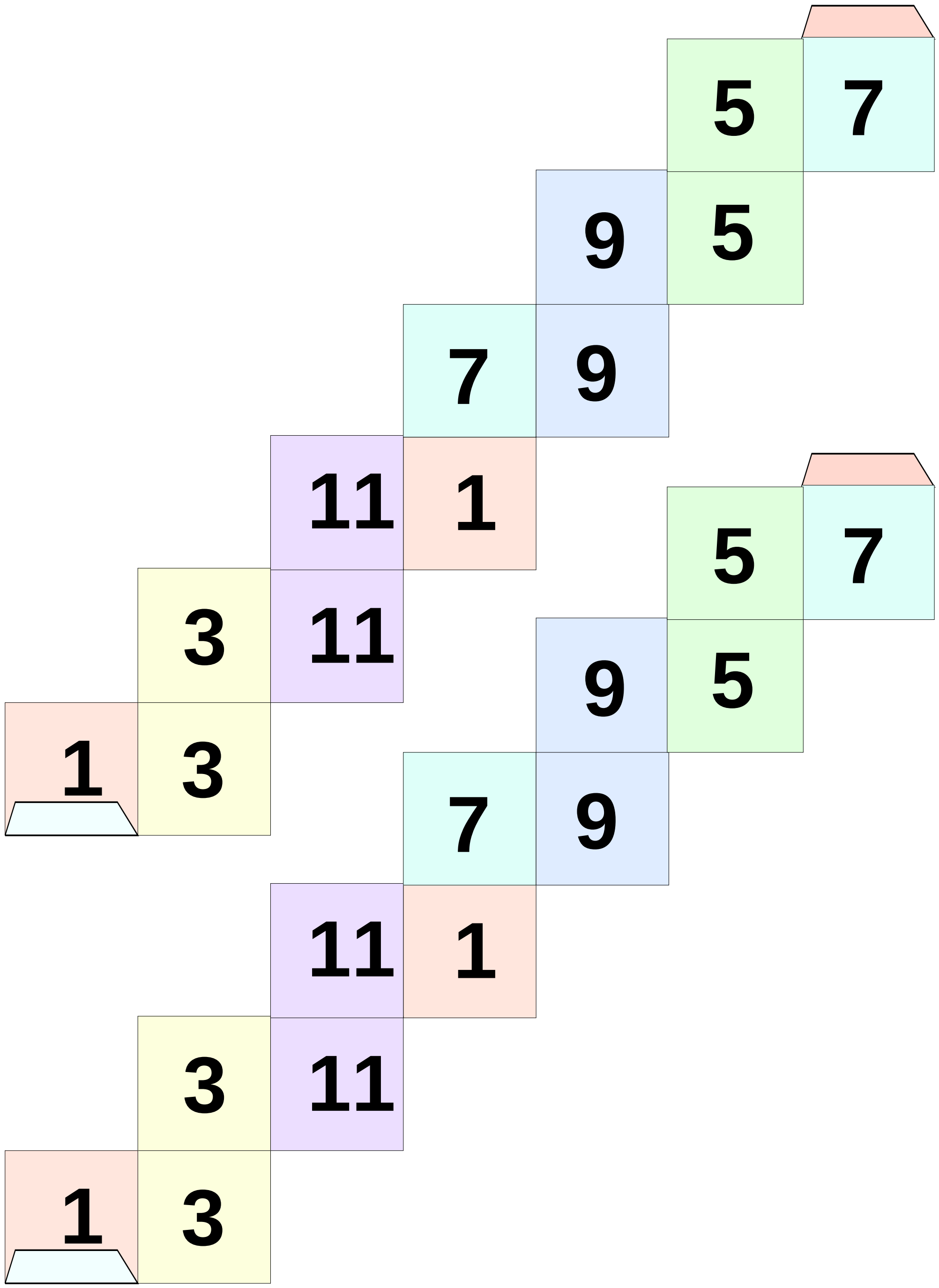


Figure 30: Strips from which the second level tetraflexagon can be folded. One strip folds into a single sector, so two are needed altogether.

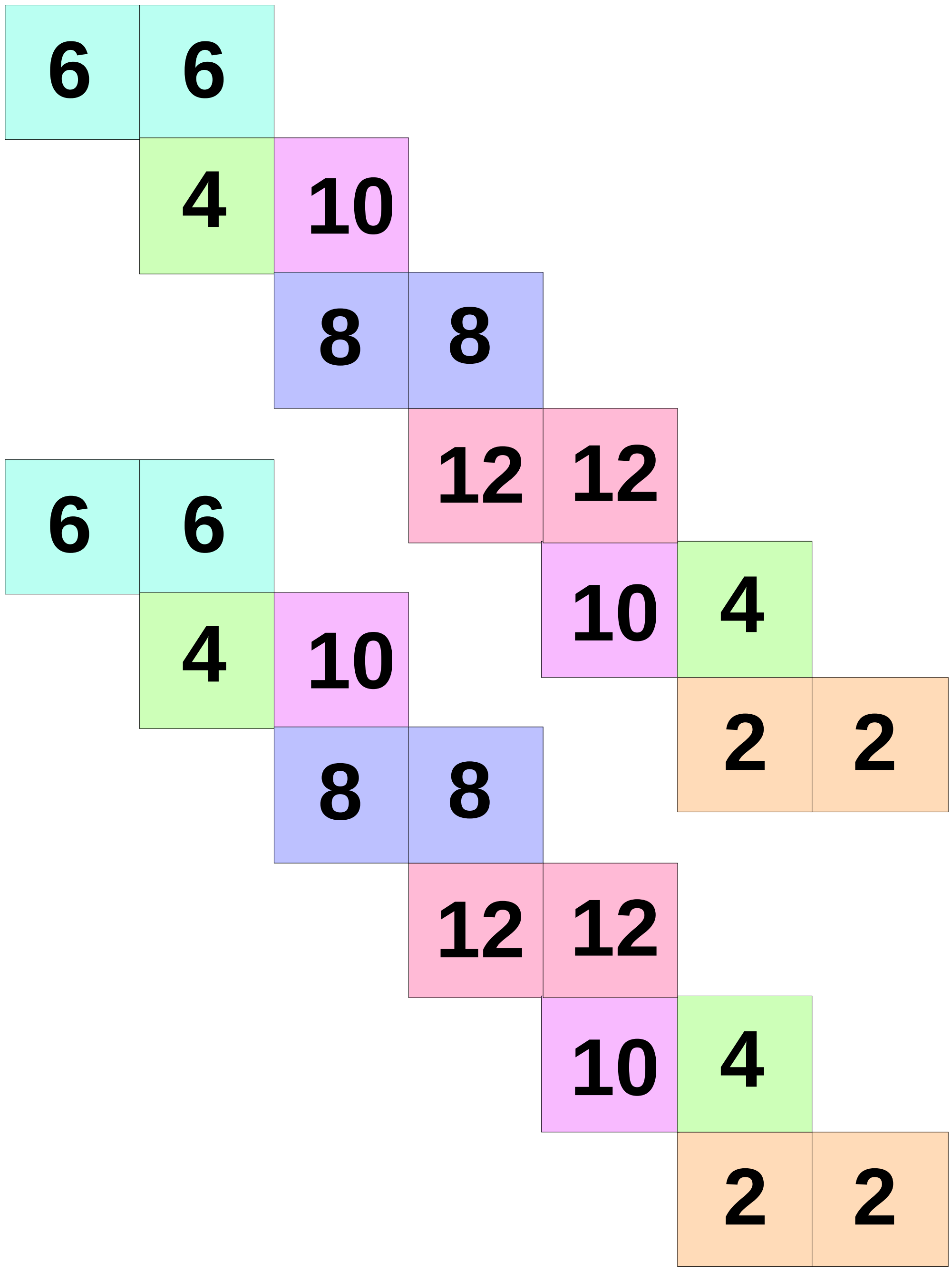
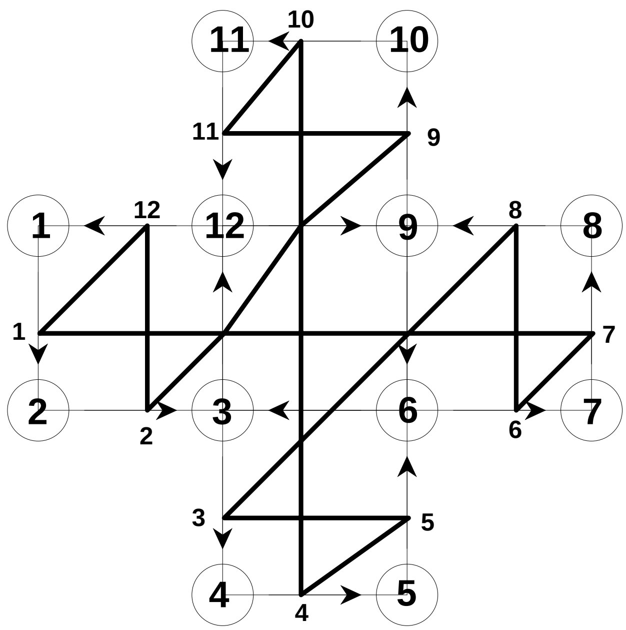


Figure 31: Back side of the two strips for the second level normal tetraflexagon. One sector per strip.

4.2 Second level tubulating tetraflexagon



+	-	-	+	+	-	-	+	+	-	-	+	+
1	8	6	9	3	6	4	11	11	10	2	1	1
2	7	7	8	4	5	5	10	12	9	3	12	2

Figure 32: The 12 vertices of the generic second level tetraflexagon can be connected to produce tubulations.

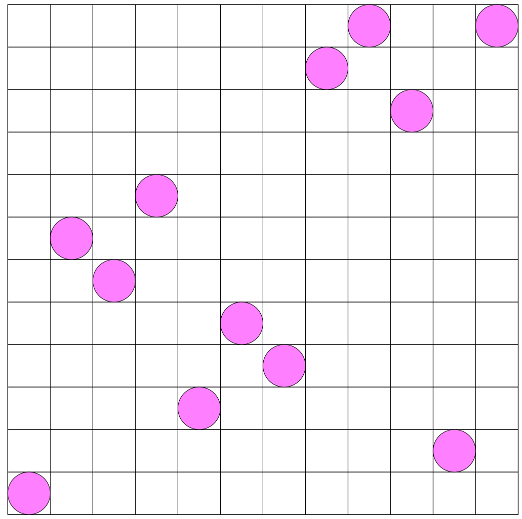


Figure 33: Permutation of the squares along the strip for a tubulating second level tetraflexagon.

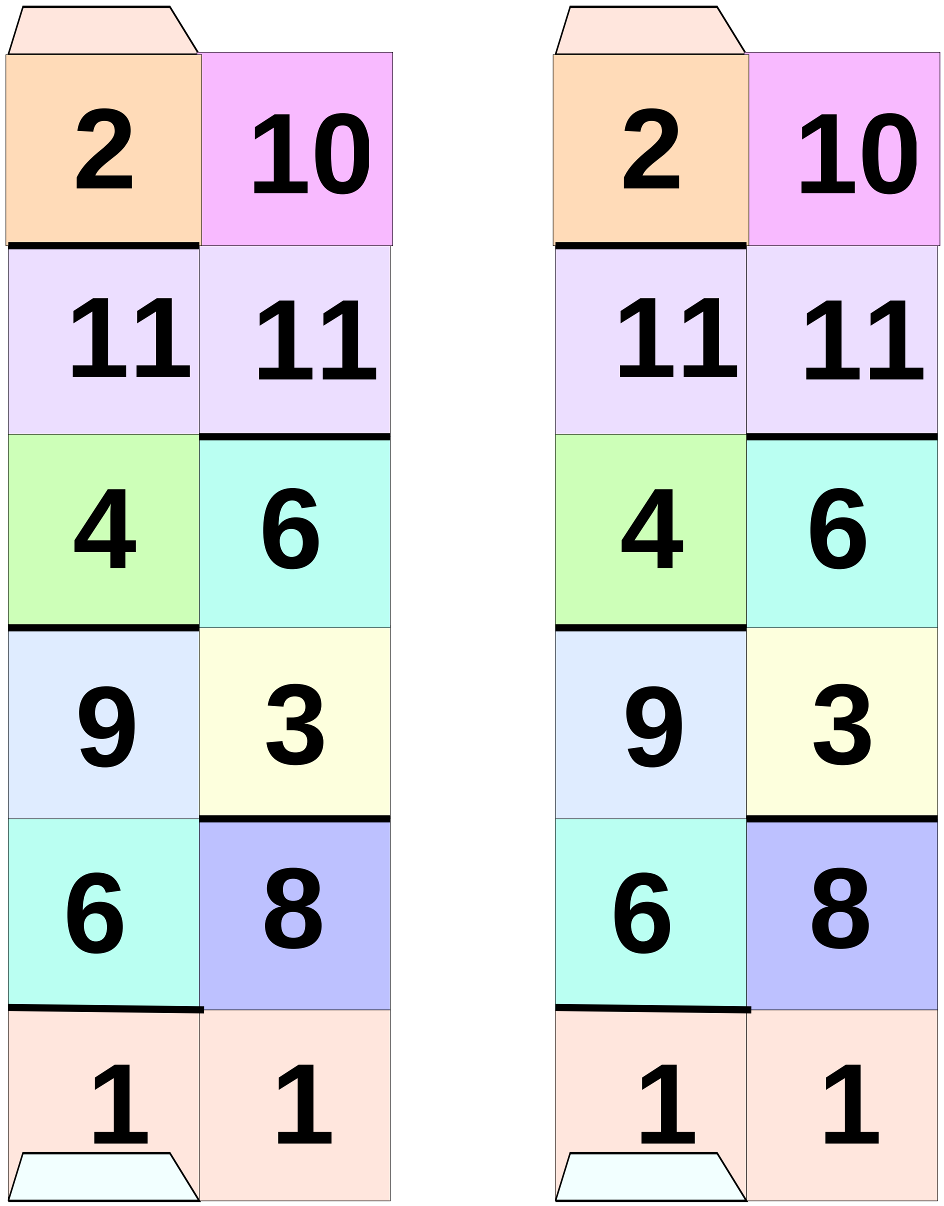


Figure 34: Top side of the tubulating second level tetraflexagon. Two strips are shown, sufficient to make both of the two sectors which the flexagon requires.

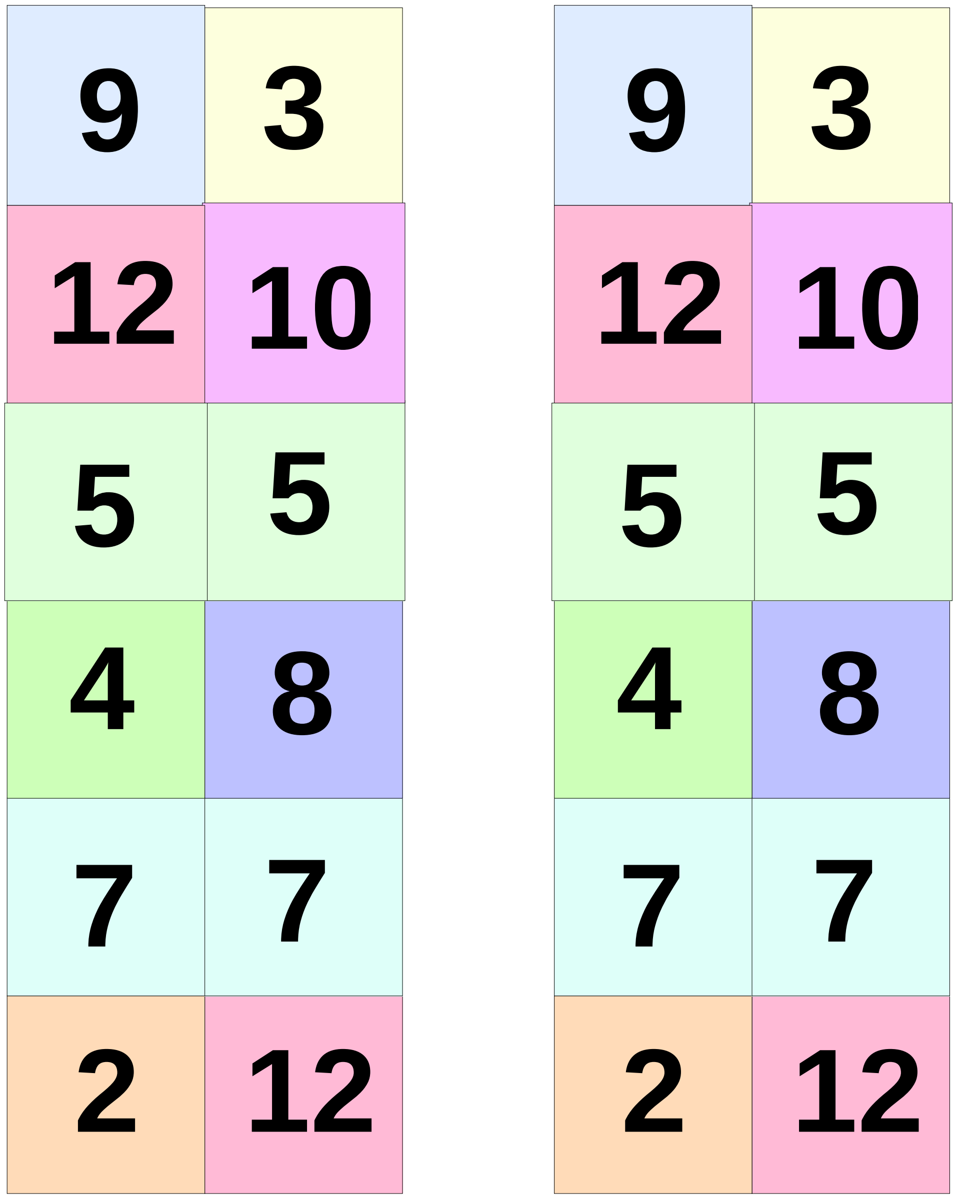


Figure 35: Bottom side of the tubulating second level tetraflexagon. Two strips are shown, sufficient to make both of the two sectors which the flexagon requires.

5 Third level tetraflexagon

The generic third level tetraflexagon has thirty six faces, and is still easy to fold up from a previously prepared strip of paper. However, rather than many pages of explicitly numbered squares, the templates described in the next section should be consulted.

The easiest way to create a generic flexagon is to prepare some uncolored and unnumbered segments corresponding to the sign sequence $(+ + + + \cdots)$ with tabs on the end. They can later be joined and labelled as required. It is especially easy with triflexagons because a long straight strip of paper only has to be marked off into triangles. Polygons with more sides still use an essentially straight strip of paper, but with little meanders which have to be incorporated since the beginning.

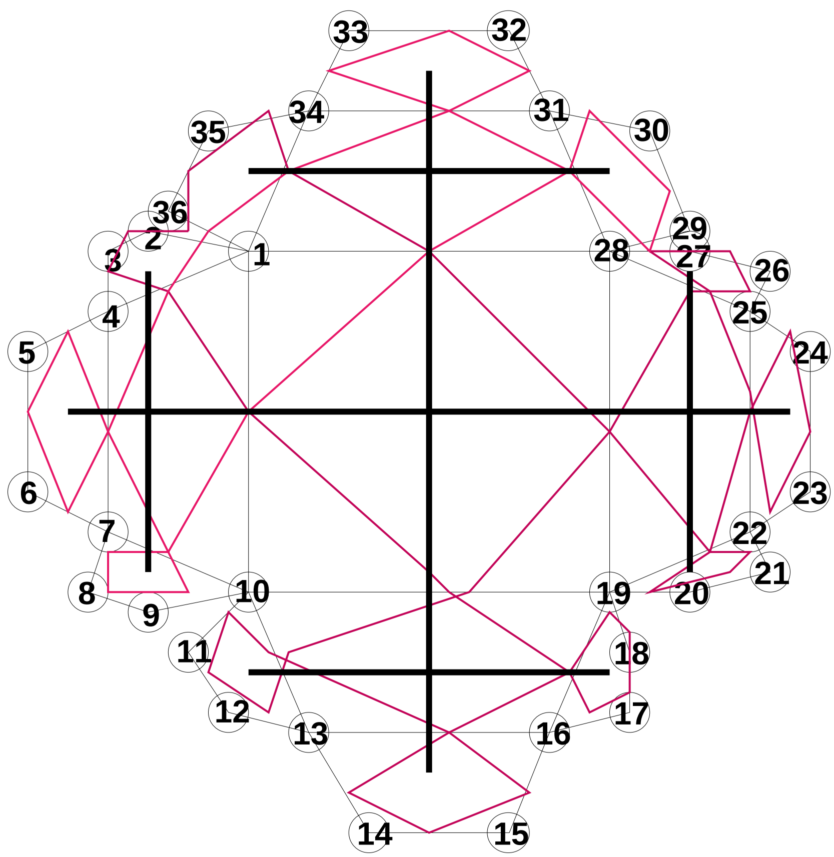


Figure 36: Since each of the twelve edges of the second level tetraflexagon spawns two new vertices, the full second level tetraflexagon has 36.

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
1	3	3	17	17	19	13	15	15	11	11	13	25	27	27	23	23	25	19	21	21	35	35	37	31	33	33	29	29	31	7	9	9	5	5	7	1
2	2	4	16	18	18	14	14	16	10	12	12	26	26	28	22	24	24	20	20	22	34	36	36	32	32	34	28	30	30	8	8	10	4	6	6	2

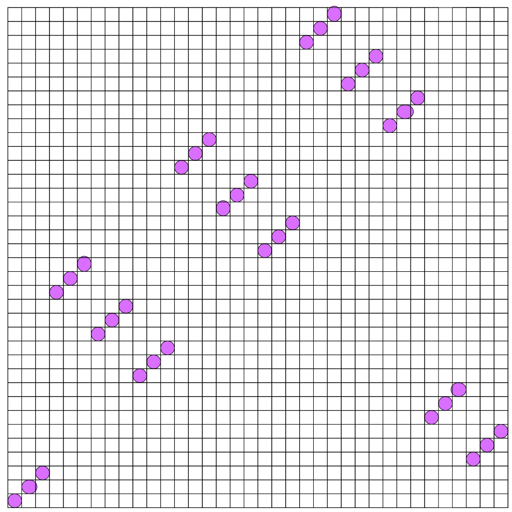


Figure 37: Permutation of the squares along the strip for a third level tetraflexagon.

October 31, 2000