RIAS

THE THEORY OF THE FLEXAGON

by

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Abstract

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Contents

THE FLEXAGON

1 The Three-Sided Flexagon

Occasionally some fascinatingly mysterious device manages to creep out of the obscurity of mathematics to claim a place for itself in the everyday world. Such an object is the flexagon. This odd contraption has emerged from one of the most intriguing fields of today's mathematics – topology[1]. Topology is that study whose play-things are one-sided bottles, one-edged paper bands, bridges, maps, etc. Now, to this list of things topology has given us, we may add the flexagon.

The flexagon has a great advantage over most scientific toys in that it requires a ridiculously meager material investment to reap immense results. All that is needed is paper, scissors, pencil, and tape. The embellishments – paint, special paper, carrying envelopes, etc. – can, of course, be allowed to raise the investment considerably. No such extravagance, though, is necessary, so dig out some scissors and paper, and let's get started.

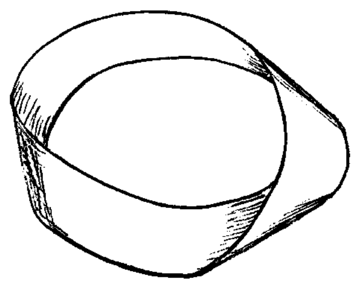

Figure 1: sample flexagon.

I say that flexagons (and there are an infinite number) belong to topology. This is because they are, basically, modified Moebius bands. We will let ourselves be introduced to the flexagon by means of the Moebius band. The Moebius band is the strip of paper mentioned previously, which has but one edge and one side (see figure 1)[2]. If the strip is given *two* half-twists, it will have two sides and two edges. That is, it is no longer a true Moebius band. However, the flexagon does not depend upon the number of sides or edges. In its case, it is the number of twists that is critical.

"And just what is a flexagon, anyway?" you ask. To this I could give you any of various answers – "A flexagon is a Moebius band of $4n - 6$ half-twists" – or – "A flexagon is a flexible hexagon" – or – "A flexagon is an ordered pair of parts" – each of which leaves us just as much in the dark as we were before. Therefore, let's begin by making a flexagon or two and then you will be able to see for yourself.

The simplest flexagon is a squashed Moebius band of three half-twists. Figure 2 A shows how the metamorphosis is enacted. The resulting hexagonal object, which is the flexagon, has to be folded along all the dotted lines. Rather than go through the squashing process, we will use a shortcut.

[1]See *Astounding Science Fiction* for March, 1954: "Topology" by J. G. Hocking.

[2]The Moebius bands named after the German topologist A. F. Moebius, is a one-sided, one-edged band formed by giving a strip of paper one half-twist before joining the ends. As used in this paper, the term refers to bands of an odd number of half-twists, which all have one side and one edge.

Rule off a straight strip of paper, about six times as long as it is wide, into nine equilateral triangles. Cut off the excess, as shown in figure 2 B. Crease the paper between each pair of triangles (see figure 2 B), then lay the paper flat again. Following the figure, fold together the top sides of triangles 2 and 3 and of triangles 8 and 9. Fold together the lower sides of triangles 5 and 6. Triangles 1 and 2 should now be in such a position that they can be taped together. The hexagonal object resulting should look like figure 2 A.

By the way, if you doubt that this band has only one side, try painting one side red and the other green.

The strip out of which the original Moebius band was built was, of course, uniformly straight. Hence it would seem that the twists could be placed anywhere along its length. Not only is this true, but the twists can be moved along the band. It is the remarkable ability of the squashed-flat band – the flexagon – to move its twists along its length that makes it so interesting. The reason for folding off equilateral triangles will now become apparent, for these folds are employed in the twist-moving, which in this case is called "flexing." The process of flexing is accomplished without bending or folding the triangles, by the method that will be shown presently.

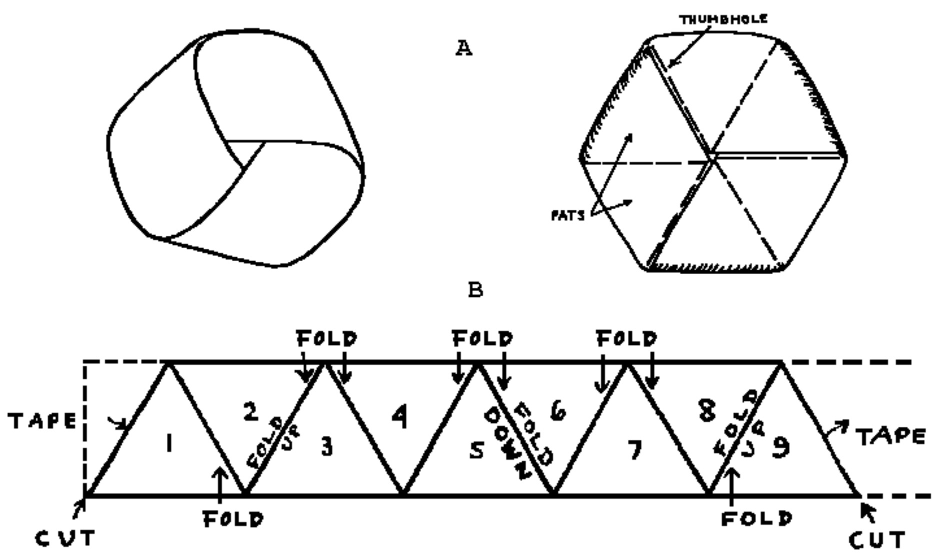

Figure 2:

First note that the hexagonal shape is divided into six piles of triangles, each of which is known as a "pat" (see figure 2 A). Those pats which enclose twists are made up of two triangles. The twist-moving turns out to be identical with the moving of one triangle in each double pat to the next pat, which would make the next pat double.

In order to flex, start by folding adjacent pats together. Triangles which were adjacent in the strip from which the flexagon was made should be folded so that they are next to one another (the whole process may be followed in figure 3). The flexagon is now in a three-bladed position, and is symmetrical in this respect: It looks exactly the same viewed from either end of the axis along which the three blades meet.

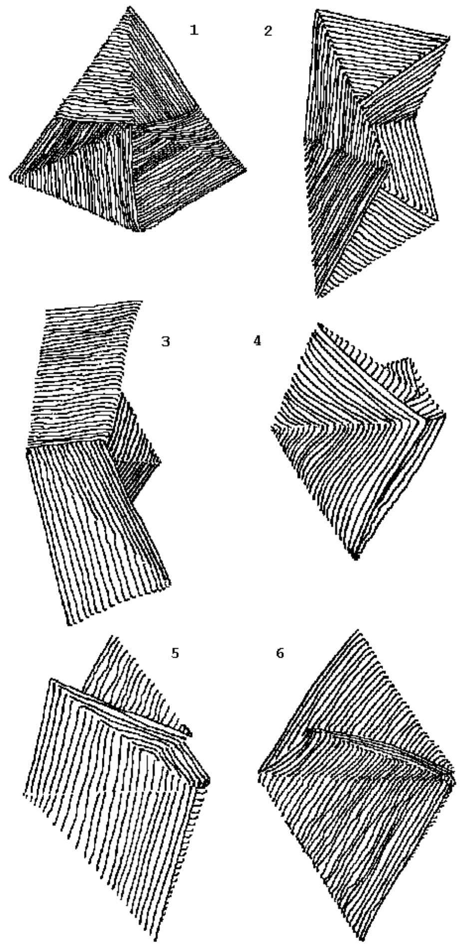

Figure 3:

The preceding process could, of course, be performed in reverse, laying the flexagon flat. But, due to the flexagon's symmetry, this can be done from either end of the axis. If we open out the flexagon from the *other* end (see figure 3), we find that the desired triangle shift has been made. Since the flexagon is now structurally the same as it was before, it may be flexed again without going back. Moreover, we can flex repeatedly, going on infinitely, without once flexing backwards.

The overall effect is one of indefinitely turning inside out. This phraseology is soon seen to be even more apt than might at first be suspected, for if we hold the flexagon flat at any given time and mark or paint the top surface, then flex, the painted part turns up on the bottom. Another flex hides it completely, and a third restores it to its original position. This suggests coloring or numbering each surface – or as it is properly called, "side" – of the flexagon when it appears. It will be found that there are three sides. These follow one another in one of two ways:

	1	2	3	1	2	3	1	2	...
or	3	2	1	3	2	1	3	2	...

depending on whether the flexagon is upside-down or not. When a flexagon is flexed, the side that last showed is then found on the underside.

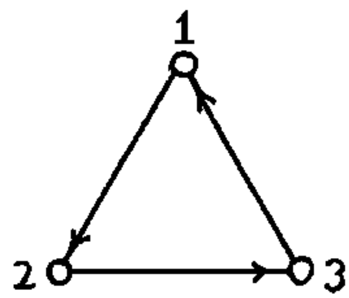

Figure 4:

To represent this cyclic change of sides, we employ what is called the "map" of the flexagon (see figure 4). If we turn the flexagon over, the directions of all the arrows are reversed.

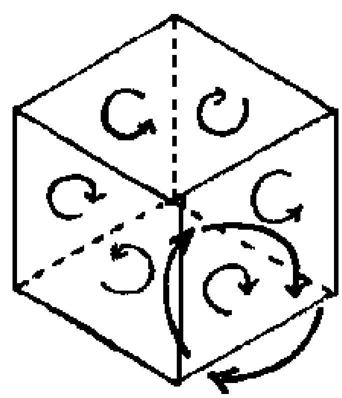

Figure 5:

Let's go back for another look at the flexagon itself. Its ability to open out at either end after having each pair of pats folded together is due to double hinging, as is found in doors that swing both ways. But the hinging edges are not parallel, as they are in the door, and therefore use of the flexagon hinge should shift the triangles 120° about their midpoints, as in figure 5. To observe this effect, fold the flexagon together into one large pile of triangles and dip each corner into colored ink. One color is used for each corner. When the flexagon is opened out, the central angles will all be the same color (see figure 6). When the flexagon is flexed, the central angle changes color. The third color appears after the second flex. After the third flex (and a rotation of $3 \times 120° = 360°$), the first color is again shown.

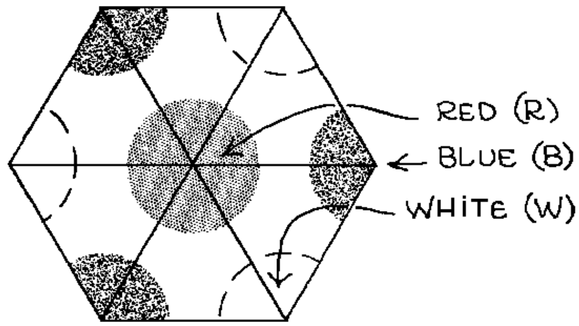

Figure 6:

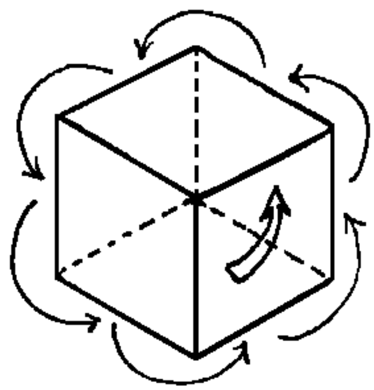

Figure 7:

2 Building More

It may be noted that in order to flex twice successively you must rotate the flexagon 60° about its center (see figure 7) before the second flex. This gets the flexagon into the right position for the next folding-together. If the flexagon is folded together without the intermediate rotation, all the hinges are folded the other way, the important symmetry is lost, and we can't flex.

What is there about the rotated flexagon that allows it to flex, while the unrotated one cannot? It is the slit between the two triangles in the double-thickness pats. This slit has been fancifully named the "thumbhole," because, naturally enough, it is the hole where the thumb may be pushed through the front of the flexagon to emerge at the back (see figure 2 A). Each thumbhole enters at one side of the double pat and leaves at the other side. What is more, all thumbholes pass through in the same direction (clockwise or counterclockwise) in any given flexagon.

If each single pat could be given an extra triangle, it would no longer be necessary to rotate between flexes. There would be a thumbhole available to make the triangle-shift no matter where the flexing folds should be made. All the pats would be the same.

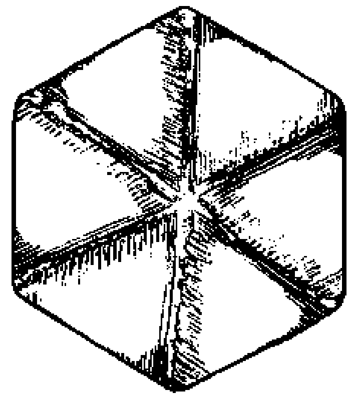

Figure 8:

Adding in these extra triangles necessarily involves adding extra twists. When we have finished the job, our flexagon resembles figure 8. Now, at last, we can perform the long sought-after flexing operation, and we are probably no longer able to be too surprised when a wholly new side is exposed.

Suppose the side exposed when the extra thumbholes were added was labeled “1.” Sides may, of course, be labeled with color, a marking code, or any similar thing, but numbers are most useful. Instead of going on to the old side, “2,” we detoured to a new side which we will call “4.” Another flexing brings us back onto familiar ground with side “3.” This is not too startling, because the position that would flex to side “2” from “1” and the position that flexed to the new side from “1” are identical. Therefore, when we rotate between each flexing, as we did in the unaltered flexagon, our new cycle operates just as the old one did:

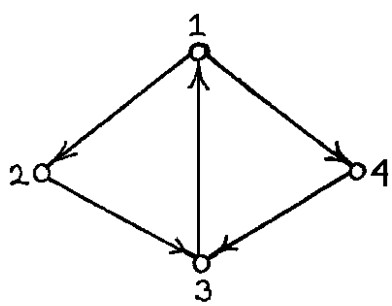

Figure 9:

	1	4	3	1	4	3	1	4	...
or	3	4	1	3	4	1	3	4	...

when turned over. Each time we pass between sides “3” and “1,” we are given a choice where to go next: “2” or “4.” All of this is concisely shown in the map of the new flexagon, which appears in figure 9.

When we use one cycle of this flexagon – either 1 2 3 ... or 1 4 3 ... – the twists and extra triangles which are used in the other cycle remain together, undisturbed. The only changes that occur in the use of one cycle are between the two halves of the thumbhole-pat in use and the other pat.

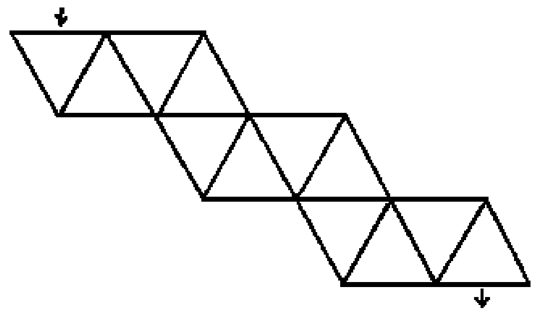

Figure 10:

We should note at this point that, if the new flexagon is unwound, it is seen to be made from a crooked strip of paper (see figure 10). This in no way impairs its function as a Moebius band, for any topological figure can be bent, squeezed, stretched, or shrunk. In this case it is actually advisable to use a crooked band, so that the finished flexagon will fit together. Otherwise, it turns out looking like an octahedron with a pair of opposite faces knocked out. The fact is that most flexagons must be made from non-straight strips, or "plans." The "plan" is the layout of the unfolded flexagon. Some plans, as we will see, take on rather amusing forms.

What can be done once can be done again. Therefore, each time a side is exposed there is a choice of which way to flex next – whether to rotate or not. And each time, if necessary, a new set of triangles can be added in to permit further flexing. Here, then, is a method of building a flexagon of any size. We can add in sides as long as we want to do so.

In order to have a thumbhole, there must be a hinge opposite the central vertex of the pat containing the thumbhole. Only in cases where there is no such hinge already in place (that is, an already existing thumbhole) can one be added. If there is no hinge here, the only hinges must be those leading to the adjoining pats, and the given pat must therefore be of a single thickness.

With this small fund of knowledge, let's return to the map. As has been seen, we have but two choices, in any case, in leaving any given side. These consist of going to one or the other of the two "sides" on either side of the map line which was last traversed (see figure 11). The rule of thumb is that the last side visited, the side next to be visited, and the side being visited must be the vertices of a triangle in the map (figure 11).

Then the positions at which we can add new sides are those at the edge of the map, where each map line is a member of only one triangle. Adding a side in the flexagon is represented by adding a new map triangle at the corresponding map line, which will be found at the edge of the map. All outer edges of the map are representations of positions in the flexagon at which three pats are of single thickness.

Suppose we want to expand the four-sided flexagon by adding in a new side, "5," between sides "1" and "4" on the map, as in figure 12. (We can omit arrows from the map henceforth, since they only clutter it up).

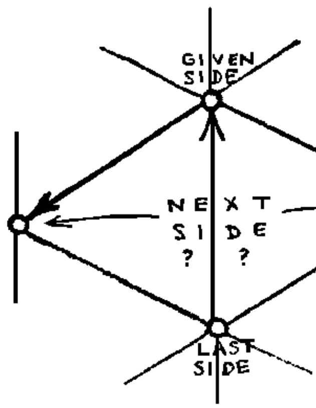

Figure 11:

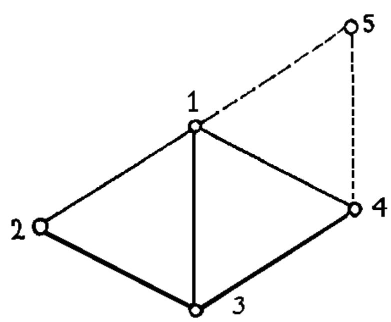

Figure 12:

The first step is to flex to side "1." Then travel along the line leading to "4." From here we should be able to reach the hypothetical side "5," according to the rule of the map. That is to say, "1," "4," and "5" form a triangle over which we can travel. Three of the pats at this position will be found to be of a single thickness. Mark them, cut open and unfold the flexagon, and make them double. This method, shown in figure 13, is about the easiest way to add in new sides.

The new side which will turn up will be formed from the newly created faces of the doubled triangles. If the old flexagon was numbered or painted, the pattern will have to be repaired, since we did not actually produce the new thumbhole by splitting a triangle. The new addition will have disturbed the coloring scheme.

To fold the flexagon back up, first fold together the new triangles, as in figure 13 B. This will require a kind of winding motion. Then, being careful to continue winding the folds in the same direction, fold the flexagon up the same way it was dismantled. If you feel at this point that the flexagon has become too messy in the doubling process, it is relatively easy to make a new five-sided flexagon plan from the pattern of the old one.

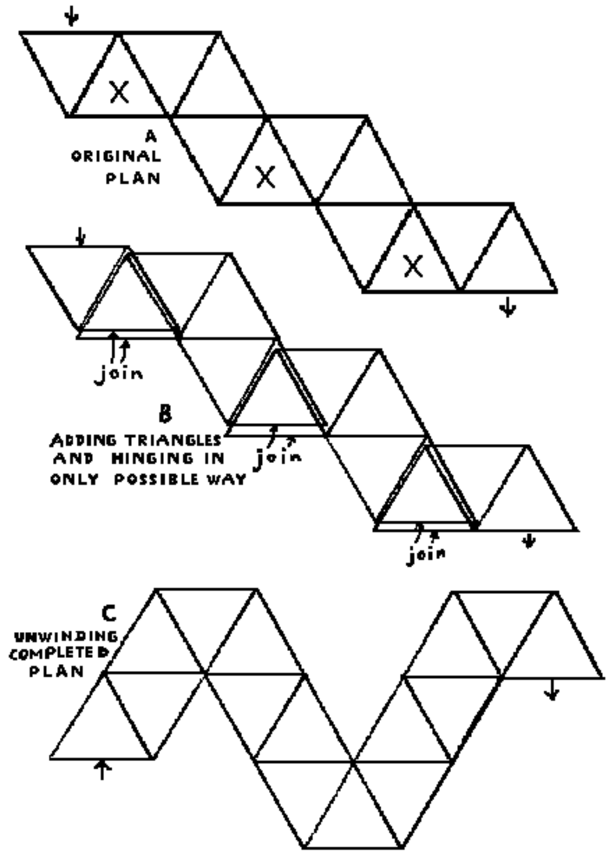

Figure 13:

3 Building Them All

The doubling method works, but it is clumsy and inefficient. In order to make the manufacture of flexagons more readily practicable, a method has been developed for building a flexagon of any size, with any given map.

So that we may learn by example, let's pick a map at random: that shown in figure 14. Notice that every side of the flexagon is touched by just two of the map's outer edges. To start the figuring, a network of lines must be drawn throughout the map, made up of the lines joining the midpoints of the map lines. One straight line is continued as long as possible, and is then broken and continued. The drawing of the network is shown in figure 15. This network is known as the Tukey triangle network, named after its inventor, John W. Tukey.

Next, each side must be numbered, in sequence, around the edge of the map, as in figure 14. Then the vertex of the Tukey triangle network between "1" and "2" is labeled "1," that between "2" and "3" is labeled "2," and generally that between "n" and "$n+1$" is labeled "n." This is demonstrated in figure 15 C.

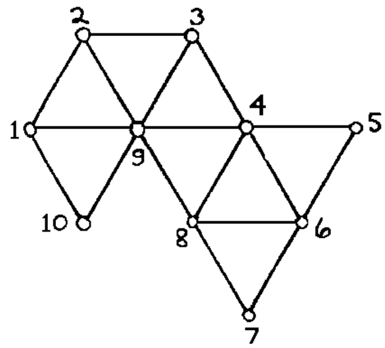

Figure 14:

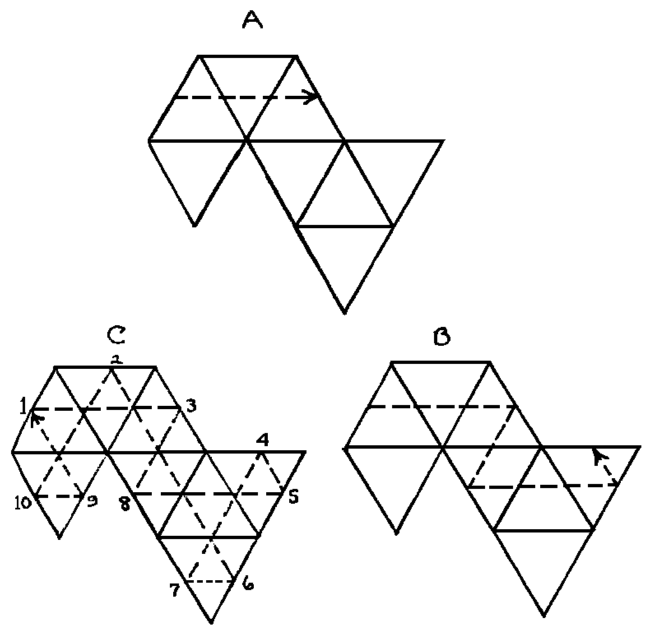

Figure 15:

Now draw a line below the map. Travel over the Tukey triangle network in the manner used in its construction, along the lines, copying down the numbers at the outer vertices alternately on one side of the line and the other, thus:

1		8		4		6		10		|	1		...
	3		5		7		2		9	|		3	...

The vertical line indicates one complete traverse of the network. Add +1 to each number of the

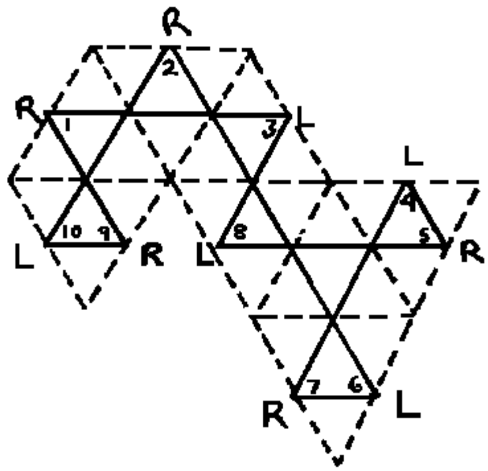

Figure 16:

sequence and place the resulting sum across the line, opposite the number to which 1 was added. Across the line from the highest number in the sequence, place the number "1." This gives us:

1	4	8	6	4	8	6	3	10	10	1	4	...
2	3	9	5	5	7	7	2	1	9	2	3	...

Every outer vertex in the network occurs at the midpoint of a map line. This line may go in one of three directions, which are spaced 120° apart. In the following discussion, the term "map direction" is used to indicate the direction of this line.

Put an "R" at vertex "1." If the map direction is the same for vertex "1" as for the next vertex along the network (which, in the example, is "3"), put an R at the second vertex. If it is different, as it is in the example, use an "L." Pass on *through* the network in this way. If the map direction changes, change from "R" to "L" or "L" to "R," as the case may be. If it remains the same, retain the sign – "L" or "R" – last used (see figure 16).

Now copy each sign beneath the numbers associated with the same network vertex:

1	4	8	6	4	8	6	3	10	10	1	4	...
2	3	9	5	5	7	7	2	1	9	2	3	...
R	L	L	R	L	R	L	R	L	R	R	L	...

We are now ready to construct the flexagon.

Make an equilateral triangle with an arrow indicating a side through which to enter. Accordingly, as the first sign is "R" or "L," add a new triangle on the right- or left- hand edge of the first triangle, as approached through the arrow. Leave the second triangle on the side indicated by the second sign, and so on (see figure 17). Repeat the sequence twice, copying down the signs anew from the network with each repetition, so that there are three like groups of triangles, tied end to end (see figure 18). Cut out this plan.

Label the triangles on one side of the strip of paper with the numbers on one side of the line and label the same triangles on the other side of the strip with the numbers on the other side of the

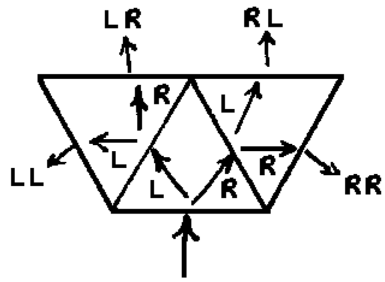

Figure 17:

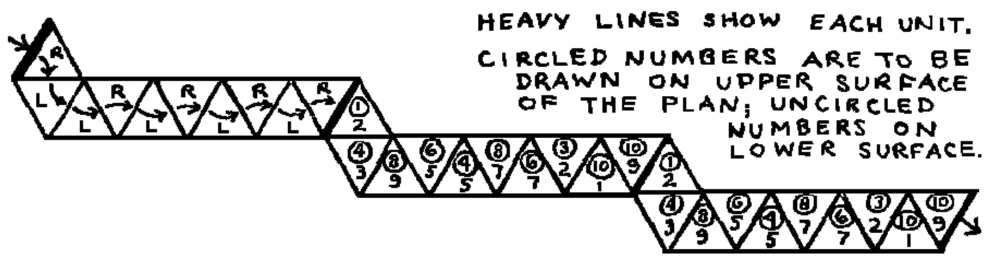

Figure 18:

line. Numbers across the line from each other should be on opposite sides of the triangle associated with the corresponding sign (again see figure 18).

To fold the flexagon up, fold together pairs of like adjacent numbers until just two different kinds of numbers show. Then tape the ends of the plan together. For example, you might fold together all the "5" 's, then all the "7" 's, then "10" 's, "6" 's, "8" 's, "4" 's, "3" 's, and "9" 's, in that order. This would leave sides "1" and "2" showing, one on the top and the other on the bottom.

Take another example of the use of this horribly involved but correspondingly efficient system. Suppose we want to make two of the *three* six-sided flexagons. (Do you find it surprising that there should be more than one? Well, *each permutation* of the map triangles produces a different flexagon. There are always two more sides than map triangles, so that, for the six-sided flexagon, we must permute four map triangles. The only possible non-equivalent permutations are those shown in figure 19 A). Right now we will make flexagons 2 and 3 (figure 19 A). First we construct the Tukey triangle network (figure 19 B), and then number about the vertices (We won't worry this time about which side on the map is labeled which way.) (see figure 19 C) and mark down the signs (see same figure). Now, copying down all this data, we get:

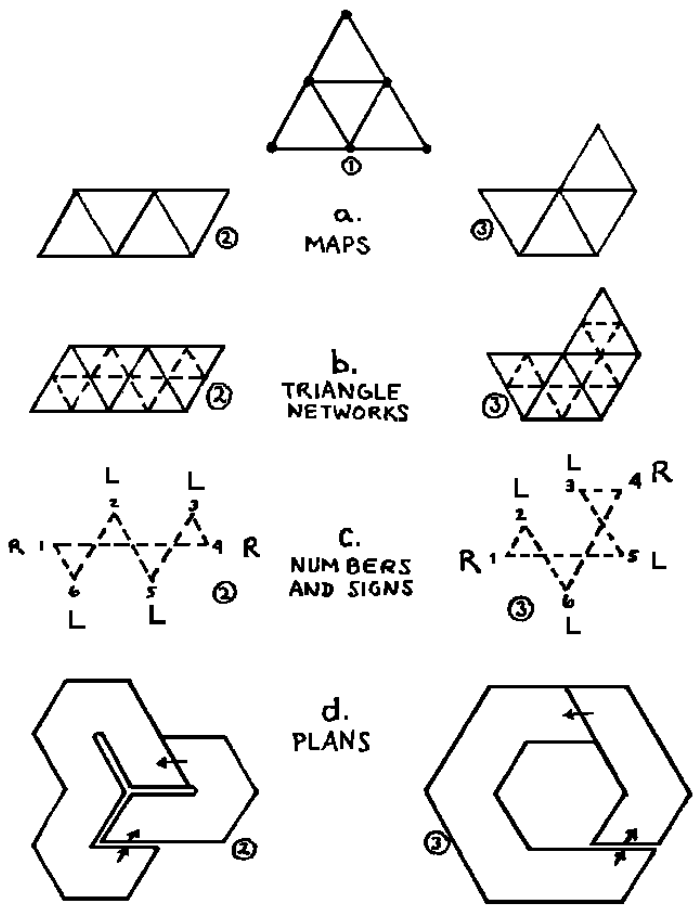

Figure 19:

1		3		2		1	
	4		5		6		4
R	R	L	L	L	L	R	R

1		6		3		1	
	2		4		5		2
R	L	L	R	L	L	R	L

and adding one to each numerical term:

1	5	3	6	2	1	1	5
2	4	4	5	3	6	2	4
R	R	L	L	L	L	R	R

1	3	6	5	3	6	1	3
2	2	1	4	4	5	2	2
R	L	L	R	L	L	R	L

Now all we have to do is cut out the plans (figure 19 D). The figure shows the entire plan, with the two repetitions. IMPORTANT: Notice that if the *number of sides had been ODD*, all the signs would have been REVERSED during the first repetition – try it yourself and see – coming out this way on the second time around the Tukey triangle network. In flexagons with an even number of sides, there is no such disturbance. As an example, look at the complete "L" - "R" sequence for the five-(odd) sided flexagon: *R L L L R L R R R L R L L L R*.

When the plans have been finished, they are numbered and folded up by folding together like adjacent numbers. It should be remarked that the *numbers* for the first repetition in *odd-sided* flexagons also come reversed, with numbers falling on opposite sides of the strip from the usual position. The numbers may be read off the map in this order. This is the reason for including a small portion of the first repetition each time we write a sequence —– odd-or-evenness must be kept in mind. Again we can use the five-sided flexagon as an example:

1	3	5	4	4	2	2	1	3	5	1	3	5	4	4
2	2	1	3	5	1	3	5	4	4	2	2	1	3	5

4 How Many? What Kinds?

ORDER	NUMBER OF FLEXAGONS
(2)	(1)
3	1
4	1
5	1
6	3
7	4
8	12
9	27
10	82
11	228
12	733
13	2,282
14	7,523
15	24,834
16	83,898
17	285,357
18	983,244

TABLE I

Now that you can make any flexagon you want, "How much," you wonder, "would it cost for adding machine tape to make them all?" It would, of course, be impossible to make *all* the flexagons, for they can be built to any desired and practically feasible size. But, considering flexagons with a given number of sides, or, as is usually said, of any given "order," there are a limited number. To find out how many we take all possible different permutation of two-less-than-the-order map triangles which works out to give the values listed in Table I. It may be found amusing to work out a few of these experimentally. As for the numbers involved, order ten seems to be a jumping-off place – beyond there the number of flexagons increases almost astronomically. The moral of this is: Don't be too ambitious; if you decide to make all the flexagons of some order n, keep n well below ten.

Some flexagon plans are more interesting than others, mostly for aesthetic reasons. We have already seen the stair-like plan of the flexagon of order four, the three-leaf clover formed by the flexagon shown in figure 19 A2, and the flexagon shown in figure 19 A3, which is the first of a whole

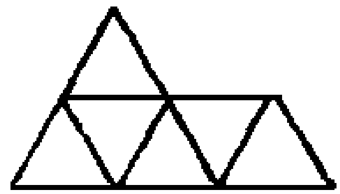

Figure 20:

family of flexagons with plans that form neatly nesting hollow hexagons. There is also a family of nesting triangles, the first member of which has the map shown in figure 20.

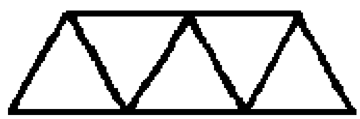

Figure 21:

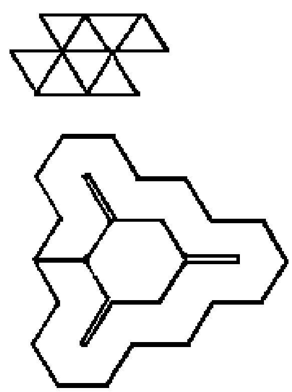

Figure 22:

The flexagon shown in figure 21 comes out in a figure eight. A flexagon of order ten has the fascinating plan seen in figure 22. Plans with twists and curlicues are plentiful. Among them are the ones shown in figure 23. Only one-third of the entire plan is shown in this figure. Certain plans even bear a limited resemblance to well-known forms. The plan of one flexagon, which is probably the most twisted plan ever made, could be interpreted as a representation of a sidewinder slaloming across a bed of nails (see figure 24). Particularly amusing to several researchers, for some reason,

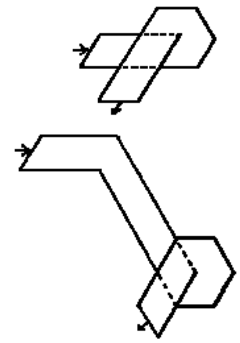

Figure 23:

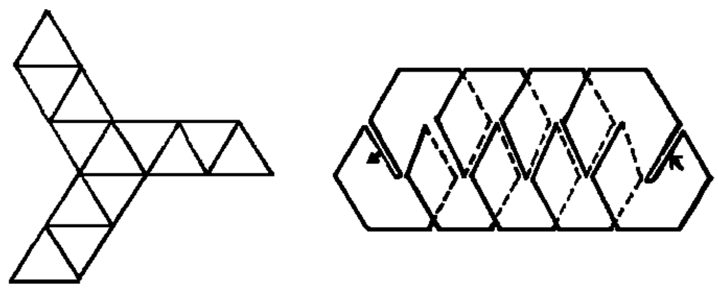

Figure 24:

is the group of flexagons whose plans take on the form of figure 25.

Perhaps the most interesting single group of flexagons comprises those made up from a straight strip of triangles, as was the original three-sided flexagon. Some of these are made by doubling the length of the strip previously used. That is, the second strip (after the three-sided flexagon) would be made up of l8 triangles and subsequent ones of $2^n \times 9$ triangles. In folding the flexagons up, the first and second triangles are folded together, then the third and fourth, then the fifth and sixth, etc. (see figure 26). When this has been done, the first and second piles of triangles are folded together, then the third and fourth etc. This is kept up over and over until the strip is nine piles long.

The strip can then be made into the flexagon rather easily. Care must be taken to fold, or rather to wind, the strip up in *one* direction so that the twists do *not* go in opposite directions.

The 18-triangle flexagon mentioned, which is the same as the first flexagon of order six (figure 19 A1), is in my opinion the best and most easily made flexagon for demonstration and study purposes. It is usable as an example in almost any discussion. As has been said, it comes in one long straight row of triangles with the sign sequence R L R L R L .... These are wound about one another in the manner previously described (follow figure 26). This results in a straight row of double thickness triangles, which is formed into a flexagon as if it were the plan for a three-sided flexagon. Remain wary of twisting or winding in different directions, which unwinds the flexagon,

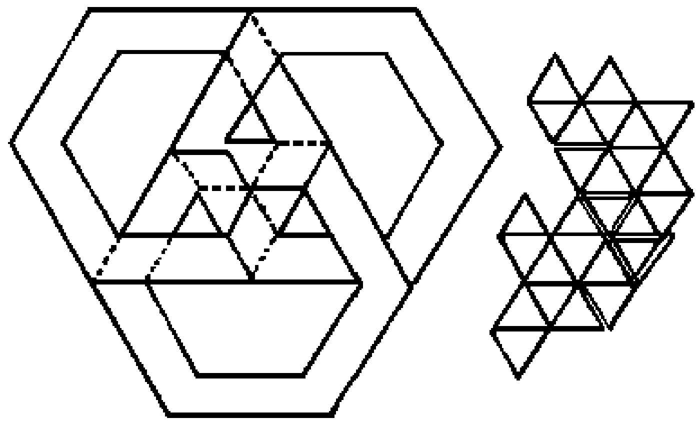

Figure 25:

causing it to fall apart.

The tri-pole symmetry of the maps of this and subsequent straight-strip flexagons produced by uniform doubling (e.g. see figure 27) provided the inspiration for the discovery of what is, in my opinion, the most meaningful system for coloring flexagons. The system may be used to great advantage in most flexagons but is basically best-suited to the evenly-doubled straight-strip flexagons.

The sides of the flexagon which are represented by the vertices of the central triangle of the map are colored red, yellow and blue – the pigment primaries. The side between red and yellow and opposite blue on the map is colored with the mixture of red and yellow – orange. The side between blue and yellow is colored green, and that between red and blue, violet. In this way we pass outward from the central triangle, mixing adjacent colors. Between red and orange comes red-orange etc. The color-map of the 24-sided straight-strip flexagon is shown in figure 27. Colors are usually not easily distinguished much beyond the second or third "mixing," but this is sufficient in most cases.

Now for the advantages of this system. It is, of course, impossible in most cases to tell at first sight just what the map of a given flexagon is. But this coloring scheme gives us a practical system for "remembering" where we are on the map of an unidentified flexagon. We know, for instance, that from the position with blue-green on one side of the flexagon and blue on the other we can go either to green, or if the flexagon is large enough, to blue-blue-green. Thus we can travel anywhere we choose without the aid of a map (although it is necessary to keep the rule of operation of the map in mind). If the map of the flexagon has actually been lost or forgotten, the coloring helps in its reconstruction.

Most people having their first experience with flexagons will find it incredibly difficult to get from, say, green to orange, because this involves a number of consecutive flexes without rotations. The tendency is to rotate consistently after every flex.

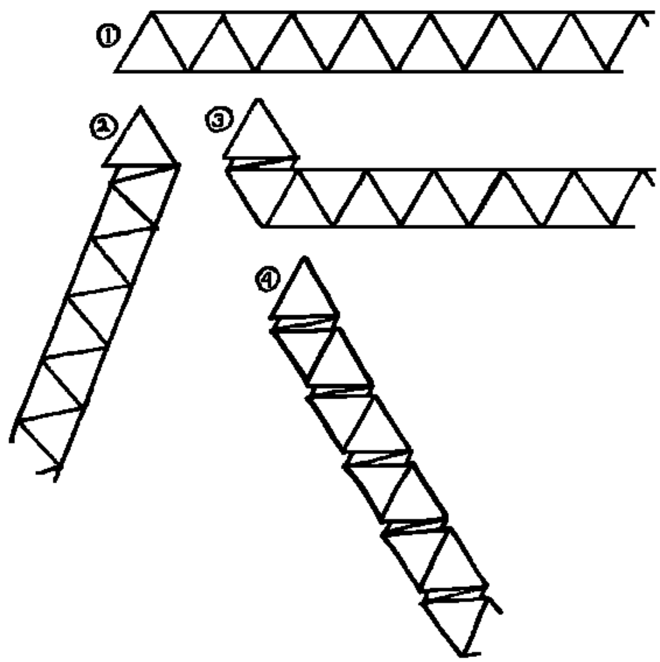

Figure 26:

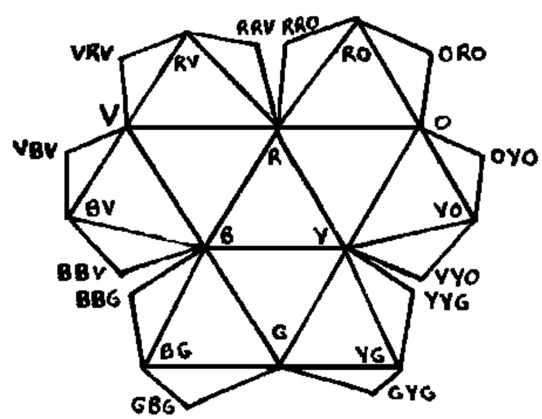

Figure 27:

5 Sides and Faces

I have said that flexagons may be made with any number of sides. Surely, however, there is a limitation due to the bulkiness of the paper.

The situation is reminiscent of an old puzzle which requires that we fold a sheet of paper over, double, ten times. The puzzle can theoretically be solved, of course, but the bulk of paper involved would make the problem physically possible only through the use of a considerable yardage of extremely thin paper.

The practical limit for ordinary flexagons is somewhere around 50 sides. Many 48 sided straight-strip flexagons have been built successfully. However, by using a special type of flexagon and by omitting just about one-half of the different flexes that would ordinarily be possible, the practical limit may be extended quite appreciably. The only flexagon making use of this principle, a giant of 658 sides, is believed to hold the world's record. It is furthermore believed that the title will rest secure for some time, for the arduous chore of constructing such a monster will no doubt dissuade most contenders.

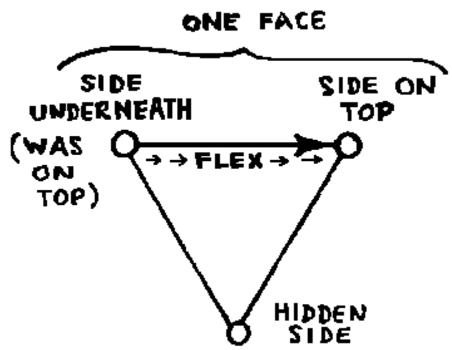

Figure 28:

This large flexagon, it was said, makes usually half of the usual number of different flexes that would be used. What is this "usual number?" To find the answer to this problem, we again look to the map. On any map a line between two points represents the flexing operation that replaces the side represented by one of the two points with the side represented by the other point. That is, in passing along a map line, the point at the end of the line toward which we travel is the side that will next be found on top of the flexagon. The point we are leaving is the side that was last on top and which will next be found on the bottom. This is all diagrammed in figure 28. Now, it is possible to travel the other way along the same path, by turning the flexagon over. Therefore, each line connecting two points of the map stands for two different flexes and also for two different positions of the flexagon. We say that each path stands for two "faces" of the flexagon.

The number of faces, f, of a flexagon with n sides is given by the equation $f = 2(2n-3) = 4n-6$.

To travel to every one of these faces in the shortest possible way, flex as long as possible without rotating, rotate, and so on, keeping this up until at least two sides repeat consecutively. Then turn the flexagon over and start again. This method takes $2(3n-6)$ flexes and $2n$ rotations. It has been named the Tuckerman traverse after its inventor (L. B. Tuckerman).

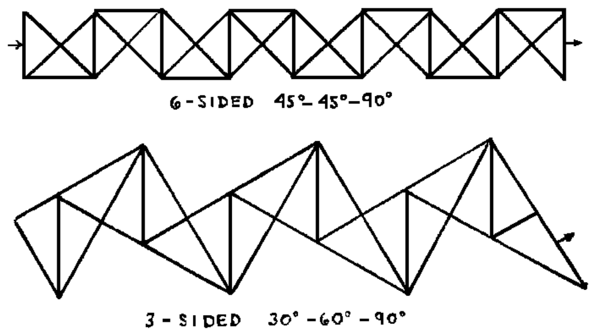

Figure 29:

6 More Flexagons

This is the flexagon. Or, rather, this is a tiny slice of the whole flexagon problem, most of which has not even been hinted at. If you enjoy the *really* obscure, and if you are willing to delve into some of the gory details, there are several variations on the theme ...

Flexagons need not be built of just three pairs of pats. Four pairs work just as well. With two pairs, most of the possible faces are eliminated. One pair can be flexed in theory only. Any number will do, but only three will lie flat; unless, of course, we change some angles.

The triangles need not be equilateral, although a convenient rule is that they must overlie one another in an even stack. A pair of interesting specimens that have been built successfully are a six-sided isosceles-right- triangle flexagon of eight pats, and a three-sided $30° - 60° - 90° -$ triangle flexagon of twelve pats, both of which lie flat when certain sides are exposed. These may be made from the plans shown in figure 29. The order of numbering the triangles in the plan is the same for these as for ordinary straight-plan six- and three- sided flexagons. You may figure out the details. By the way, flexagons made up from odd-shaped triangles tend to fall apart easily so that the practical limit is lowered enormously hovering just above three sides.

To break another rule, flexagons need not be built of triangles. Square flexagons, called tetraflexagons, are planned and built with little more difficulty than flexagons made of triangles. Only two pairs of pats need be used. To make the simplest tetraflexagon follow the plan in figure 30. Notice the square map.

As a general example of flexagons of higher "class" than three; i.e., made up of polygons with more than three sides, we will build a pentaflexagon (made up of pentagons). There is a connection, which we will not bother to prove here, between the "class" (number of sides of the polygons in the *plan*) and the "cycle" (number of sides of the polygons in the *map*). They are usually the same. Thus the pentaflexagon we will build has the pentagonal map shown in figure 31. First make the Tukey triangle network by joining midpoints of sides as in figure 32. Then number the vertices as shown, and copy down the number sequence:

1		3		8		6		1		...
	2		4		7		5		2	...

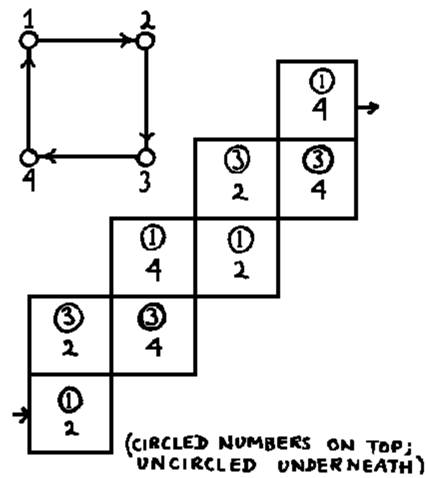

Figure 30:

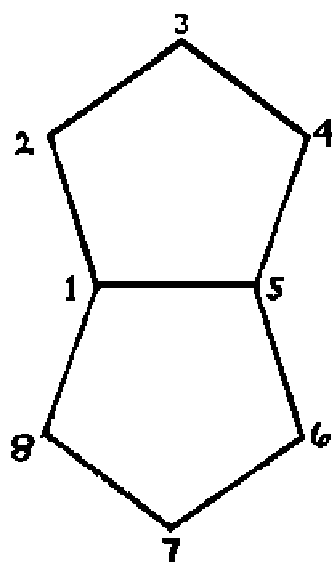

Figure 31:

In finding the signs, there are extra directions involved, but these are merely treated in the usual way. Copying the two sequences in final form we have:

1	3	3	5	8	8	6	6	1	3	...
2	2	4	4	1	7	7	5	2	2	...
R	L	R	L	L	R	L	R	R	L	...

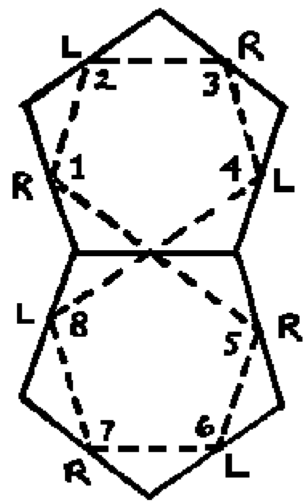

Figure 32:

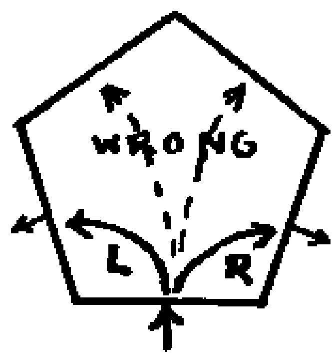

Figure 33:

To build the plan, always leave a given leaf as soon as possible on the right or left. Don't take the second side in a given direction in either case (see figure 33). The plan we make will look like figure 34, except that in the figure all the signs have been changed to show that this does not influence the final flexagon.

Two pairs of pats are needed; therefore, this sequence of pentagons will be repeated. The final flexagon, which does not lie flat, is seen in figure 35. It is not difficult to operate once you become accustomed to it, and a little experimentation with flexagons such as this produces fascinating results.

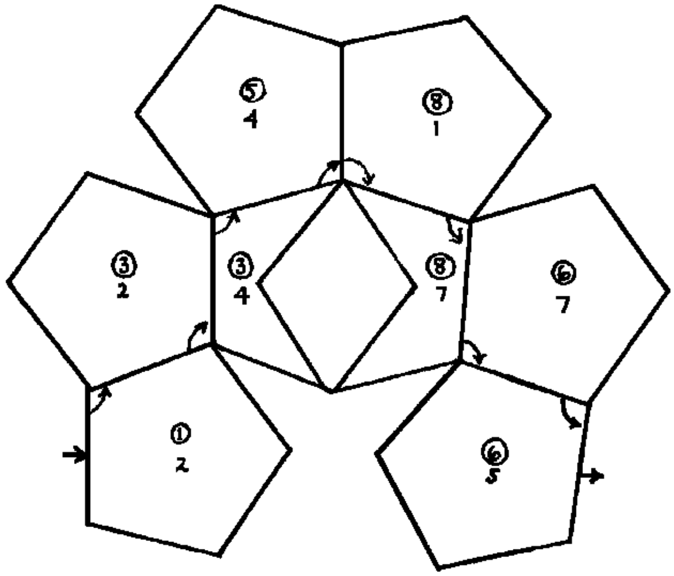

Figure 34:

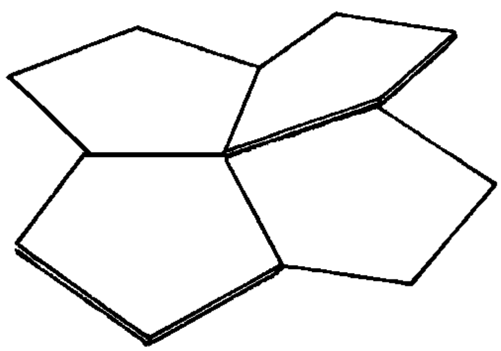

Figure 35:

References

[1] Martin Gardner, "Flexagons," *Scientific American*, December. 1956.

[2] C. O. Oakley and R. J. Wisner, "Flexagons," *American Mathematical Monthly*, March 1957.