Optimal design of axially loaded non-prismatic columns via genetic algorithms

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ABSTRACT : We present a method for optimizing the design of axially loaded non-prismatic columns using a search technique based on the mechanics of natural selection, called the genetic algorithm. The design problem was formulated as an optimization problem in which the objective function is to minimize the volume of a column under a given load by changing its shape, subject to both buckling and strength constraints. Both floating point representation and binary representation (with and without Gray coding) were used and compared against a mathematical programming method based on the generalized reduced gradient method. Our results show that the floating point representation scheme provides the best solutions, both in terms of precision and in terms of computing time.

1 INTRODUCTION

The optimization of structural members subject to compression forces has been of great interest for a long time. Leonhard Euler derived first the formula for the critical buckling load of an ideal slender column (Euler 1960b), and was also the first to solve the problem of inextensible elastica (i.e., for constant modulus of elasticity and modulus of inertia, he found the curve of prescribed length with prescribed terminal displacements and slopes and minimum stored energy). Euler solved the case of a column that has the lower end fixed and the upper end free. Later he extended (Euler 1960a) his work on columns, and even today it has a great influence in every strength of materials textbook. In fact there were very few contributions to his work until Lamarle (1845) noticed that Euler's formula should be used only for slenderness ratios over a certain limit, and that the experimental data should be applied only to small ratios. In 1889 the French engineer Considère (1891) performed a set of 32 tests on columns, establishing the so-called theory of the reduced module. During that same year, the German engineer F. Engesser (1889) independently suggested the theory of tangential module. From these two theories, the first one dominated until 1946, when American professor F. R. Shanley indicated the logical paradoxes of both theories. In a remarkable paper of only one page (Shanley 1946) he explained not only what was wrong with both theories, but he also proposed his own theory that solved the paradoxes.

On the other hand, the problem of non-prismatic columns (i.e., those with a variable cross-section area) has been studied more recently. A. N. Dinnik (1932) discussed the design of columns in which the moment of intertia of the cross-section areas varies according to a power of the distance along the member axis. Keller (1960) and later Tadjbakhsh and Keller (1962) derived optimal solutions to the strongest-column problem, which was characterized in the following way: "For a column of given length and volume of material, determine the column shape for which the Euler buckling load is maximum". In their analysis, Tadjbakhsh and Keller established the necessary conditions for a maximum by performing variations on the differential equations of equilibrium and associated boundary conditions, and the constraint of constant volume.

J. Taylor (1967) studied the same problem using an energy approach, and presented a method to calculate a lower bound to the maximum eigenvalue. Spillers and Levy extended Keller's solution for the optimal design of columns to the case of plates (Spillers & Levy 1990) and later, for axisymmetric cylindrical shells (Spillers & Levy 1991). However, in all these works, only the constraint of constant volume was considered and, as Fu and Ren (1992) point out, in a practical design, material strength constraints are equally important. With that in mind, these last two authors added such constraint to the optimization problem and used an algorithm called the generalized reduced gradient method (Reklaitis et al. 1983) to select the design variables at nodal points. The results that they obtained are very reasonable and verifiable. The generalized reduced gradient method linearizes the non-linear constraints of this problem, and uses the convex simplex method to select the best direction of search from all the candidate directions which are both feasible and descent. Then the search for the optimum is started from the feasible initial point. Newton iteration is employed to adjust the basic variable to maintain feasibility. The convergence is accelerated by incorporating conjugate direction or quasi-Newton constructions. Naturally, the existence of continuous differentiability of the problem functions is a fundamental requirement for using the method.

Our work followed this last approach, and applied the genetic algorithm (GA) instead of the generalized reduced gradient method. Some problems had to be faced, though, namely the representation scheme and the parameters to be optimized. However, our results are practically as good as those found by Fu and Ren (1992), and in at least one case we found a better solution than them. Binary and floating point representation schemes were tried, since this problem has a continuous search space. Nevertheless, the search space can be easily discretized, since in real designs there is always a lower and an upper bound on the designs of the column.

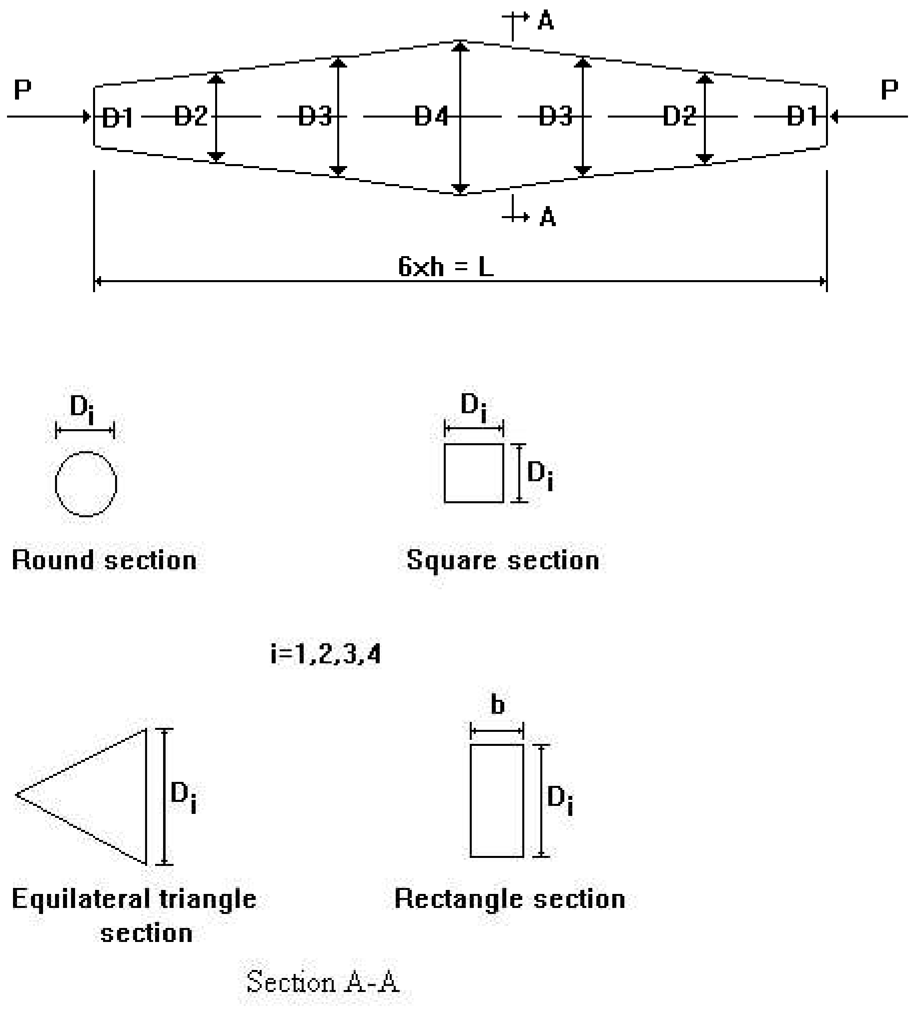

Figure 1 : Column elevation and sections.

2 STATEMENT OF THE PROBLEM

Given a column subject to axial load along the horizontal direction, the governing differential equation is

$$EIy'' + Py = 0 \qquad (2.1)$$

where E is the modulus of elasticity, I is the moment of inertia, P is the axial load, and y is the function that represents the slenderness of the column. Let' s assume that the column that we are going to study is divided into 6 equal-sized segments throughout its length (see Figure 1). Then, Equation (2.1) may be expressed in a finite difference form, as (Fu & Ren 1992):

$$\frac{E}{h^2}\begin{bmatrix} -2I_2 & I_2 & 0 \\ I_3 & -2I_3 & I_3 \\ 0 & 2I_4 & -2I_4 \end{bmatrix}\begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} + P\begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.2}$$

For non-trivial solution, the determinant must vanish, namely

$$\begin{vmatrix} \left(-2+\frac{Ph^2}{EI_2}\right) & 1 & 0 \\ 1 & \left(-2+\frac{Ph^2}{EI_3}\right) & 1 \\ 0 & 2 & \left(-2+\frac{Ph^2}{EI_4}\right) \end{vmatrix} = 0 \tag{2.3}$$

or, in linear form:

$$\frac{P^3h^6}{E^3I_2I_3I_4} - 2\frac{P^2h^4}{E^2}\left(\frac{1}{I_2I_3}+\frac{1}{I_3I_4}+\frac{1}{I_2I_4}\right) + \frac{Ph^2}{E}\left(\frac{2}{I_2}+\frac{4}{I_3}+\frac{3}{I_4}\right) - 2 = 0 \tag{2.4}$$

where, for round and regular polygonal sections, the moments of inertia will be given by

$$I_i = \alpha D_i^4 \tag{2.5}$$

D_i is the diameter for round sections, or the side length for regular polygonal sections. Table 1 shows the values of α for the most commonly used cross-sections.

Table 1. Value of α for the most common sections.

Round section	Square section	Triangular section*
$\frac{\pi}{64}$	$\frac{1}{12}$	$\frac{\sqrt{3}}{96}$

*We assume an equilateral triangle.

In general, for an n-sided regular polygon, α may be derived as

$$\alpha = \frac{n}{192}Cot\frac{\pi}{n}\left(3Cot^2\frac{\pi}{n}+1\right) \tag{2.6}$$

For rectangular sections where width b is assumed to be constant throughout the length of the column,

$$I_i = \frac{bD_i^3}{12}, \quad i = 2,3,4 \tag{2.7}$$

In a column design, Equation (2.4) represents a buckling constraint. Furthermore, a compressive strength constraint must also be satisfied, that is

$$\frac{P}{A_1} \le \sigma_y \tag{2.8}$$

where A_1 is a function of D_1. Since P and σ_y are always given values, we can compute the minimum D_1 or A_1 for each particular problem. This means that we may express the compressive strength constraint only in terms of D_1 or A_1.

Now, we have all the necessary elements to express the column design problem as an optimization problem. If we assume that P, h and σ_y are given, the objective is to minimize the volume of the column. Therefore, we' ll consider 2 cases:

a. Square or round columns: The objective function may be stated as (Fu & Ren 1992)

Minimize:

$$V_s = K(D_1^2 + 2D_2^2 + 2D_3^2 + D_4^2 + D_1D_2 + D_2D_3 + D_3D_4) \tag{2.9}$$

where K is a constant defined according to Table 2, and V_c is the volume of the round or square column.

Table 2. Value of K for the most common sections.

Round section	Square section	Triangular section*
$\frac{\pi l}{36}$	$\frac{l}{9}$	$\frac{l\sqrt{3}}{36}$

*We assume an equilateral triangle.

The objective function is subjected to the equality constraint defined by Equation (2.4), and the following additional inequality constraints:

$$C_l < D_i < C_\mu, \quad i = 1,2,3,4 \tag{2.10}$$

where D_i are the design variables; C_l and C_μ are, respectively, the lower and upper bounds of the design variables.

b. Rectangular columns: The objective function will be (Fu & Ren 1992):

Minimize:

$$V_r = \frac{bl}{9}(D_1 + 2D_2 + 2D_3 + D_4 + \sqrt{D_1D_2} + \sqrt{D_2D_3} + \sqrt{D_3D_4}) \tag{2.11}$$

where V_r is the volume of the rectangular column.

The objective function is subjected to the equality constraint defined by Equation (2.4), and the following similar equation that is derived on the basis of buckling in the orthogonal direction:

$$\frac{P^3h^6}{E^3I_2^{'}I_3^{'}I_4^{'}}-2\frac{P^2h^4}{E^2}\left(\frac{1}{I_2^{'}I_3^{'}}+\frac{1}{I_3^{'}I_4^{'}}+\frac{1}{I_2^{'}I_4^{'}}\right)+\frac{Ph^2}{E}\left(\frac{2}{I_2^{'}}+\frac{4}{I_3^{'}}+\frac{3}{I_4^{'}}\right)-2=0 \tag{2.12}$$

where

$$I_i^{'}=\frac{D_ib^3}{12}, \quad i=1,2,3,4 \tag{2.13}$$

Furthermore, an additional set of inequality constraints have to be satisfied

$$\left.\begin{array}{l} b\times D_1 > A_1 \\ C_l < D_i < C_\mu \\ C_l < b < C_\mu \end{array}\right\} i=1,2,3,4 \tag{2.14}$$

where $A_1=\dfrac{P}{\sigma_y}$.

3 USE OF THE GENETIC ALGORITHM

To solve this problem, we used the Simple Genetic Algorithm (SGA) proposed by Goldberg (1989). An issue in this application is the representation scheme, because we are dealing with real-valued parameters, and therefore it is necessary to use some kind of discretization, so that we can apply a binary representation scheme. We used a linear discretization that uses the upper and lower bounds given by the user to compute the decoded value with a 3 decimal precision.

We also tried to use Gray codes as suggested by Goldberg (1989). The Gray code representation has the property that any two points next to each other in the problem space differ by only one bit (Michalewicz 1992). In other words, an increase of one step in the parameter value corresponds to a change of a single bit in the code. This is a well known technique used to reduce the distance of two points in the problem space, and it is argued to bring some benefit because of their adjacency property, and the small perturbation caused by many single mutations. However, the use of Gray codes didn' t help much in this particular application, as we' ll see in the next section.

Finally, we used a floating point representation, since it is conceptually closest to the problem space (Michalewicz 1992), and allows the easy and efficient implementation of closed and dynamic operators. We' ll see how this last approach provided the best results, both in terms of the precision obtained and in terms of the computation time needed.

The fitness function that we used is illustrated by the following algorithm:

check1 = Error in Equation (2.4)
If $P/(A_1\times\sigma_y)-1.0>0.0$ then *check2* = 1.0
else *check2* = 0.0
$fitness=1.0/(vol\times(check1+check2)+1.0)$

As we can see, if our answer violates the constaint imposed by Equation (2.4) then the fitness function will be penalized by the error produced. On the other hand, if it violates the stress constraint (i.e., $P/A_1\le\sigma_y$), then the penalty is 1.0. For rectangular columns, the constraint is that $b\times D_1>P/\sigma_y$. In this last case, we must also check the orthogonal direction, so that we have three penalty values instead of two. These values are added and the result is multiplied by 1000—we magnify the error—so that we "punish" our result. Note that when there are no violations to any of the constraints the fitness function returns the inverse of the volume.

As we mentioned before, we used an SGA as described by Goldberg (1989), but with some modifications: we used two-point crossover and binary tournament selection. The four diameters that we want to find were represented by consecutive binary or floating point strings of the same length. The halting criteria was through a maximum number of generations. The GA was implemented in Turbo Pascal 7.0 using the technique proposed by Porter (1988) for dynamic memory management. We found experimentally that the following parameters seem to give the best results:

Population size	$\Rightarrow$ 400
Crossover probability	$\Rightarrow$ 0.80
Mutation probability	$\Rightarrow$ 0.01

4 EXAMPLES

The following examples were taken from Fu and Ren (1992):

a. Example 1: Select the best diameters at nodal points for a steel round column of 10' (3.048 m) length which is subjected to an axial load of 400 kips (181.437 Ton). The modulus of elasticity is E=30$\times 10^6$ psi (2109.209 Ton/cm^2) and the yield strength, σ_y is 60,000 psi (4.218 Ton/cm^2). Thus, the minimum diameter may be computed as 2.914" (7.402 cm). The design variables are the diameters at nodal point, D_1, D_2, D_3 and D_4. The lower and upper bounds, C_l and C_μ are 2.914" (7.402 cm) and 20" (50.8 cm), respectively. The size of the search space for this problem is $(20000\text{-}2914)^4 \cong 8.52\times10^{16}$.

b. Example 2: Select the best side-widths at nodal points for a steel square column of 10' (3.048 m) length which is subjected to an axial load of 400 kips (181.437 Ton). The modulus of elasticity E=30×10⁶ psi (2109.209 Ton/cm²) and the yield strength, σ_y is 60,000 psi (4.218 Ton/cm²). Thus, the minimum diameter may be computed as 2.582" (6.558 cm). The design variables are the diameters at nodal point, D_1, D_2, D_3 and D_4. The lower and upper bounds, C_l and C_μ are 2.582" (6.558 cm) and 20" (50.8 cm), respectively. The size of the search space for this problem is $(20000\text{-}2582)^4 \cong 9.20\times10^{16}$.

c. Example 3 : Select the best side-widths at nodal points for a steel equilateral triangle column of 10' (3.048 m) length which is subjected to an axial load of 400 kips (181.437 Ton). The modulus of elasticity $E=30\times10^6$ psi (2109.209 Ton/cm²) and the yield strength, σ_y is 60,000 psi (4.218 Ton/cm²). Thus, the minimum diameter may be computed as 3.924" (9.967 cm). The design variables are the diameters at nodal point, D_1, D_2, D_3 and D_4. The lower and upper bounds, C_l and C_μ are 3.924" (9.967 cm) and 20" (50.8 cm), respectively. The size of the search space for this problem is $(20000\text{-}3924)^4 \cong 6.70\times10^{16}$.

d. Example 4 : Select the best side-widths at nodal points for a steel rectangular column of 10' (3.048 m) length which is subjected to an axial load of 400 kips (181.437 Ton). The modulus of elasticity $E=30\times10^6$ psi (2109.209 Ton/cm²) and the yield strength, σ_y is 60,000 psi (4.218 Ton/cm²). Thus, the minimum diameter may be computed as 1.500" (3.810 cm). The design variables are the diameters at nodal point, D_1, D_2, D_3 and D_4. The lower and upper bounds, C_l and C_μ are 1.500" (3.810 cm) and 20" (50.8 cm), respectively. The size of the search space for this problem is $(20000\text{-}1500)^4 \cong 1.20\times10^{17}$. In this problem we had to use a larger population than in the others (500 chromosomes as compared to the 400 used in the others), and we ran the algorithm for 100 generations, instead of the 50 generations used before. The reason for this change of parameters was the extra length added to our chromosomic strings in this case (we have 15 extra bits in the binary representation and 5 extra digits in the floating point representation) because of the extra parameter needed (the width b of the column).

Table 3 shows the final results of our experiments (notice that we used English units for the tests, but the results are displayed in SI units). We can see how, in general, the floating point representation gave better results, and the binary representation using Gray coding produced the worst. This wasn't only in terms of the volume, but also in terms of the convergence time and the maximum error produced. The differences between our approach and that used by Fu and Ren will look bigger because of the units used, but in cubic inches (units used for the computations) is minimal. Fu and Ren (1992) don't report the times that their approach took, but only mention the number of cycles required. In our case, we can say that in all tests, the computing time didn't exceed one minute, in an IBM PC 486/DX running at 66 MHz. For details, refer to our technical report (Coello & Christiansen 1995).

Table 3. Comparison of final volumes for the four examples.

Method	Volume 1 (cm^3)	Volume 2 (cm^3)	Volume 3 (cm^3)	Volume 4 (cm^3)
Fu & Ren	26914.173	26355.373	24695.360	26513.541
GA (FP)*	26918.434	26363.403	24740.080	19891.596
GA (B)**	26919.302	26367.286	24742.292	26657.469
GA (G)***	27005.777	26897.081	24853.708	28227.829

*Floating point representation.
**Binary representation.
***Binary representation with Gray coding.

5 FUTURE WORK

Currently, we are interested on extending this technique to any kind of columns, including those made of composite materials. This will complicate the analysis a bit more, but it will make the system more useful for real world applications. Also, we are starting to develop applications to deal with structural optimization problems in which we have more than one objective function at a time (multiobjective optimization), both with conflicting objectives and equality and inequality constraints. The GA seems very suitable for multiobjective optimization, and we expect to come up with a new system able to handle such kind of problems in a near future.

6 CONCLUSIONS

We have shown another successful application of the GA to an structural optimization problem. This system is not an isolated entity, but instead it should be seen as a part of the set of GA-based structural optimization tools that we have developed in the last two years. So far, we have been able to successfully design plane and space trusses, rectangular beams and columns, and we'll keep working with the remaining framed structures (i.e., plane grids, plane and space frames). One important lesson learned during the development of this application was the importance of using floating point representation when dealing with a continuous search space. This representation not only generates superior solutions than the binary representation scheme (with or without Gray coding) in terms of the precision obtained, but also in terms of the speed. Because we can use the same genetic operators with only slight modifications, the convergence will be faster since the chromosomes are of shorter length. This problem is an interesting one because, even when its analysis is very simple, it normally has fairly large search spaces and several constraints. The penalty technique that we used has proved to be useful incorporating the constraints into the fitness function for this particular application, as can be seen from our results.

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