Trigonal Flexagons

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Abstract

Maps and cutouts for a variety of flexagons are presented, emphasizing those which can be cut out, mostly from single sheets of paper. Since TeX may not align front and back images, and in any event if cutting up the booklet is not desired, the .eps files can be printed directly to get sheets suitable for cutting. In the same spirit, only those sheets which are going to be used right away need be printed.

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1 Introduction

Flexagons can become fairly complicated. The ones based on triangles are most conveniently made from long strips of paper; a roll of adding machine or calculator tape is ideal for this purpose given its convenient width. Crooked strips can be gotten by gluing faces of straight strips together, or just cutting out prepared segments and then joining them together. Leaving one extra triangle in each segment for overlapping and later gluing leads to efficient constructions.

Coloring the triangles is another problem, which can be done with crayons or markers once it is known which colors ought to be used. Aside from copying an already existent design, this is best done by drawing the Tukey triangles and then lettering or numbering the triangles in the strip. That information is sufficient to fold up the strip, since pairs of consecutive numbers are to be hidden by folding them together. Painting can be done before folding by following a color code for the numbers, or after the folding is done, when the faces can be painted wholesale, or even embellished with designs.

Making pages of cutouts, even the ones folded from "straight" strips, require a higher degree of preparation to fit them into the rectangular format of a page, although it is relatively easy to assemble a collection of primitive components which later can be glued together according to the necessities of the individual flexagon.

Although flexagons can be created from any convex polygon, the variety of possibilities to be explored is greatly reduced by using tringles. Even so, there are many possibilities to try out, but the following list shows up some of the more important regularities.

1. The classical flexagon is built from equilateral triangles, so that all thre angles are the same and equal to $60°$. The plans of the homogeneous normal flexagons, with $9, 18, 36, 72, \ldots$ triangles and $3, 6, 12, 24, \ldots$ faces, their genotypes are all straight strips as shown in Figure 1.

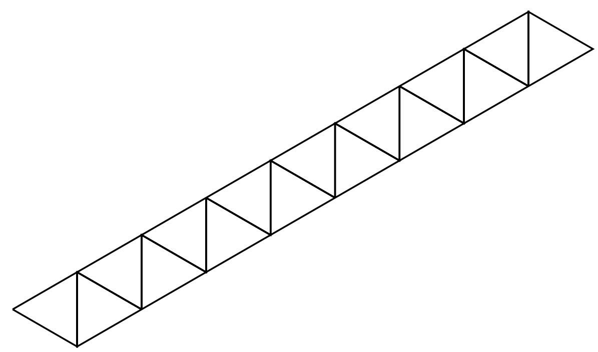

Figure 1: The classical flexagon.

Because the angle between every triangle and its reflection will be the same $120°$, the smallest combination which will fill out $360°$ requires three sectors. Trying to make do with only two sectors will result in one flexacup or another, none of which can be flexed through a full cycle although they will offer different color combinations to view. Of course more than three sectors could be used, but such excessive flexagons are always rather ungainly.

2. The next most classical flexagon is the one made from isosceles right triangles, meaning angles of 45°, 45°, and 90°, its fame resting mostly on the flexatube.

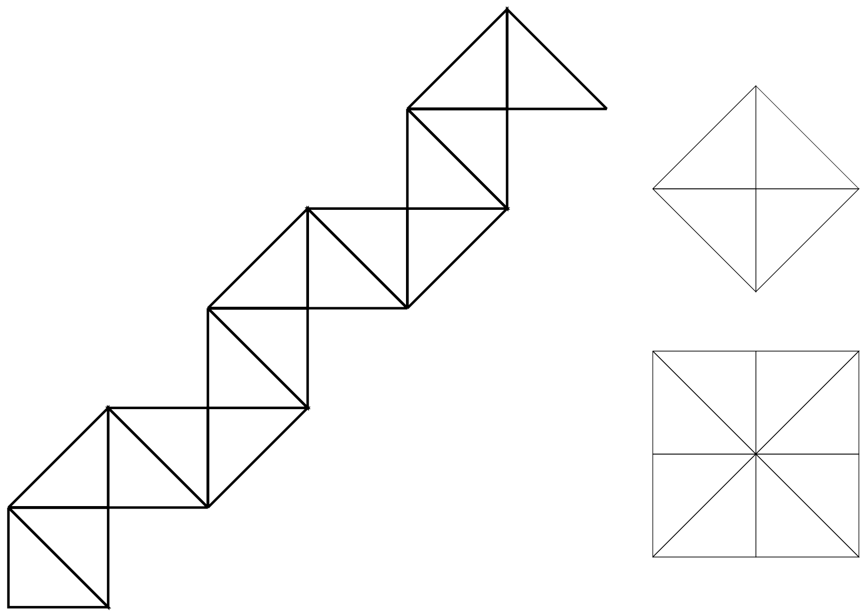

Figure 2: The flexatube precursor. One way to get the actual flexatube is to fill in the missing notches and work with diagonally creased squares.

Depending on where the stack of triangles is opened, the sectorial angle can be either 45° or 90°. Two sectors of the formar will only make a flexacup; three sectors don't work very well, but four sectors will accomodate the 45° angles quite nicely. They show square phenotypes during two phases of the flexing cycle, with a third phase which can be skipped over by skipping a leaf during flexing and oscillating between the two flat forms.

In the two-sector flexacup version, there is a temptation to place additional creases in the faces to obtain the flexatube configuration. Nevertheless, it doesn't quite work out, although it can be successfully performed, without any additional creasing, in the four-sector variant.

3. Half of an equilateral triangle is a skewed right triangle, having angles 30°, 60°, and 90°. All three divide 180° evenly and so will lie flat using six, three, or two sectors respectively. However, to avoid the flexacup phenomonon, the smallest angle needs to be accomodated, leading to a six-pointed star in the phenotype.

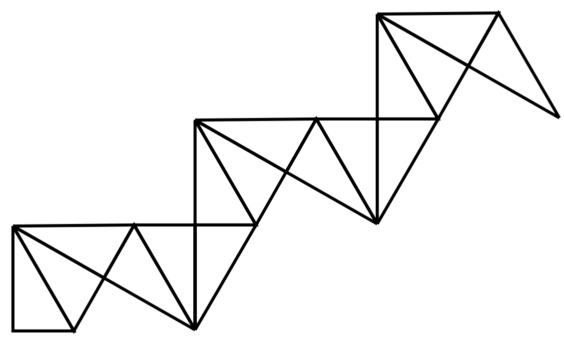

Figure 3: The frieze which will fold up into a 30° - 60° - 90° triagonal flexagon.

Two sectors will accomodate the 90° right triangle but the result is a flexacup which will not progress further. Likewise three sectors will permit the 60° angles to surround the center and even allow a puckered arrangement of the 90° angles, but the best that can be done with the 30° angles is to form a flexacup.

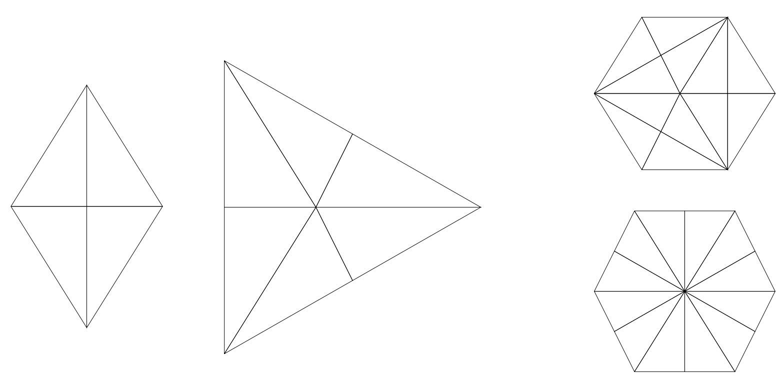

Figure 4: Different phenotypes result from joining respectively two, three, or six sectors, each of which is shown in its planar phase.

Although six sectors surround a center with 30° angles quite nicely, the space for the other vertices is quite cramped. By opening out every second leaf space can be made for the 60∘ angles but there is still an excess of faces to get a strictly flat figure. However, some rearrangement of perimeter will produce another hexagonal figure which lies flat.

4. A third of an equilateral triangle is an obtuse isosceles triangle with angles 30°, 30°, and 120°.

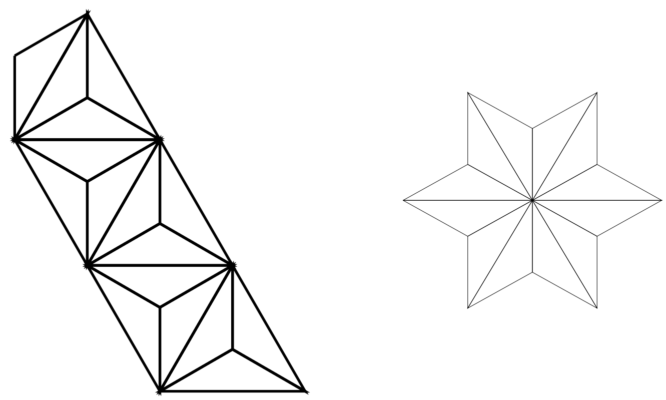

Figure 5: With six sectors, there are two phases of the 30° - 30° - 120° flexagon which will lie flat, the phenotype being a six-pointed star.

In this form the figure will lie flat during two phases of the flexing cycle.

5. Just to prove that it can be done, a five pointed phenotype can be constructed from an 36° - 36° - 108° isosceles triangle.

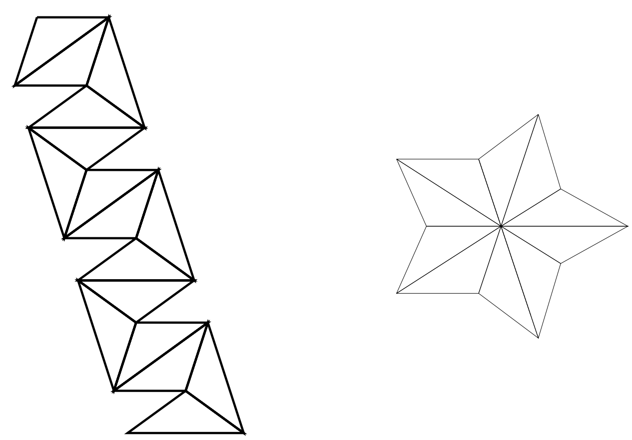

Figure 6: By using angles of 36° - 36° - 108° five sectors will produce a flat flexagon. Left: the genotypic plan as a frieze. Right: the resulting phenotype when the central angle is 36°.

2 First Level Triflexagon

2.1 Tukey triangle

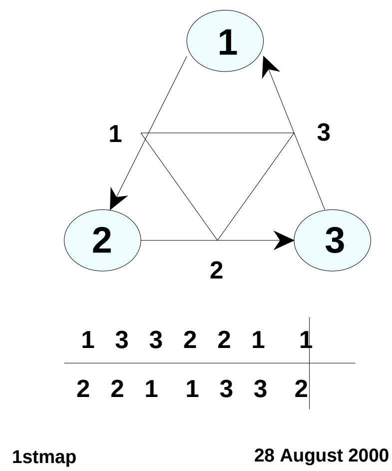

Figure 7: Left: Tukey triangle map for the first level triflexagon, which consists of a single cycle of three colors.

2.2 Flexagon permutations

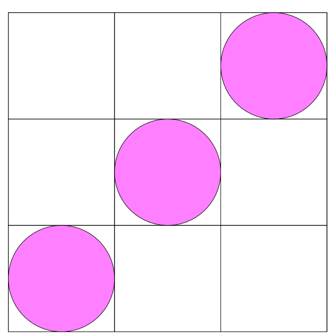

Figure 8: Permutation of the triangles along the strip for a normal triflexagon. They run in order, subject to being flipped over, so the permutation is the identity.

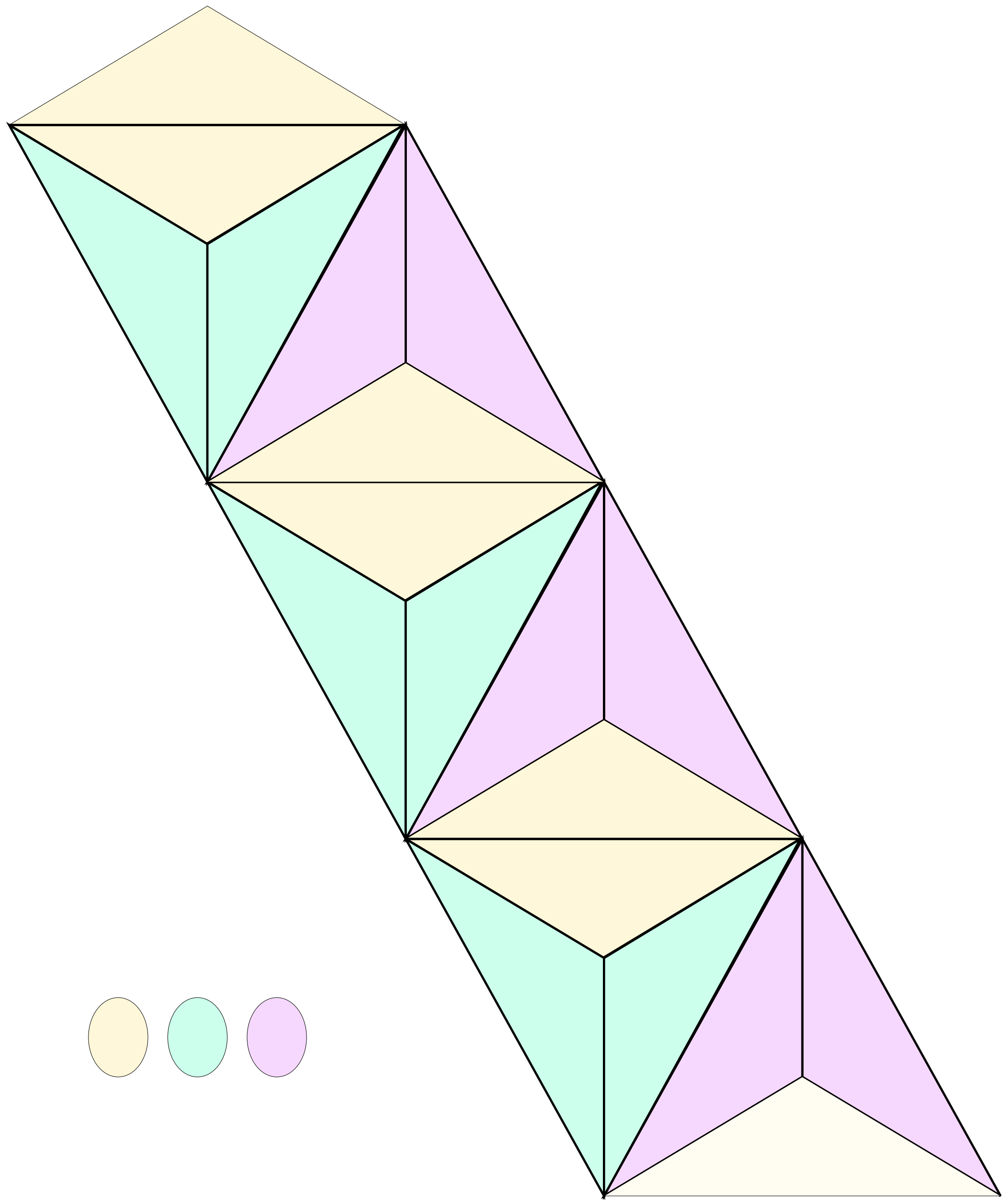

Figure 9: Top side of the $30^\circ - 30^\circ - 120^\circ$ triangle cutout. Together with its backside, the figure displays one single flexagon with six sectors, the minimumum requirement to lie flat on account of the 30° angle.

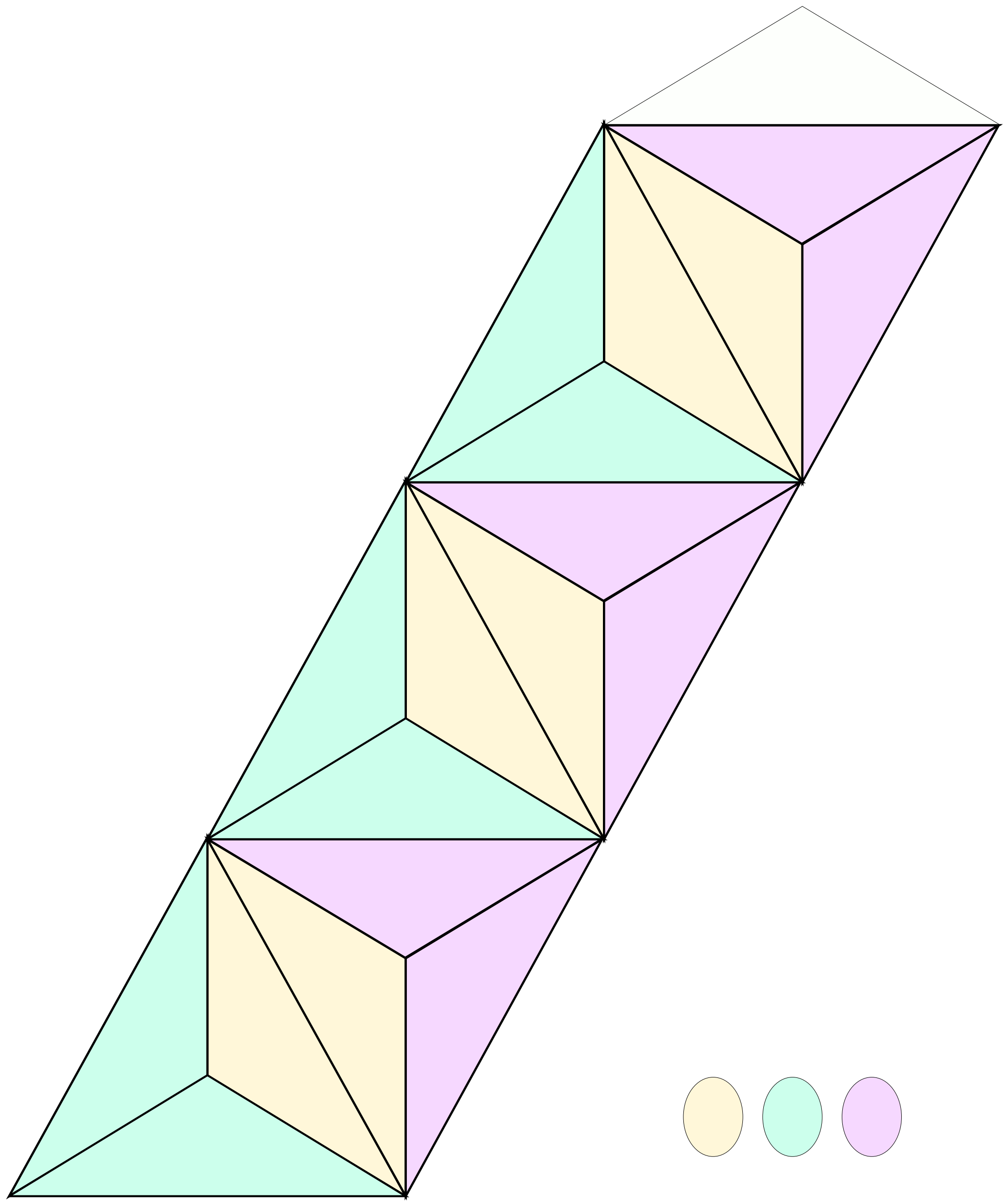

Figure 10: Bottom side of the $30° - 30° - 120°$ triangle cutout.

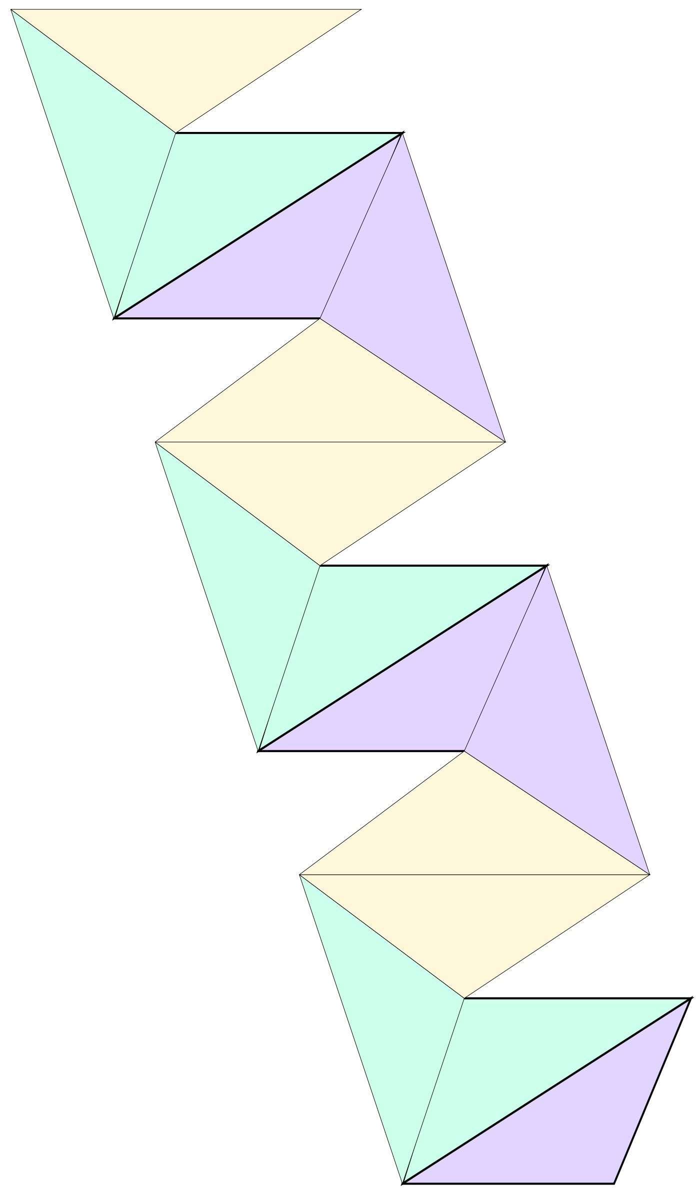

Figure 11: Top side of the $36° - 36° - 108°$ triangle cutout. Together with its backside, the figure displays one single flexagon with five sectors, the minimumum requirement to lie flat on account of the $36°$ angle.

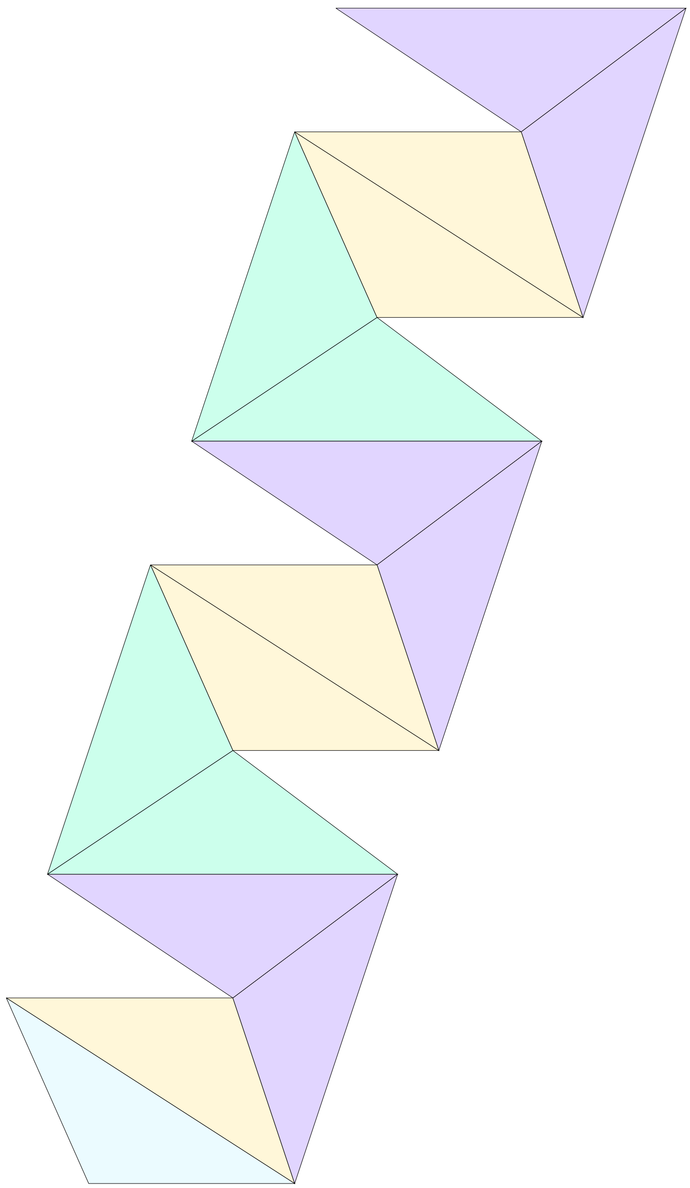

Figure 12: Bottom side of the $36^\circ - 36^\circ - 108^\circ$ triangle cutout.

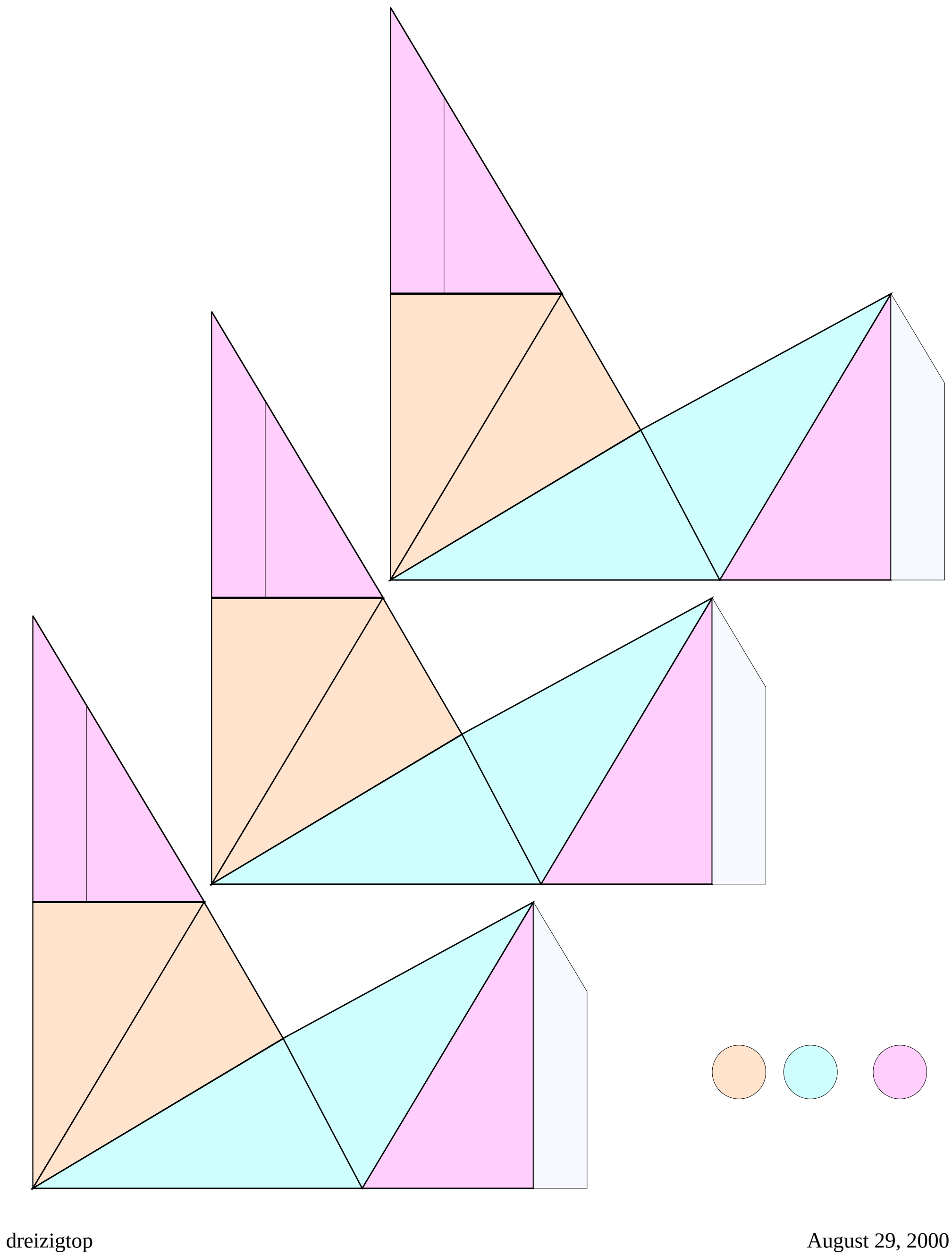

Figure 13: Top side of the $30° - 60° - 90°$ triangle cutout. Together with its backside, the figure displays one single flexagon with six sectors, the minimumum requirement to lie flat on account of the $30°$ angle.

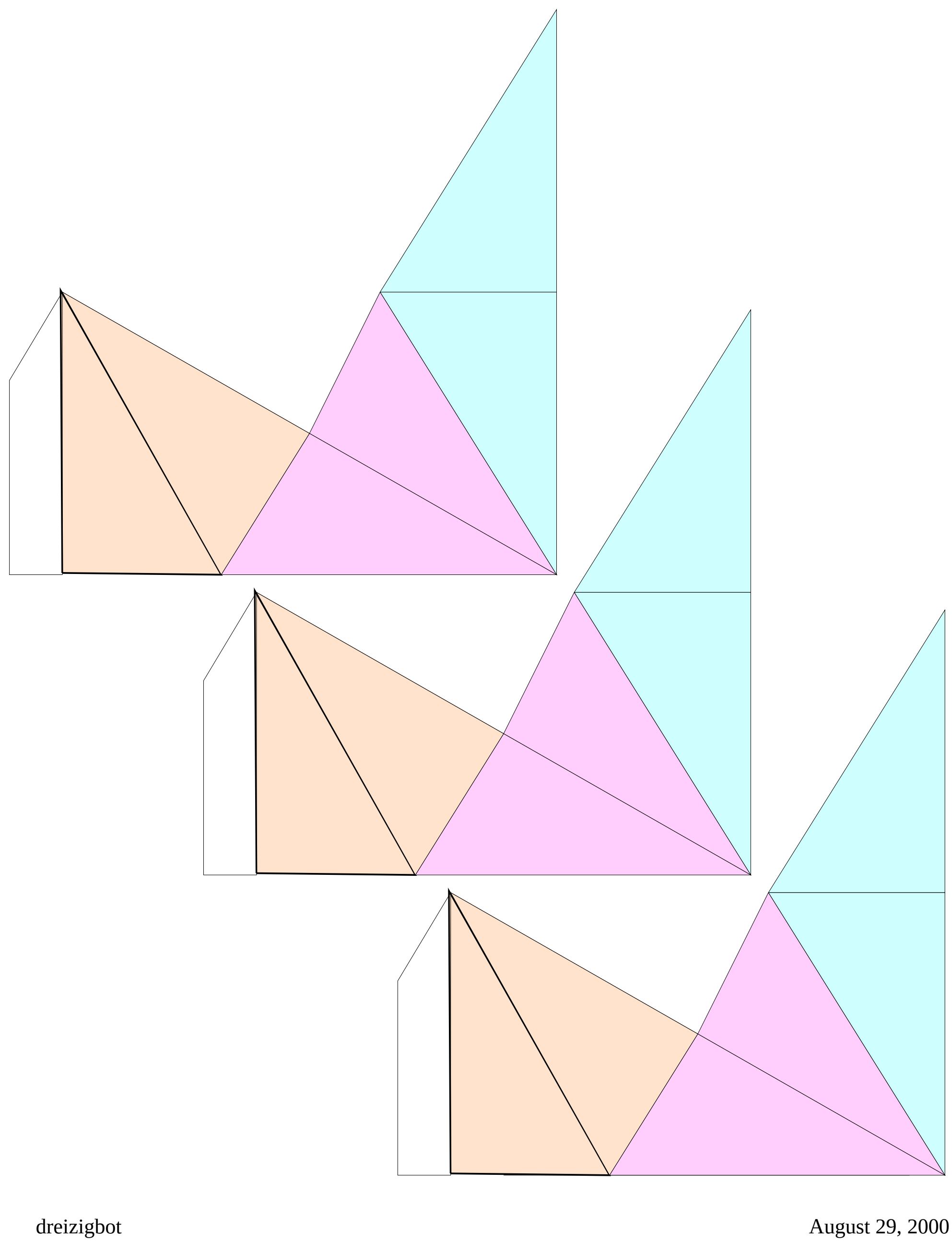

Figure 14: Bottom side of the $30^\circ - 60^\circ - 90^\circ$ triangle cutout.

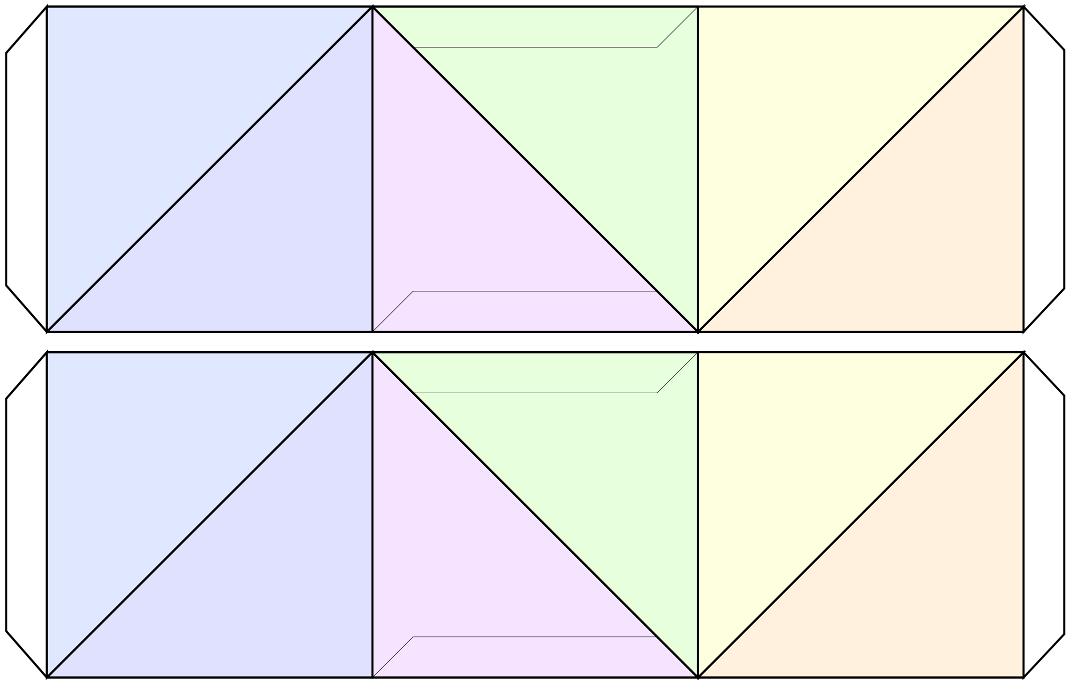

Figure 15: Top side of the $45° - 45° - 90°$ triangle cutout. Together with its backside, the figure displays one single flexagon with four sectors, the minimumum requirement to lie flat on account of the $45°$ angle.

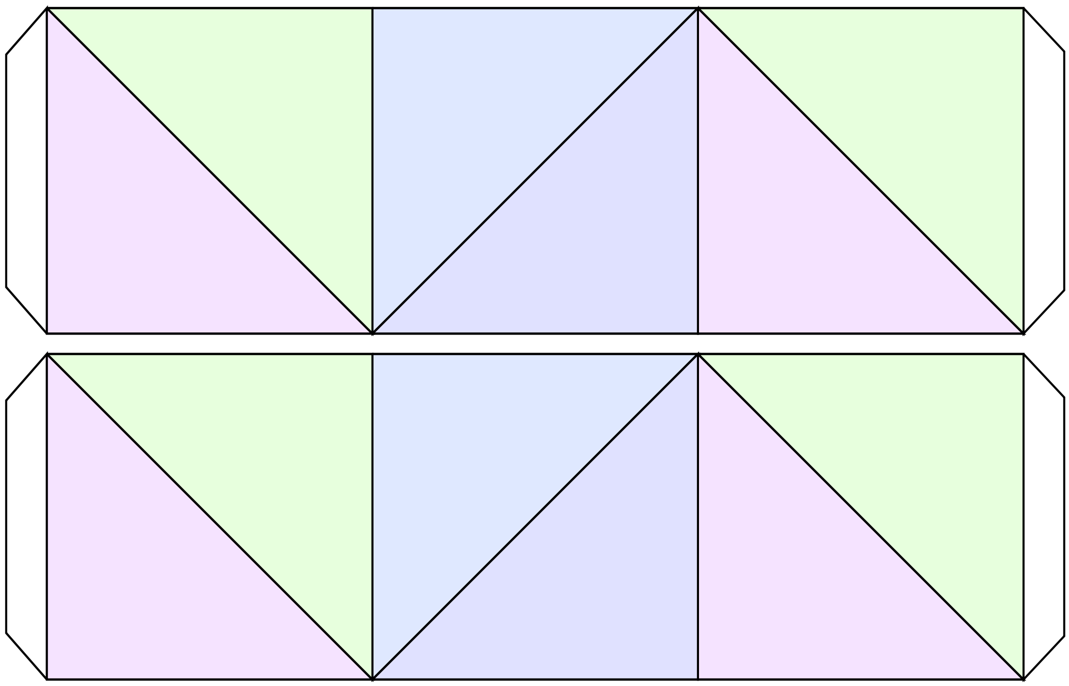

Figure 16: Bottom side of the $450^\circ - 45^\circ - 90^\circ$ triangle cutout.

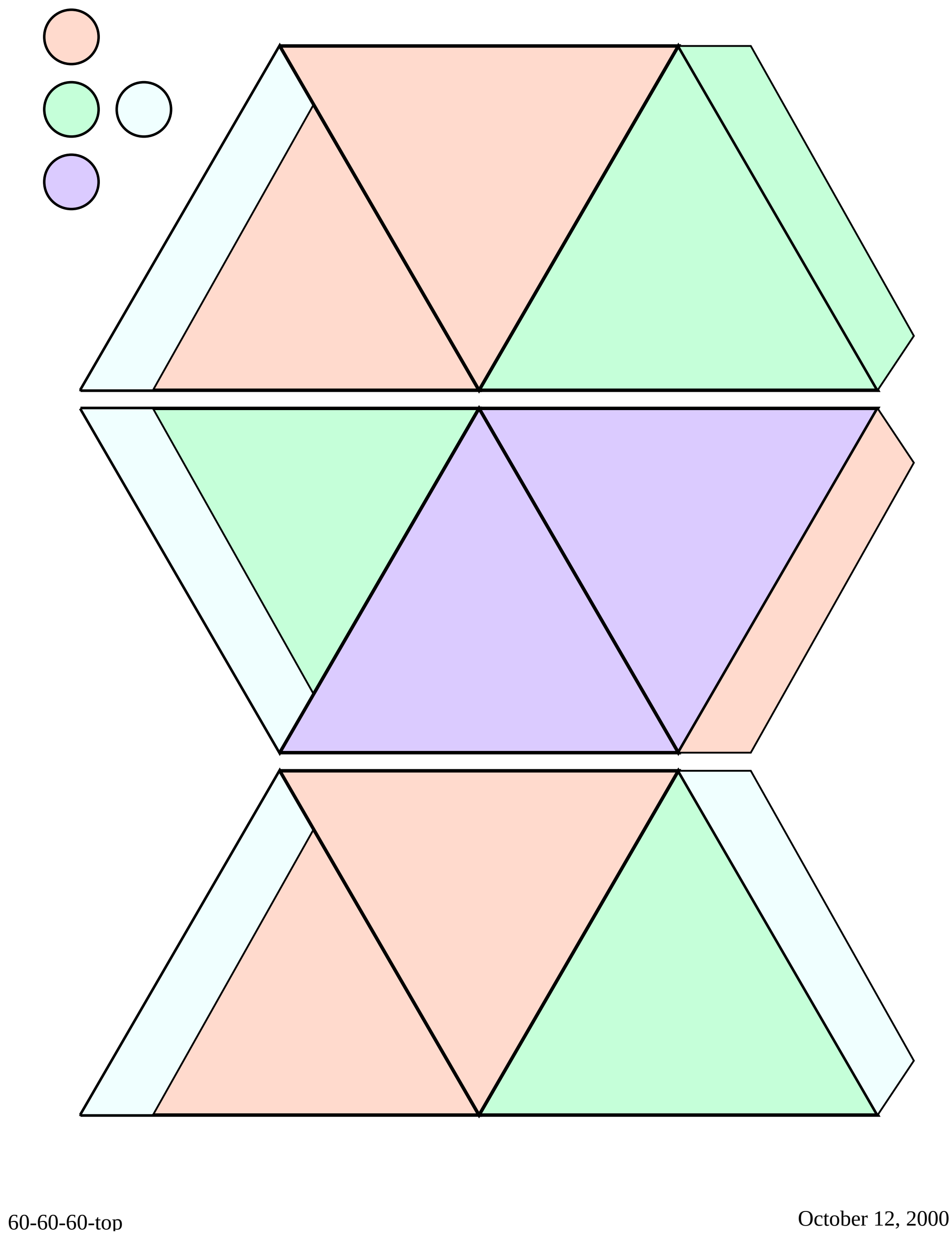

Figure 17: Top side of the first level normal triflexagon cutout. Together with its backside, the figure contains one single triflexagon.

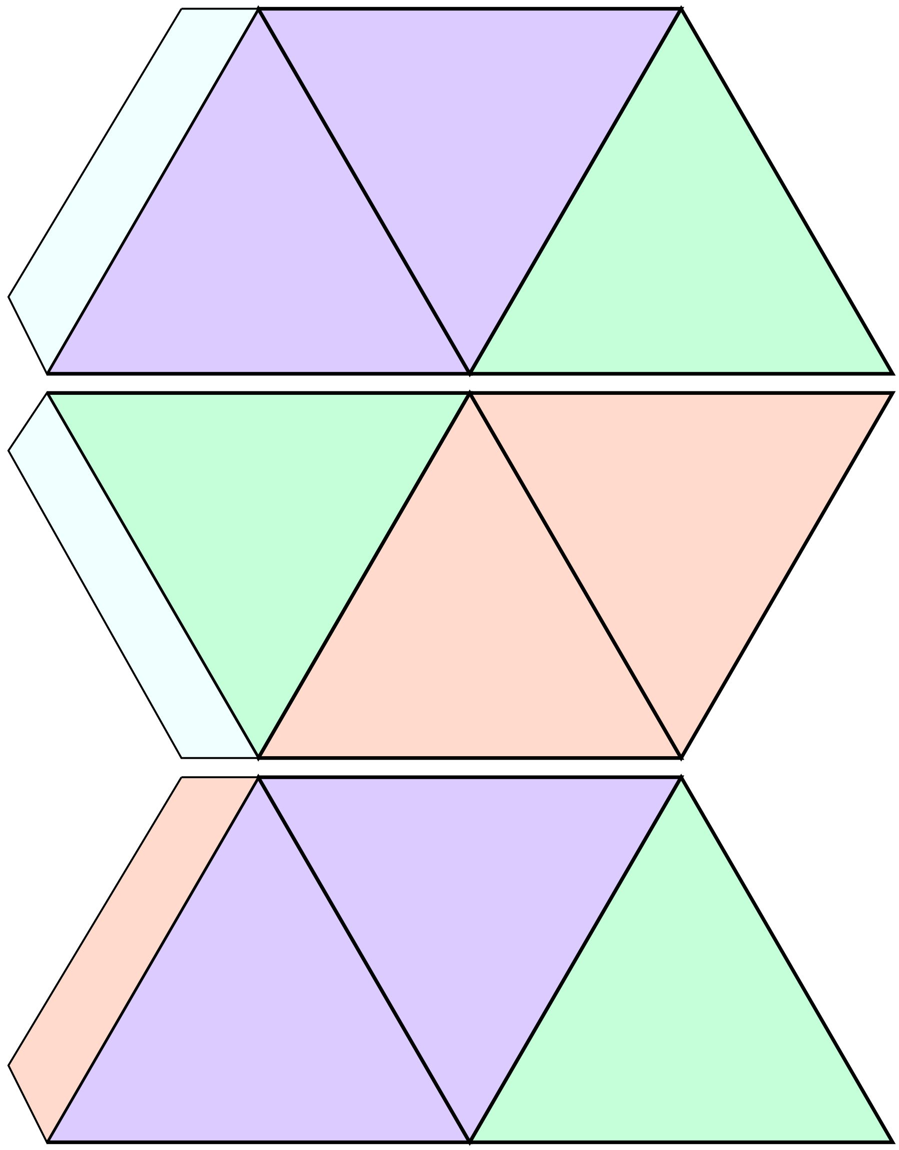

Figure 18: Bottom side of the first level normal triflexagon cutout. Together with its backside, the figure contains three triflexagons.

3 Binary Triflexagon Cutouts

In the recursive elaboration of a flexagon, a single n-gon is replaced by a counterrotating package of $(n-1)$ n-gons. If the n-gon is not regular, a single substitution will produce different plans according to the edge involved. The ultimate flexagon should look and act the same irrespective of the place where the substitution is made, but the plans from which they are folded may overlap differently, requiring artistic anjustments in the actual frieze which will be laid out.

3.1 binary Tukey triangles

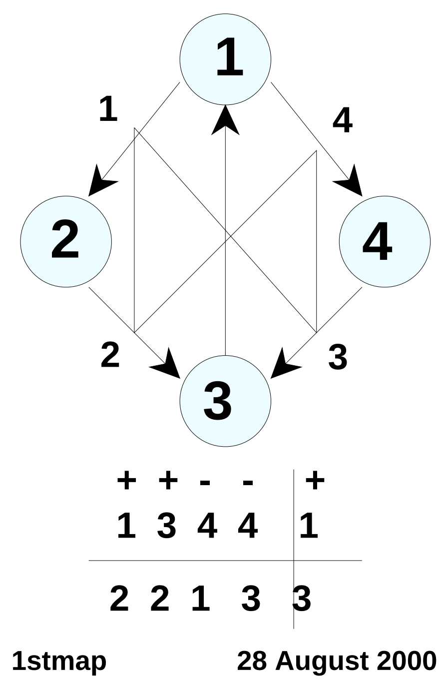

Figure 19: Left: Tukey triangle map for the binary triflexagons, which consist of a two tangent cycles with four colors.

3.2 binary flexagon permutations

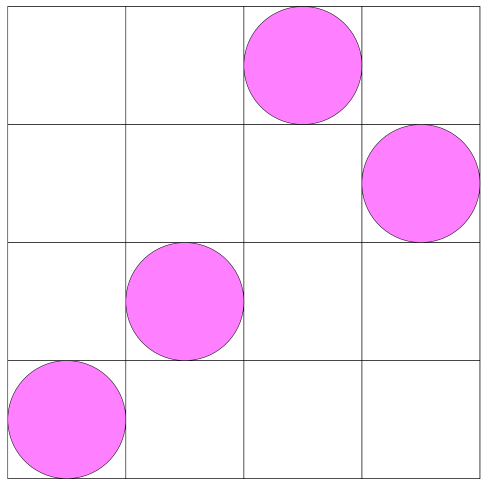

Figure 20: Permutation of the triangles along the strip for a binary triflexagon. A reversed pair of faces takes the place of one single face in the first level flexagon.

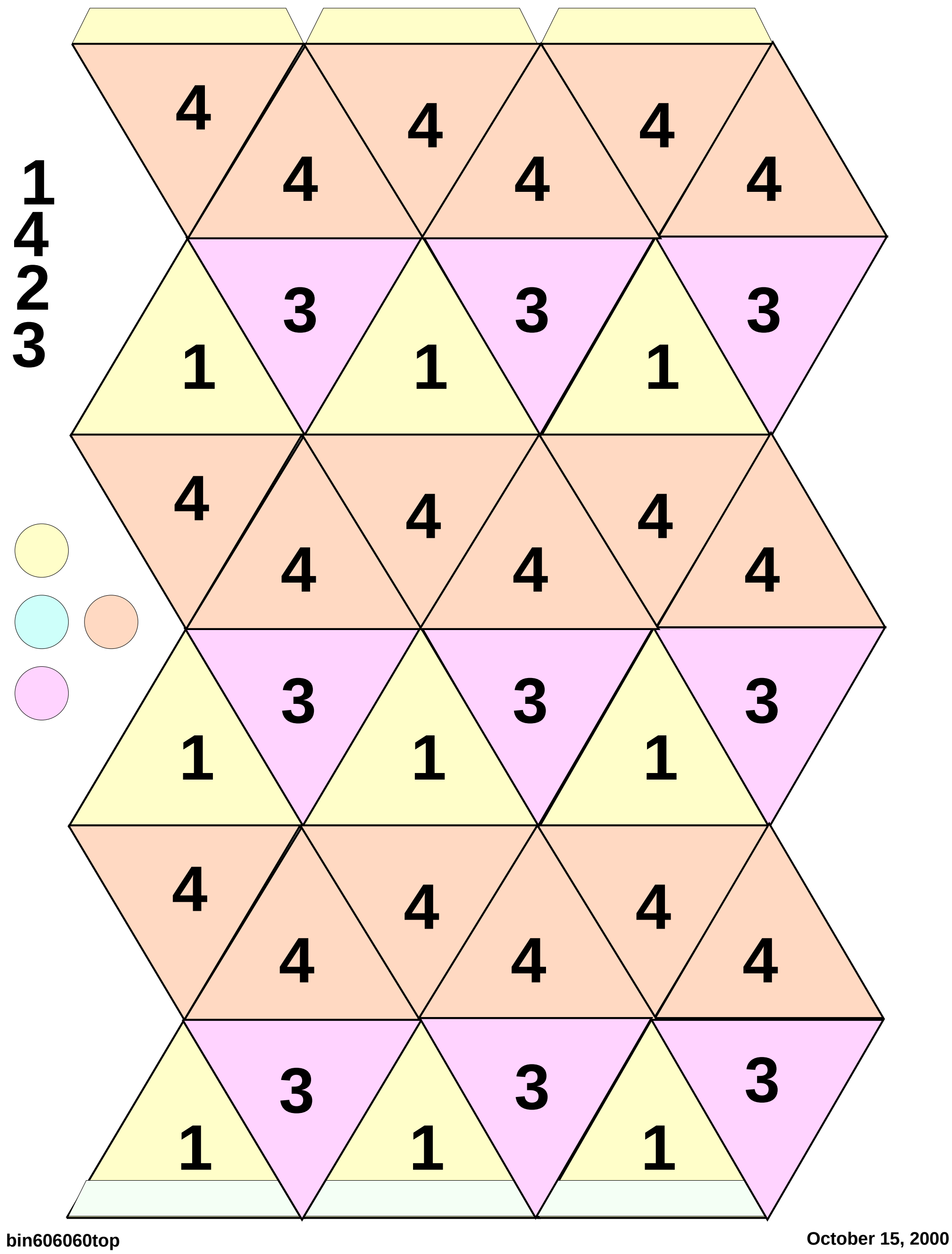

Figure 21: Top side of the first level binary triflexagon cutout. Together with its backside, the figure contains three binary triflexagons which are laid out side by side, given that there is enough space to accomodate them.

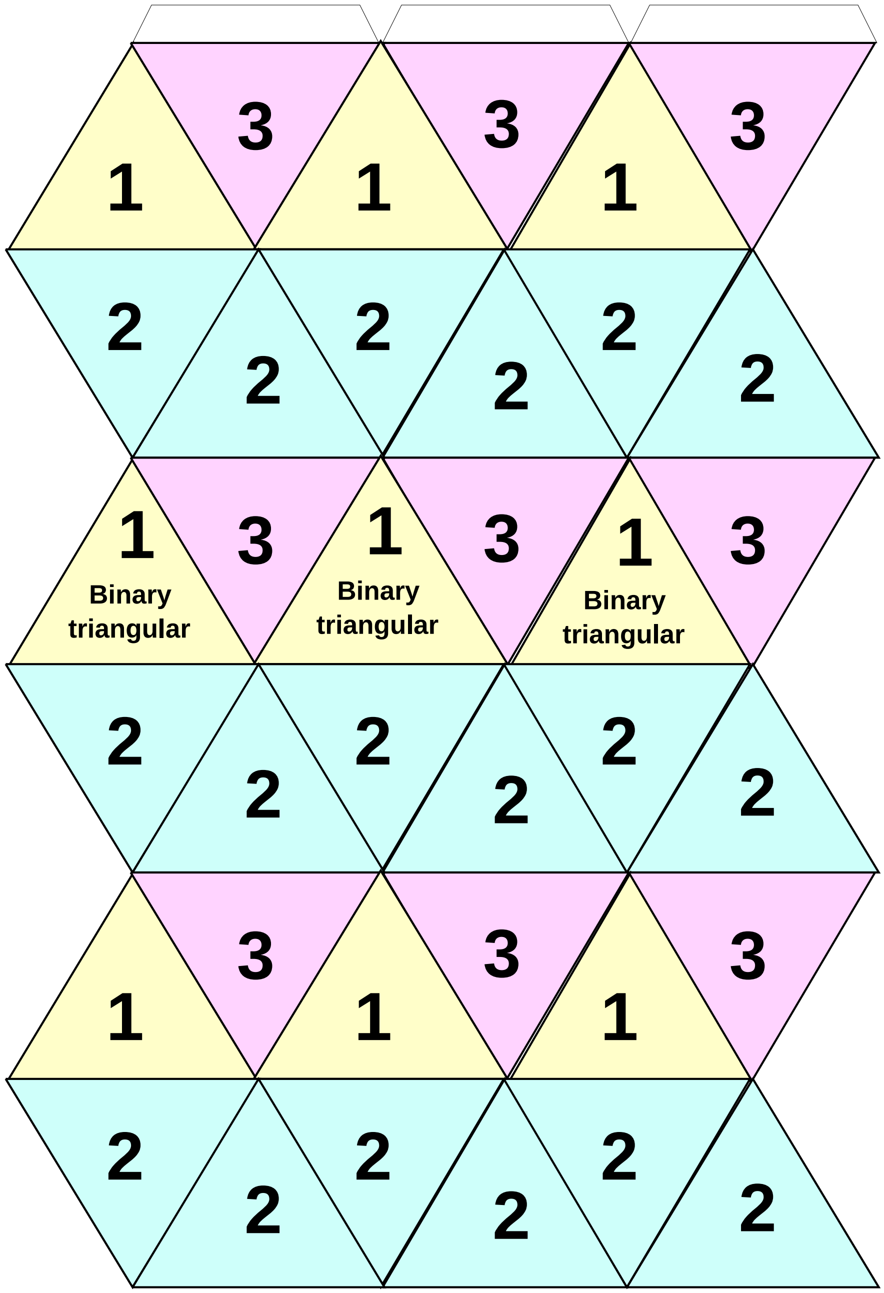

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Figure 22: Bottom side of the binary flexagon cutout. Together with its backside, the figure contains three triflexagons.

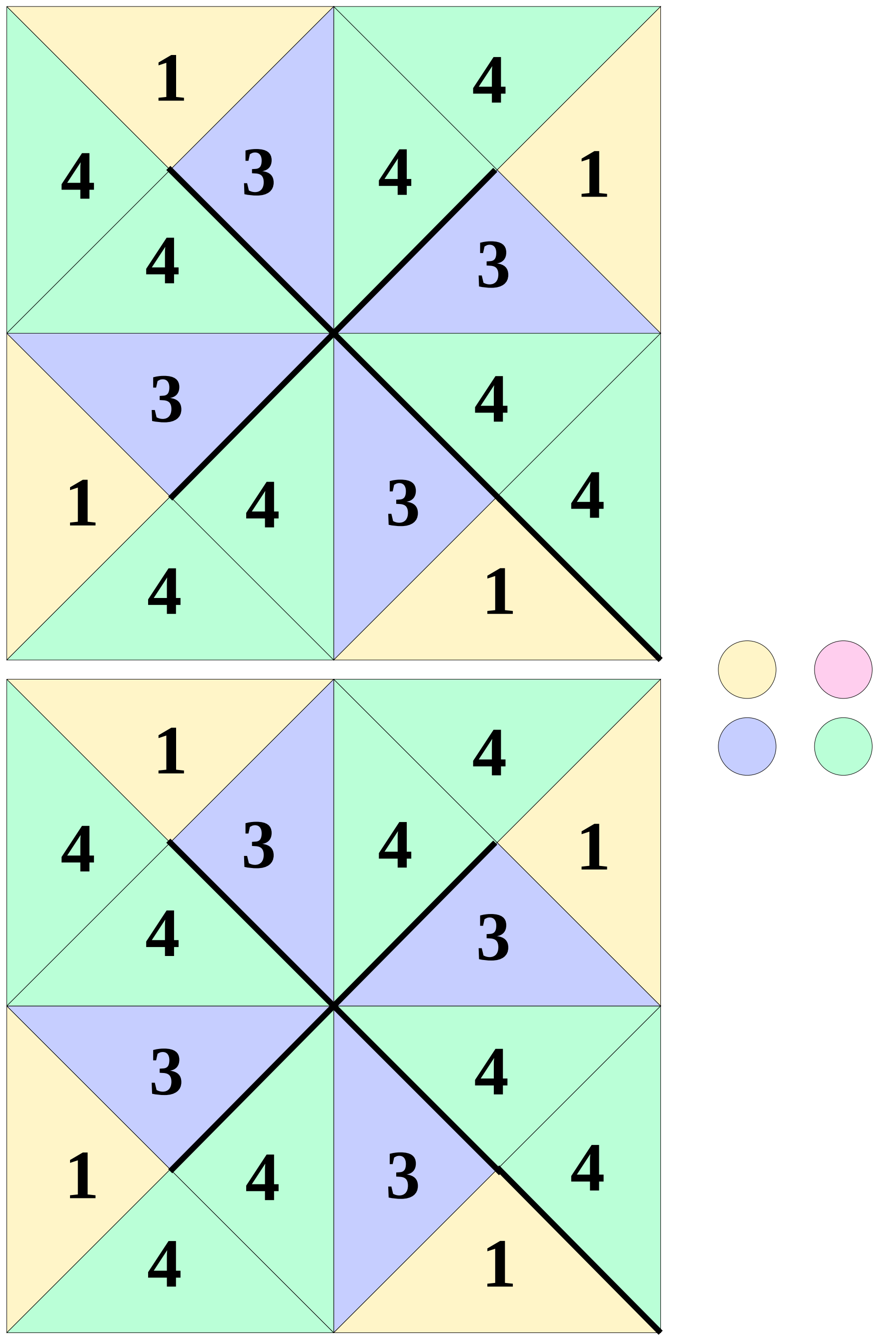

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Figure 23: Top side of a binary $45° - 45° - 90°$ triflexagon cutout. Together with its backside, the figure contains one single triflexagon.

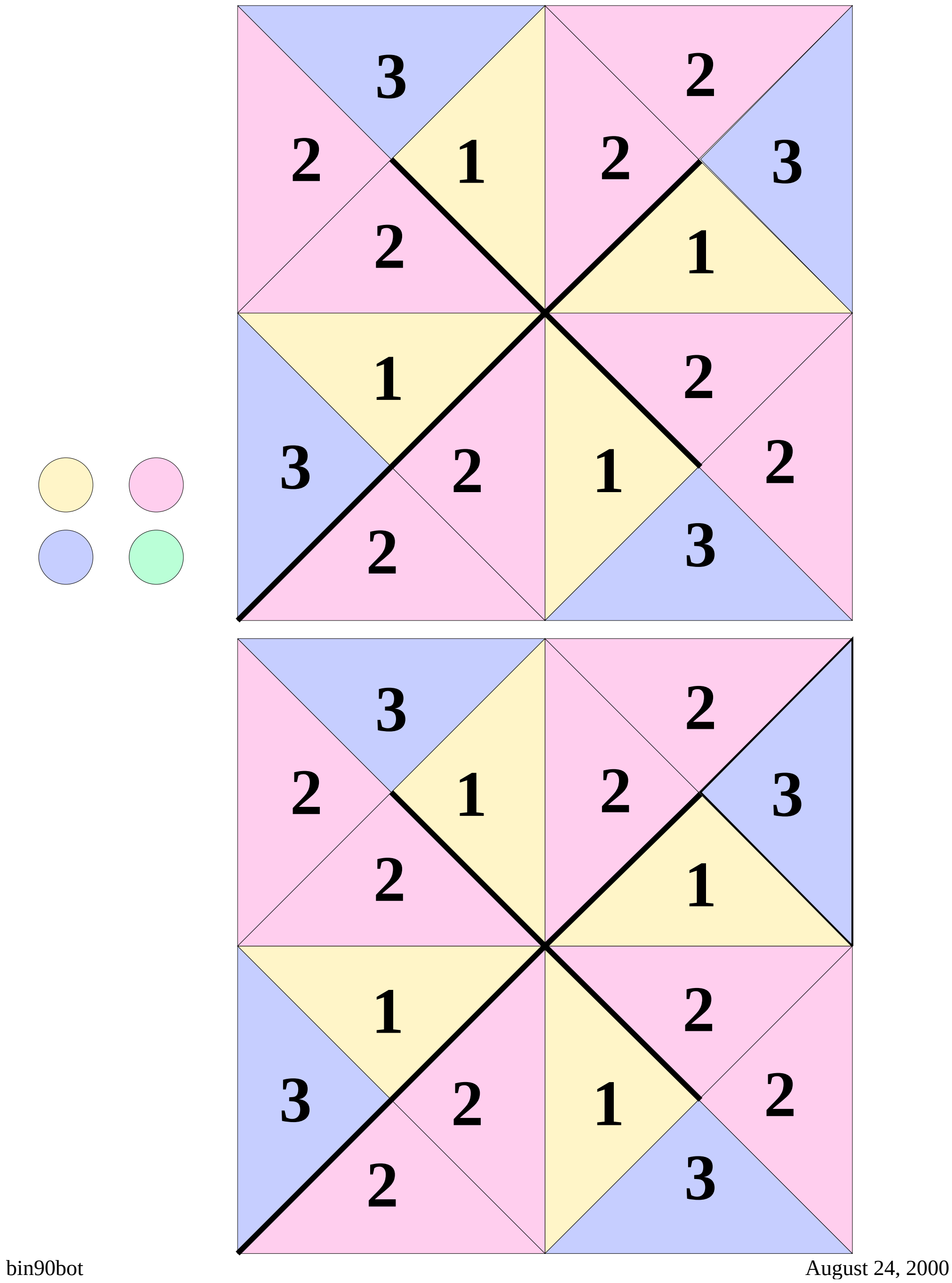

Figure 24: Bottom side of a binary 45-45-90 triflexagon cutout. Together with its backside, the figure contains three triflexagons.

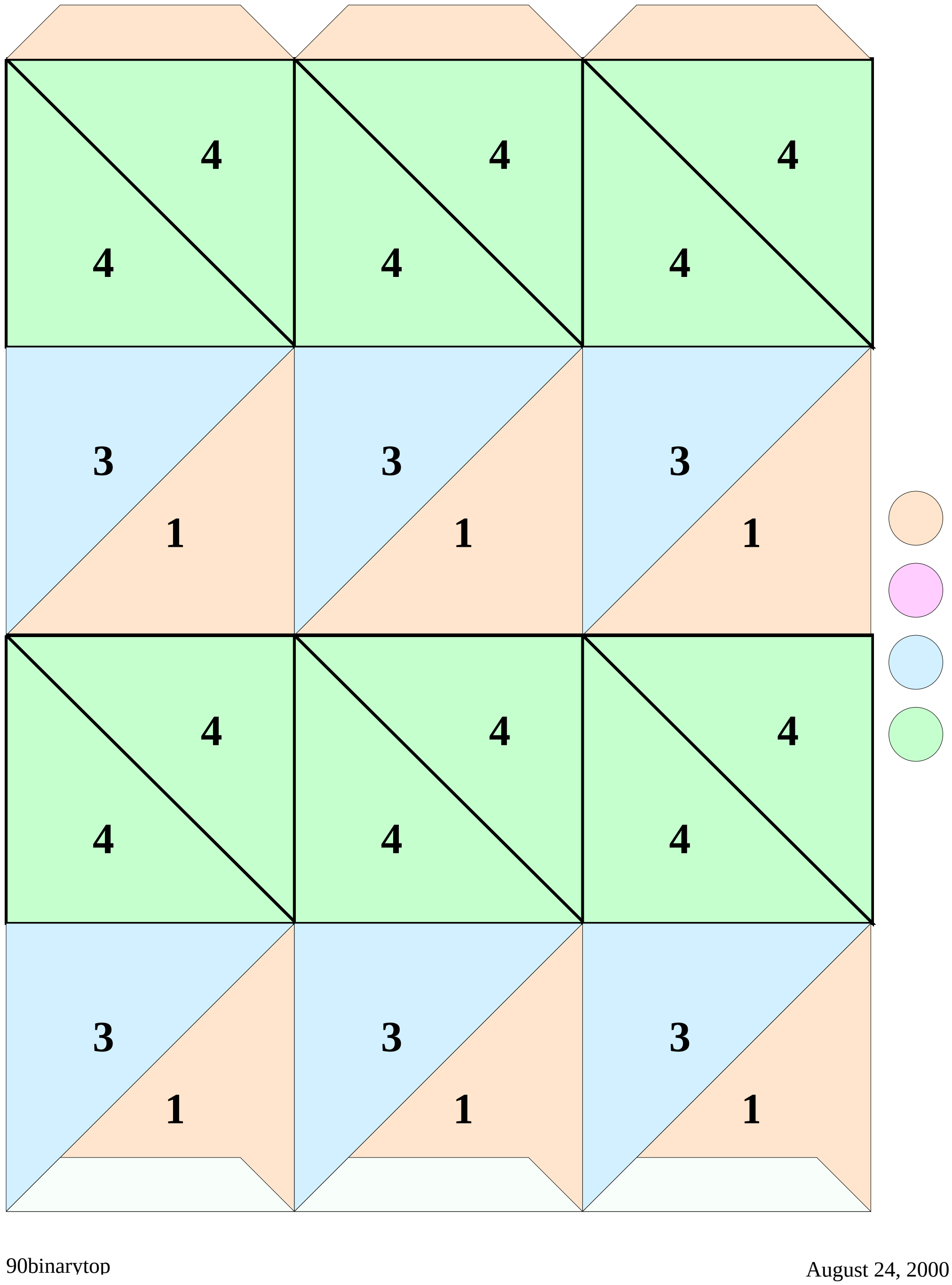

Figure 25: Top side of a binary $45° - 45° - 90°$ triflexagon cutout. Together with its backside, the figure contains one single triflexagon.

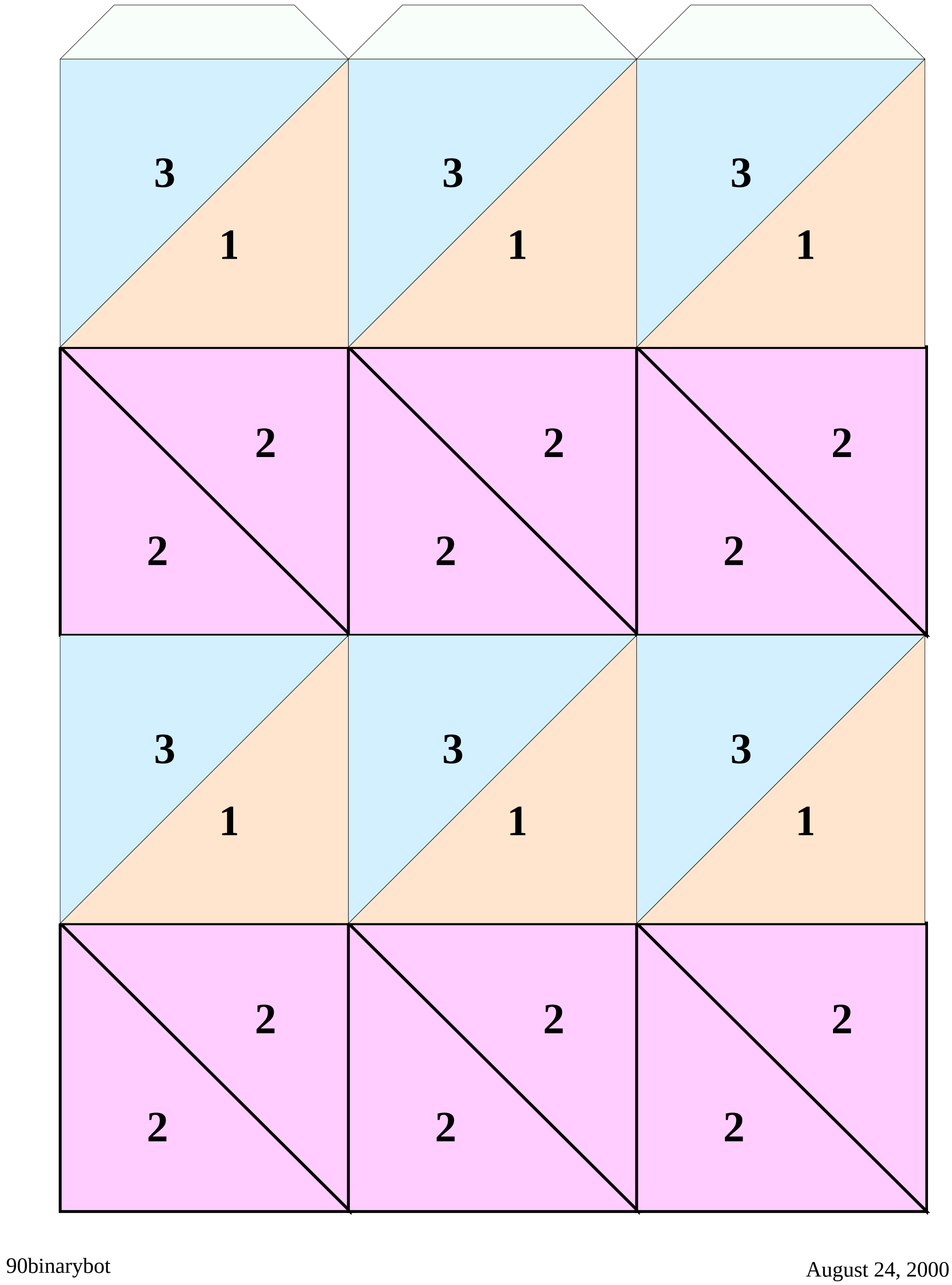

Figure 26: Bottom side of a binary 45-45-90 triflexagon cutout. Together with its backside, the figure contains three triflexagons.

4 Second Level Triflexagon Cutouts

4.1 Tukey triangles

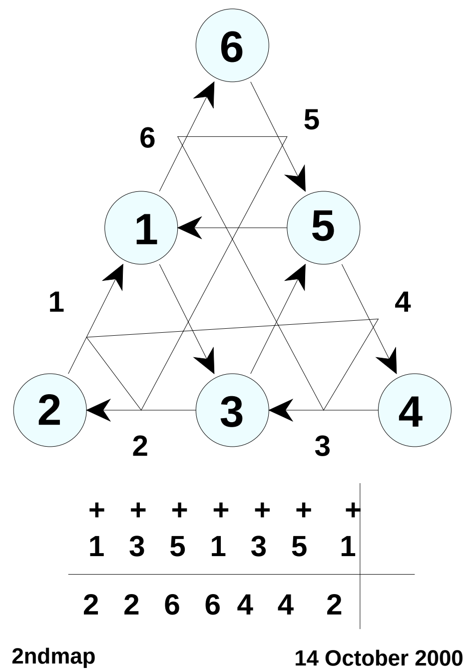

Figure 27: Tukey triangle map for the second level triflexagon, which consists of a cluster of three triangles surrounding a single central cycle.

4.2 Flexagon permutations

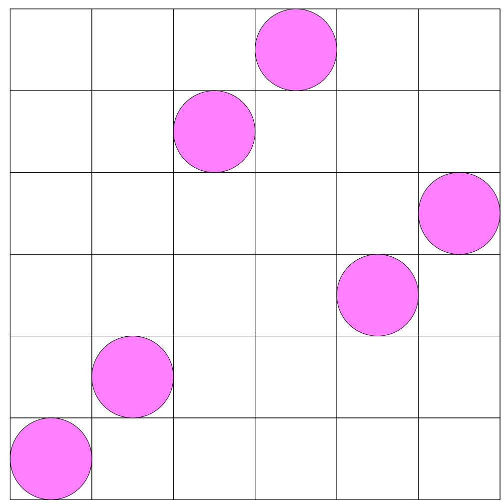

Figure 28: Left: Permutation of the triangles along the strip for a second level normal flexagon.

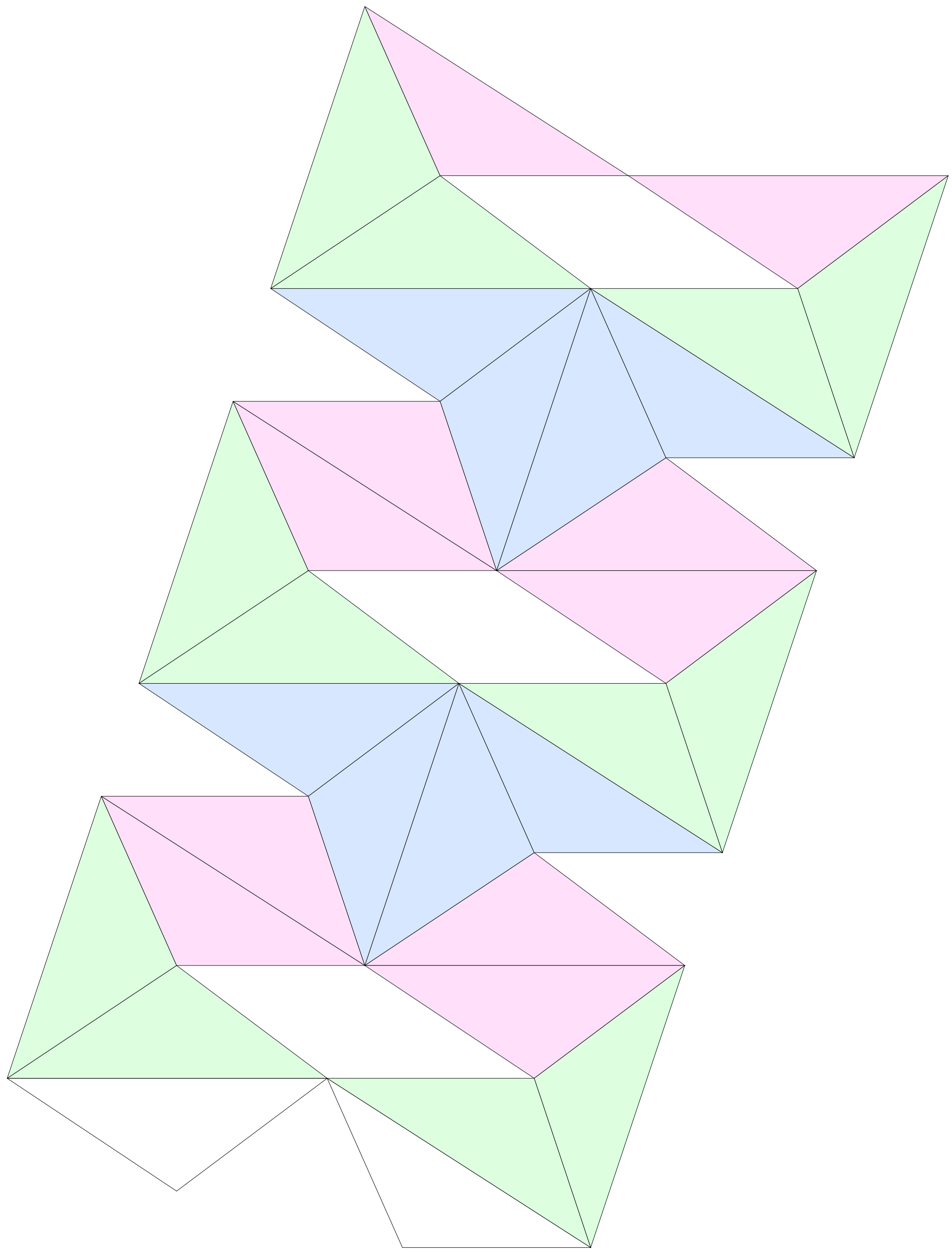

Figure 29: Top side of the second level $36° - 36° - 108°$ triangle cutout. Together with its backside, the figure displays one single flexagon with five sectors, the minimumum requirement to lie flat on account of the $36°$ angle.

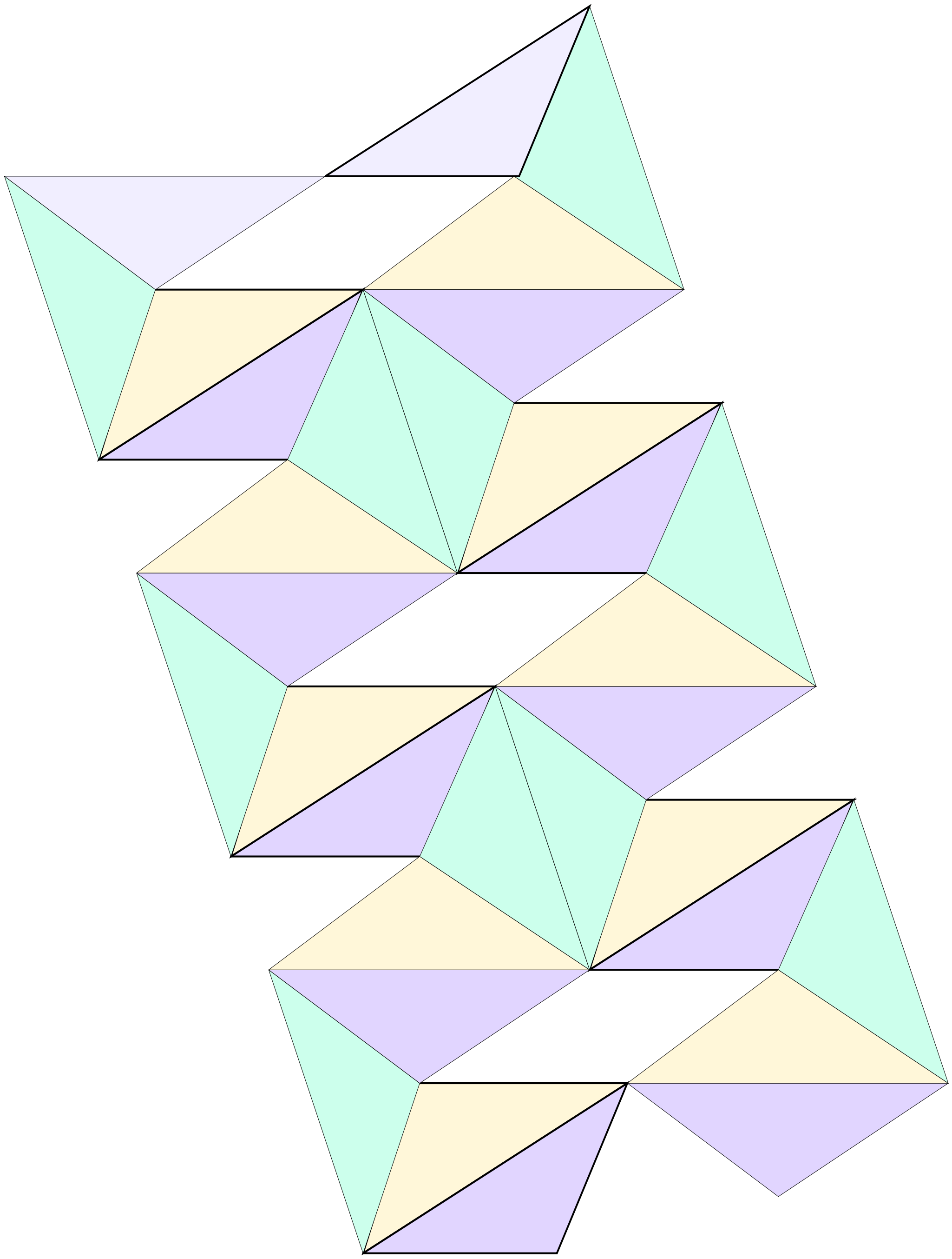

Figure 30: Bottom side of the second level $36° - 36° - 108°$ triangle cutout.

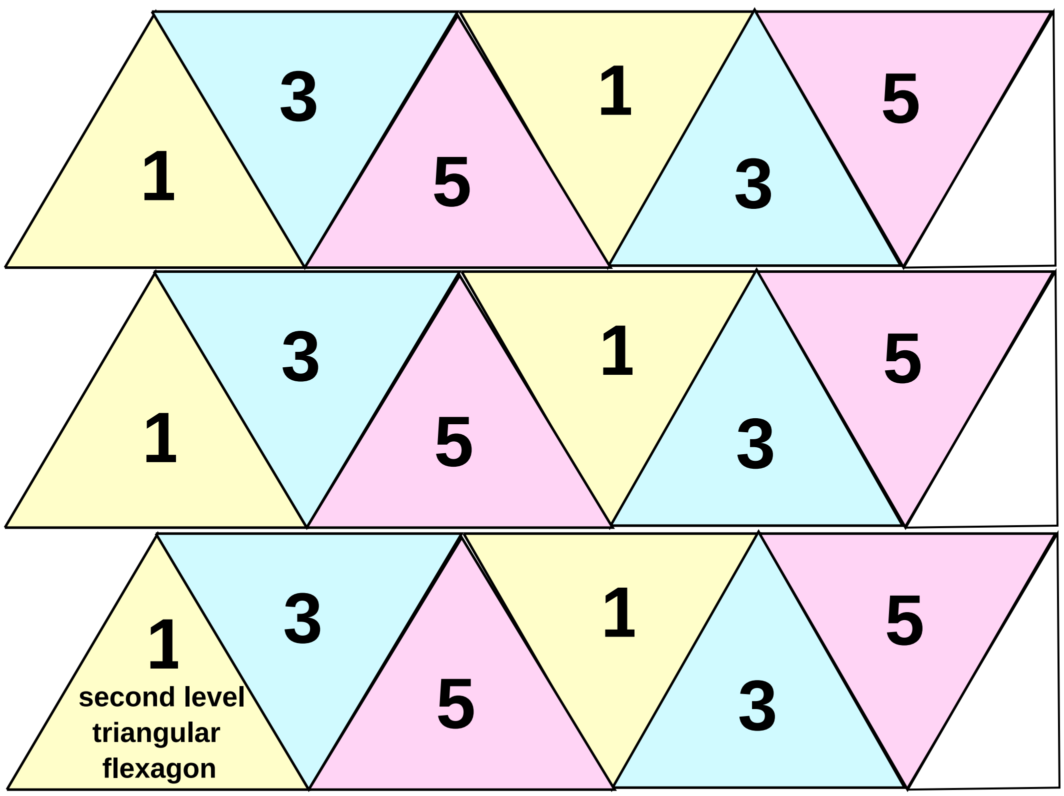

Figure 31: Bottom side of three sectors of the second level normal triflexagon cutout. Together with its backside, the figure makes one single triflexagon because three sectors are needed to make it lie flat.

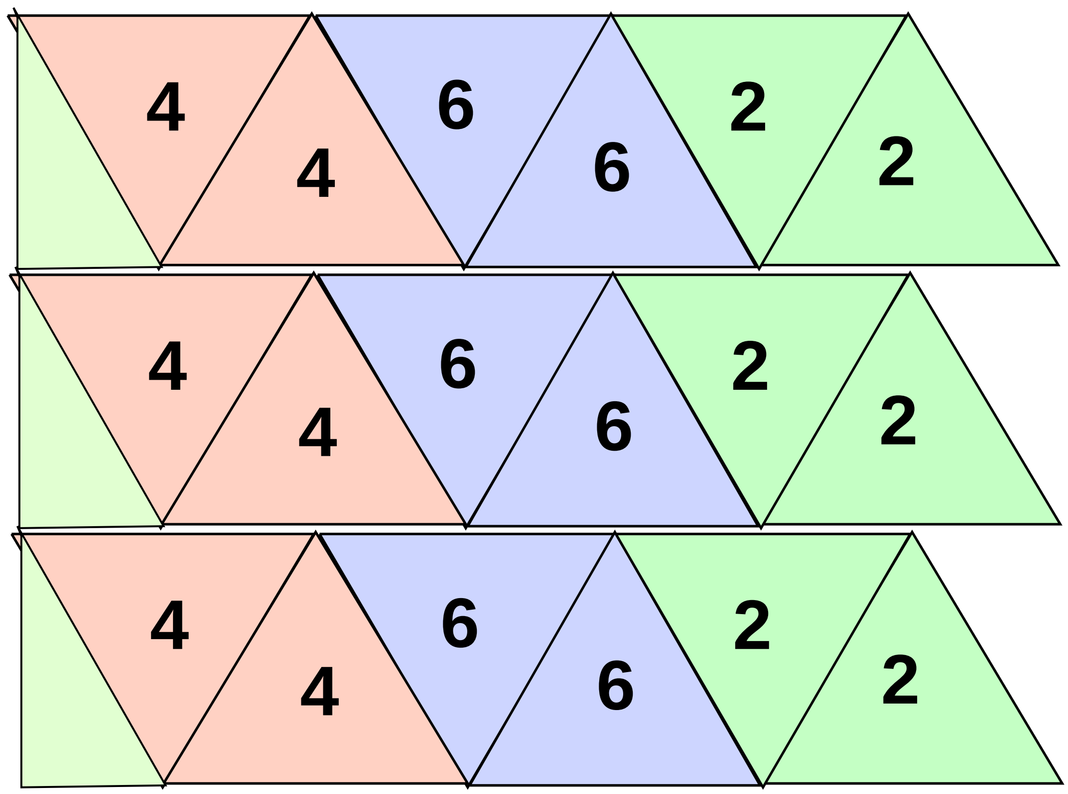

Figure 32: Bottom side of three sectors of the second level normal triflexagon cutout. Together with its backside, the figure makes one single triflexagon because three sectors are needed to make it lie flat.