A Novel Performance Indicator based on the Linear Assignment Problem

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Abstract. Evaluating the performance of Multi-Objective Evolutionary Algorithms is complex since we have to assess different characteristics of the approximation sets that they generate. Over the years, a variety of performance indicators have been proposed to fulfill this task. One of the most popular performance indicators has been the hypervolume because it can assess both convergence and spread of a set of solutions and it is fully Pareto compliant. However, its computational cost grows exponentially with the number of objectives. A good alternative is the $R2$ indicator which has a similar behavior but a much lower computational cost. Nevertheless, $R2$ sometimes is unable to differentiate two sets with different distributions. In this work, we propose a novel performance indicator based on the linear assignment problem called "ILAP", which offers advantages over $R2$. To illustrate this, we include an example in which the ILAP can differentiate two sets when the $R2$ indicator cannot do it. Furthermore, our experimental analysis shows that our proposed indicator correctly ranks solution sets with different distributions and shapes.

Keywords: Indicator · Linear Assignment Problem · Multi-Objective Optimization

1 Introduction

The solution to a multi-objective optimization problem consists of a set of non-dominated solutions which can not be easily evaluated as in the case of single-objective problems. Therefore, the performance assessment of Multi-Objective Evolutionary Algorithms (MOEAs) is an essential research topic. Over the years, a variety of indicators have been proposed to assess different characteristics of the Pareto Front approximations [12, 1, 7]. One of the most popular indicators has been the hypervolume [12], which measures the space covered by an approximation set given a reference point. This indicator is Pareto compliant and can assess both convergence and spread of the approximations produced by a MOEA. However, its computational cost becomes unaffordable as the number of objectives increases.

Another commonly used performance indicator is $R2$ [1]. This indicator can assess the convergence and diversity of the solutions by using a set of weight vectors and a scalarizing function. Moreover, the behavior of the $R2$ indicator is similar to that of the hypervolume (although $R2$ is weakly Pareto compliant) but has a significantly lower computational cost [1]. Nevertheless, as we will see later on, the $R2$ indicator may obtain the same value for approximation sets with different distributions.

This work introduces a performance indicator based on the linear assignment problem [2]: ILAP. In a linear assignment problem, we have to assign a set of agents to a set of tasks. The assignment of an agent to a task corresponds to a cost. Therefore, the aim is to find an assignment with the lowest cost. In the case of the ILAP, we use a set of weight vectors and a set of individuals, where the assignment cost is computed using a scalarizing function. Thus, we use the cost of the best assignment as the indicator value.

Our experimental results show that ILAP correctly ranks solution sets with different distributions and shapes. Moreover, we present an example in which ILAP distinguishes two approximation sets in a better way than the $R2$ indicator.

The remainder of the paper is organized in the following way. First, we introduce the necessary concepts to understand this paper in Section 2. Then, we present in Section 3 our proposed indicator. After that, in Section 4, we discuss the differences and similarities between our approach and the $R2$ indicator. In Section 5, we evaluate our proposed ILAP. Finally, we present the conclusions and some possible paths for future research in Section 6.

2 Background

2.1 Multi-objective Optimization

A Multi-objective Optimization Problem (MOP) is defined as follows[1]:

$$\begin{aligned} \text{minimize} \quad & F(\boldsymbol{x}) = [f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x})]^T && (1) \\ \text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0 \quad i = 1, \ldots, p, && (2) \\ & h_j(\boldsymbol{x}) = 0 \quad j = 1, \ldots, q && (3) \end{aligned}$$

where $\boldsymbol{x} = [x_1, \ldots, x_n]$ is the vector of decision variables, $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ are the objective functions, and $g_i, h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, p$, $j = 1, \ldots, q$ are the constraints of the problem. We denote Ω as the decision space and $\mathcal{F}$ as the feasible region.

In a MOP, we cannot easily compare the solutions because the objective functions are usually in conflict with each other. Therefore, we use the Pareto dominance relation to define a partial order of the solutions:

Definition 1. *A vector $\boldsymbol{x} \in \Omega$ is said to dominate $\boldsymbol{y} \in \Omega$ (denoted as $\boldsymbol{x} \prec \boldsymbol{y}$), if $f_i(\boldsymbol{x}) \leq f_i(\boldsymbol{y})$ for all $i = 1, \ldots, m$, and $f_j(\boldsymbol{x}) < f_j(\boldsymbol{y})$ in at least one j.*

[1] without loss of generality, we assume minimization problems

Pareto optimality, which is the most commonly used notion of optimality adopted in multi-objective optimization, is formally defined as follows:

Definition 2. *A vector $\boldsymbol{x} \in \mathcal{F}$ is Pareto optimal if there does not exist another vector $\boldsymbol{y} \in \mathcal{F}$ such that $\boldsymbol{y} \prec \boldsymbol{x}$.*

Moreover, we also adopt the following definitions commonly used in multi-objective optimization:

Definition 3. *The Pareto Optimal Set $\mathcal{P}^*$ is defined as:*

$$\mathcal{P}^* := \{\boldsymbol{x} \mid \boldsymbol{x} \textit{ is Pareto optimal}\}$$

Definition 4. *The Pareto Optimal Front $\mathcal{PF}^*$ is defined as:*

$$\mathcal{PF}^* := \{\boldsymbol{F}(\boldsymbol{x}) \in \mathbb{R}^m \mid \boldsymbol{x} \in \mathcal{P}^*\}$$

Definition 5. *Given a predefined weight vector $\boldsymbol{w} \in \mathbb{R}^m$, a scalarizing function s transforms a multi-objective problem into a single-objective problem of the following form:*

$$\textit{minimize} \quad s(f'(\boldsymbol{x}), \boldsymbol{w}) \tag{4}$$

$$\textit{subject to} \quad \boldsymbol{x} \in \mathcal{F}, \tag{5}$$

where $\boldsymbol{x}$ is the decision vector, $\mathcal{F} \in \mathbb{R}^n$ is the feasible region, $\boldsymbol{f} \in \mathbb{R}^m$ is the vector of m objective functions, $\boldsymbol{f}'(\boldsymbol{x}) := \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{z}$, and $\boldsymbol{z} \in \mathbb{R}^m$ is a reference point.

2.2 Linear Assignment Problem

A Linear Assignment Problem (LAP) comprises a set of agents and a set of tasks where assigning an agent to a task involves a cost. Therefore, the aim is to find an assignment that minimizes the overall cost. Formally, a LAP can be formulated as follows.

Definition 6. *Given a set of agents $A = \{a_1, \ldots, a_n\}$, a set of tasks $T = \{t_1, \ldots, t_n\}$, and the cost function $C : A \times T \to \mathbb{R}$. Let $\Phi : A \to T$ the set of all possible bijections between A and T, the linear assignment problem (LAP) can then be stated as*

$$\underset{\phi \in \Phi}{\textit{minimize}} \sum_{a \in A} C(a, \phi(a)) \tag{6}$$

Usually, the cost function can be expressed as a real-valued matrix C with elements $C_{ij} = C(a_i, t_j)$. Moreover, the set Φ of all possible bijections can be viewed as a set of permutation matrices $\mathcal{X}$ where each matrix $x \in \mathcal{X}$ holds

$$x_{ij} = \begin{cases} 1 & \text{if agent } i \text{ is assigned to task } j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, a LAP can be modeled as [2]:

$$\underset{x\in\mathcal{X}}{\text{minimize}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}x_{ij} \tag{7}$$

$$\text{such that} \quad \sum_{j=1}^{n} x_{ij} = 1 \quad (i = 1, 2, \ldots, n), \tag{8}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad (j = 1, 2, \ldots, n), \tag{9}$$

$$x_{ij} \in \{0, 1\} \quad (i, j = 1, 2, \ldots, n). \tag{10}$$

This problem can be solved using the so-called Hungarian algorithm, which has an $O(n^3)$ computational complexity [2].

3 Our Proposed Indicator

Molinet Berenguer and Coello Coello [9] transformed the selection process of a MOEA into a LAP. In this proposal, the authors consider a set of individuals and a set of weight vectors representing different regions of the Pareto front. Moreover, the cost of assigning an individual to a weight vector is computed using a scalarizing function. Therefore, after solving the LAP, the individuals assigned to a weight vector are selected for the next generation. The authors proposed an algorithm called Hungarian Differential Evolution (HDE) that uses the LAP selection scheme. The experimental results show that the HDE is very competitive with respect to state-of-the-art algorithms.

In the case of the LAP selection process, the size of the set of individuals is bigger than the set of weight vectors. Therefore, the Hungarian algorithm finds the subset of individuals that minimizes the overall assignment cost and discards the subsets with the worst values. Hence, we can deduce that the minimum overall assignment cost gives us an estimation of how good or bad a set is. Using this idea, we propose an indicator based on the LAP. The ILAP indicator is defined in the following.

Definition 7. *Given a set of uniformly distributed weight vectors* $W = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}$, *an approximation set* $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$, *and a cost matrix* C *such that* $C_{ij} = s(\boldsymbol{w}_i, \boldsymbol{a}_j)$ *where* s *is a scalarizing function. Then, the ILAP is defined as:*

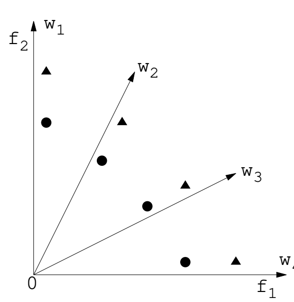
$$I_{LAP} = \frac{1}{n} \min_{x\in\mathcal{X}} \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}x_{ij} \right\} \tag{11}$$

where $\mathcal{X}$ *is the set of permutation matrices.*

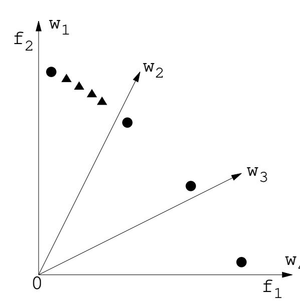
We compute the ILAP by obtaining a cost matrix C using s, A, and W. Then, we solve the LAP defined by C employing the Hungarian algorithm. Finally, the indicator value is the best assignment's cost divided by n. The cost matrix computation is performed in $O(mn^2)$, where m is the number of objectives. Moreover, the LAP problem is solved in $O(n^3)$. Therefore the computational complexity of computing the ILAP is $O(mn^2 + n^3)$.

In the ILAP, each weight vector must be assigned to a currently unassigned solution while minimizing the cost. In the ideal case, each weight vector is assigned to a solution where it obtains its lowest cost. However, let's assume that more than one weight vector obtains its lowest cost with the same solution. In that case, the indicator will assign the solution to the vector with the lowest value and will use the second-best solutions for the remaining vectors.

This process allows the ILAP to assess convergence and diversity at the same time. On the one hand, it measures convergence by always considering the best values of the scalarizing functions. On the other hand, it measures diversity because it tries to quantify how much the solutions cover the regions of the weight vectors. Examples of these two cases are shown in Fig. 1a and Fig. 1b, where the ILAP successfully ranks the sets. We used the Achievement Scalarizing Function (ASF) [10] for these examples and for the rest of the paper.



(a) Measuring convergence. $I_{LAP} = 25000.3375$ for circles' set, and $I_{LAP} = 25000.4375$ for triangles' set.



(b) Measuring diversity. $I_{LAP} = 25000.4375$ for circles' set, and $I_{LAP} = 197500.675$ for triangles' set.

Fig. 1: Examples where the ILAP assesses both convergence and diversity. A lower value is preferred; therefore, the ILAP ranks the sets correctly in both cases.

4 Comparison between Our Approach and the *R*2-indicator

The $R2$ indicator is a performance indicator that assesses the convergence and the diversity of a solution set. It assesses performance by mapping the candidate solutions from objective space into utility space. Given a reference set A, a set of reference vectors V, and a scalarizing function[2] s. The $R2$ indicator is defined as follows [1]:

$$R2(A, V) = \frac{1}{|V|} \sum_{\boldsymbol{v} \in V} \min_{\boldsymbol{a} \in A} \{s(\boldsymbol{a}, \boldsymbol{v})\}$$

The ILAP and $R2$ indicators have some similarities. Both use scalarizing functions and weight vectors to assess the performance of an approximation set. Moreover, the indicators will obtain the same value when the regions given by the weight vectors are equally covered (as shown in Fig. 2).

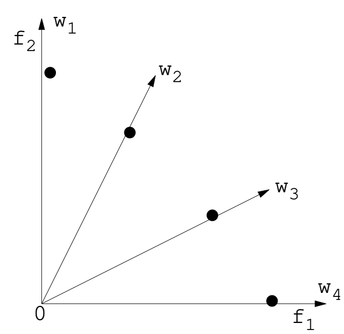


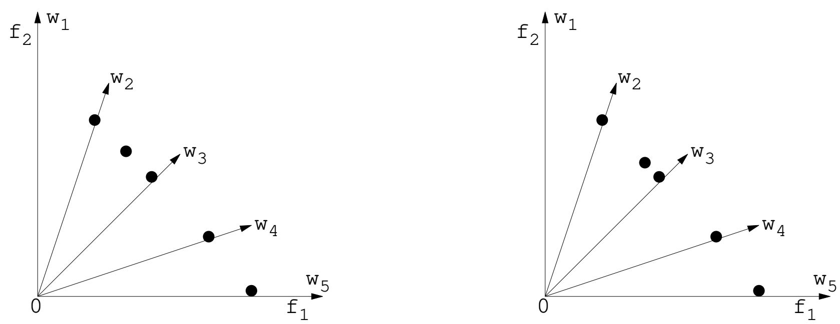
Fig. 2: Example of a case where ILAP and $R2$ obtain the same values: $I_{LAP} = R2 = 10000.3875$

However, the $R2$ indicator only considers the solutions with the best values of the scalarizing function, discarding the information provided by the solutions with the worst values. Therefore, the $R2$ indicator may not evaluate the performance of the whole set and may obtain the same value for two different approximations. On the other hand, the ILAP indicator considers the whole set since it assigns each weight vector with a different solution and obtains the indicator's value from this assignment.

An example of the previous situation is shown in Fig. 3a and Fig. 3b. Given two different approximation sets, the $R2$ indicator obtains the same value, while

[2] also known as utility function

the ILAP obtains distinct values. Furthermore, the ILAP prefers the approximation set with a solution nearer an uncovered vector.



(a) R2=44000.50133, I_{LAP}=44000.584 (b) R2=44000.50133, I_{LAP}=44000.616

Fig. 3: The $R2$ indicator obtains the same value for two sets with distinct distributions, while the ILAP indicator obtains different values.

5 Experimental Analysis

5.1 Evaluation in Artificial Many-Objective Pareto Fronts

In this section, we study the performance of the ILAP in artificial Pareto fronts. We employed three types of solutions sets generated in a unit m-simplex:

- C1. The solutions are concentrated in one corner of the simplex.
- C2. The solutions are randomly generated.
- C3. The solutions are uniformly distributed. We employ the method proposed in [5] for this type of set.

Moreover, the set size for each dimension is shown in Table 1, and Fig. 4a to Fig. 5l show the parallel coordinates graphs of the sets. Regarding the ILAP, we use the ASF, and the Uniform Design with the Hammersley method (UDH) [9] for generating the weight vectors.

m	3	4	5	6	7	8	9	10
Set size	100	110	120	130	140	150	160	170

Table 1: Set size for each dimension

The results are shown in Tables 2a and 2b. Moreover, we include the results of the hypervolume indicator (HV) [12] as a reference. We can observe that the ILAP consistently ranks the C3 sets in first place, the C2 sets in second place, and the C1 sets in last place. Furthermore, HV obtains the same ranking. Therefore, the ILAP can correctly rank a set of solutions in 3 to 10 dimensions.

m	C1	C2	C3
3	5.0453	1.5322	1.1263
4	5.6031	1.8407	1.3157
5	5.9506	2.3113	1.5775
6	5.8495	2.6022	1.9827
7	6.0938	2.9235	2.1356
8	5.9524	3.0488	2.4045
9	5.8106	3.2949	2.9855
10	5.5003	3.4681	3.1862

(a) ILAP

m	C1	C2	C3
3	0.77462	1.076862	1.11977
4	0.906105	1.32058	1.369026
5	1.036677	1.49916	1.560266
6	1.197589	1.659354	1.737507
7	1.284942	1.862622	1.920832
8	1.400768	2.057528	2.111709
9	1.586619	2.252491	2.328822
10	1.805137	2.501373	2.563038

(b) HV

Table 2: ILAP and HV values of the sets C1, C2, and C3 for each dimension m. Darker cells imply better values.

5.2 Evaluation in Pareto Front Approximations

In this section, we use the ILAP, the hypervolume, and the $R2$ indicator to evaluate the performance of two well-known MOEAs: the NSGA-II [3] and the MOEA/D [11]. For this purpose, we ran each algorithm 30 times using different problems. We adopted the DTLZ1, DTLZ2, and DTLZ7 problems from the Deb-Thiele-Laumanns-Zitzler (DTLZ) [4] test suite, the DTLZ1^{-1} from the Minus-DTLZ test problems [8], and the WFG1-WFG3 from the Walking-Fish-Group (WFG) [6] test suite with $m = 3$, 5, 8, and 10 objectives. Regarding the DTLZ problems, we set the number of decision variables to $n = m + k - 1$, where $k = 5$ for DTLZ1, $k = 10$ for DTLZ2, and $k = 20$ for DTLZ7. In the case of the WFG problems, we set the position-related parameters to $2 \times (m - 1)$ and the distance-related parameters to 20. Finally, we use the same configuration of DTLZ1 for DTLZ1^{-1}.

In the case of the algorithm's parameters, we set the population sizes to 100 for three objectives, 120 for five, 140 for eight, and 160 for ten. We set the crossover and mutation parameters to $pc = 1.0$, $pm = 1/\text{number of variables}$, $n_c = 20$, and $n_m = 20$. Regarding the MOEA/D parameters, we used a neighborhood size $T = 20$, the ASF function, and the UDH weight vectors. Finally, the ILAP and the $R2$ indicators adopted the ASF function and UDH weight vectors.

Tables 3a, 3b, and 3c display the average and the standard deviation of each indicator. We can observe that the three indicators obtain the same results for

DTLZ1, DTLZ2, DTLZ7, WFG1, WFG2, and DTLZ1^{-1}. In the case of the WFG3 problem, the $R2$ and the hypervolume get the same rank in 5 and 8 objectives. In contrast, the ILAP and the hypervolume get the same rank in 3 and 5 objectives. This situation could happen because the WFG3 is a linear problem that hardly fits the shape of a simplex. Therefore, the R2 and ILAP indicators may have some trouble with the performance assessment because they employ reference vectors sampled in a simplex.

6 Conclusions and Future Work

We proposed a novel performance indicator based on the Linear Assignment Problem, called ILAP. The experimental results showed that our proposed ILAP could successfully rank the solutions sets using different distributions and Pareto Front shapes of many-objective problems. Moreover, we described an example where the $R2$ indicator (the performance indicator with the most significant similarity with the ILAP) can not distinguish between two different approximation sets. And our proposed ILAP can differentiate them and prefers the one with a solution near an uncovered region. As part of our future work, we would like to analyze the mathematical properties of our proposed indicator.

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Table 3: Average and standard deviation of the hypervolume, $R2$, and ILAP indicators. Gray cells imply better values. Moreover, the symbol "*" represents that the algorithm is statistically better according to the Wilcoxon rank sum test.

(a) HV

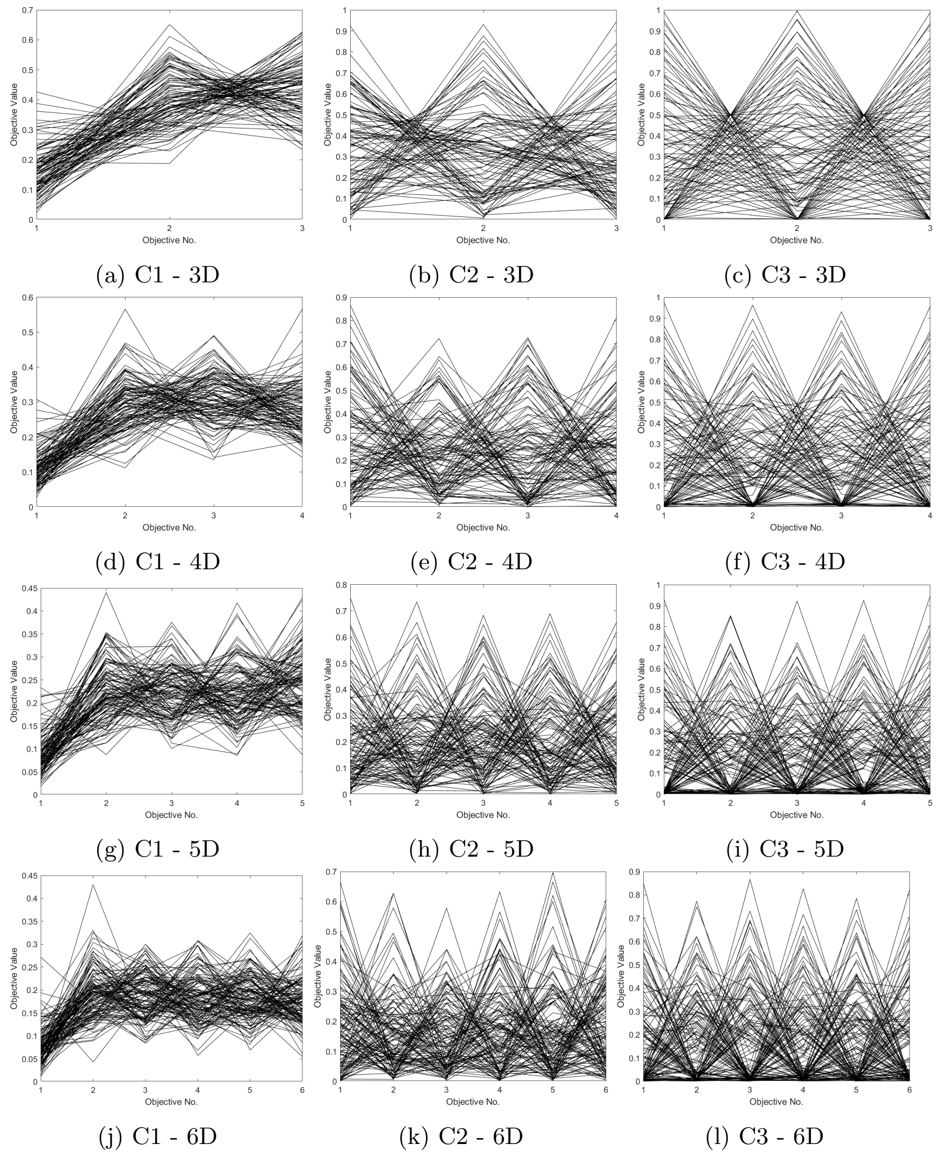
	M	MOEAD	NSGA-II
DTLZ1	3	1.324e+0 (8.5e-4)	*1.331e+0 (3.1e-6)
	5	*1.610e+0 (7.5e-6)	1.610e+0 (1.4e-4)
	8	*2.144e+0 (6.2e-6)	2.143e+0 (4.4e-4)
	10	*2.594e+0 (8.3e-6)	2.593e+0 (1.4e-4)
DTLZ2	3	*8.330e-1 (9.3e-4)	8.169e-1 (5.1e-3)
	5	*1.593e+0 (2.1e-3)	1.584e+0 (5.5e-3)
	8	*2.139e+0 (1.2e-3)	2.002e+0 (4.2e-2)
	10	*2.589e+0 (1.2e-3)	2.454e+0 (3.9e-2)
DTLZ7	3	6.436e-1 (5.e-2)	*6.908e-1 (2.6e-2)
	5	6.136e-1 (5.6e-2)	*8.222e-1 (1.9e-2)
	8	2.e-1 (1.2e-1)	*7.608e-1 (9.4e-2)
	10	1.102e-1 (9.4e-2)	*5.378e-1 (1.3e-1)
WFG1	3	*1.197e+0 (2.7e-2)	1.108e+0 (2.5e-2)
	5	*1.526e+0 (3.6e-2)	1.273e+0 (2.9e-2)
	8	*2.055e+0 (5.0e-2)	1.495e+0 (3.6e-2)
	10	*2.465e+0 (3.5e-2)	1.378e+0 (3.8e-2)
WFG2	3	1.072e+0 (8.e-2)	*1.155e+0 (8.2e-2)
	5	1.328e+0 (1.1e-1)	*1.58e+0 (6.9e-3)
	8	1.711e+0 (1.4e-1)	*2.129e+0 (6.3e-3)
	10	2.074e+0 (2.1e-1)	*2.576e+0 (7.8e-3)
WFG3	3	8.132e-1 (7.8e-3)	8.162e-1 (4.e-3)
	5	1.140e+0 (1.4e-2)	1.134e+0 (1.4e-2)
	8	1.472e+0 (2.9e-2)	*1.501e+0 (2.7e-2)
	10	1.518e+0 (5.1e-2)	*1.798e+0 (3.5e-2)
DTLZ1^{-1}	3	*2.775e-1 (2.9e-5)	2.733e-1 (2.2e-3)
	5	1.224e-2 (9.e-5)	1.215e-2 (5.3e-4)
	8	*3.472e-5 (1.3e-6)	3.167e-5 (2.3e-6)
	10	4.988e-7 (3.8e-8)	4.924e-7 (3.6e-8)

(b) R2

	M	MOEAD	NSGA-II
DTLZ1	3	7.692e-2 (7.e-3)	*2.856e-2 (6.e-4)
	5	*1.155e-3 (3.1e-5)	2.388e-1 (1.2e-1)
	8	*1.318e-3 (5.5e-5)	8.6e-1 (3.1e-1)
	10	*1.574e-3 (9.5e-5)	1.165e+0 (2.6e-1)
DTLZ2	3	*1.360e+0 (2.5e-4)	1.446e+0 (3.8e-2)
	5	*8.176e-1 (5.2e-3)	1.196e+0 (5.9e-2)
	8	*6.764e-1 (2.4e-2)	2.832e+0 (2.4e-1)
	10	*8.836e-1 (1.0e-1)	3.216e+0 (2.0e-1)
DTLZ7	3	3.517e+0 (1.1e+0)	*3.074e+0 (5.8e-1)
	5	7.344e+0 (9.4e-1)	*6.085e+0 (1.7e-1)
	8	1.702e+1 (2.2e+0)	*1.098e+1 (4.8e-1)
	10	2.339e+1 (3.5e+0)	*1.487e+1 (9.1e-1)
WFG1	3	*1.076e+0 (1.6e-1)	1.493e+0 (2.7e-1)
	5	*1.375e+0 (2.2e-1)	2.922e+0 (2.6e-1)
	8	*1.412e+0 (2.3e-1)	4.859e+0 (2.3e-1)
	10	*1.797e+0 (2.1e-1)	8.452e+0 (2.7e-1)
WFG2	3	2.126e+0 (7.0e-1)	*1.488e+0 (7.6e-1)
	5	3.305e+0 (1.1e+0)	*1.174e+0 (5.4e-2)
	8	5.121e+0 (1.5e+0)	*1.425e+0 (5.4e-2)
	10	5.187e+0 (2.3e+0)	*1.533e+0 (6.7e-2)
WFG3	3	*2.667e+0 (2.5e-2)	2.673e+0 (2.0e-2)
	5	*3.724e+0 (6.2e-2)	3.796e+0 (6.2e-2)
	8	*5.617e+0 (8.1e-2)	5.711e+0 (7.7e-2)
	10	6.985e+0 (2.2e-1)	*6.594e+0 (1.1e-1)
DTLZ1^{-1}	3	*5.014e+0 (2.0e-4)	5.46e+0 (6.4e-2)
	5	*1.349e+1 (3.2e-2)	1.581e+1 (1.4e-1)
	8	*3.073e+1 (8.9e-2)	3.825e+1 (3.5e-1)
	10	*4.034e+1 (1.2e-1)	5.187e+1 (4.4e-1)

(c) ILAP

	M	MOEAD	NSGA-II
DTLZ1	3	6.127e-1 (1.0e-3)	*5.061e-1 (5.9e-2)
	5	*1.155e-3 (3.0e-5)	9.361e-1 (1.0e-1)
	8	*1.318e-3 (5.6e-5)	1.800e+0 (1.1e-1)
	10	*1.577e-3 (9.8e-5)	2.105e+0 (6.0e-2)
DTLZ2	3	*1.363e+0 (3.4e-4)	1.958e+0 (7.2e-2)
	5	*8.181e-1 (5.2e-3)	1.478e+0 (6.5e-2)
	8	*6.765e-1 (2.5e-2)	4.292e+0 (2.2e-1)
	10	*8.851e-1 (1.1e-1)	5.212e+0 (1.8e-1)
DTLZ7	3	4.797e+0 (1.1e+0)	*3.298e+0 (6.3e-1)
	5	1.257e+1 (1.2e+0)	*6.904e+0 (1.7e-1)
	8	2.834e+1 (3.0e+0)	*1.333e+1 (4.4e-1)
	10	3.711e+1 (6.0e+0)	*1.825e+1 (6.9e-1)
WFG1	3	*1.359e+0 (1.7e-1)	1.721e+0 (3.8e-1)
	5	*1.62e+0 (3.1e-1)	3.338e+0 (3.3e-1)
	8	*1.727e+0 (2.9e-1)	5.834e+0 (4.5e-1)
	10	*2.213e+0 (2.8e-1)	1.021e+1 (4.6e-1)
WFG2	3	2.377e+0 (6.3e-1)	*1.75e+0 (8.5e-1)
	5	3.568e+0 (1.1e+0)	*1.954e+0 (2.3e-1)
	8	5.302e+0 (1.5e+0)	*2.953e+0 (2.5e-1)
	10	5.314e+0 (2.2e+0)	*3.271e+0 (3.0e-1)
WFG3	3	3.276e+0 (3.7e-2)	*2.929e+0 (5.6e-2)
	5	*4.172e+0 (1.2e-1)	4.804e+0 (1.3e-1)
	8	*6.833e+0 (1.6e-1)	7.874e+0 (2.3e-1)
	10	*9.165e+0 (2.5e-1)	9.488e+0 (2.6e-1)
DTLZ1^{-1}	3	*5.014e+0 (2.0e-4)	5.627e+0 (6.7e-2)
	5	*1.349e+1 (3.2e-2)	1.637e+1 (1.9e-1)
	8	*3.076e+1 (7.4e-2)	3.915e+1 (3.e-1)
	10	*4.037e+1 (1.2e-1)	5.325e+1 (3.7e-1)



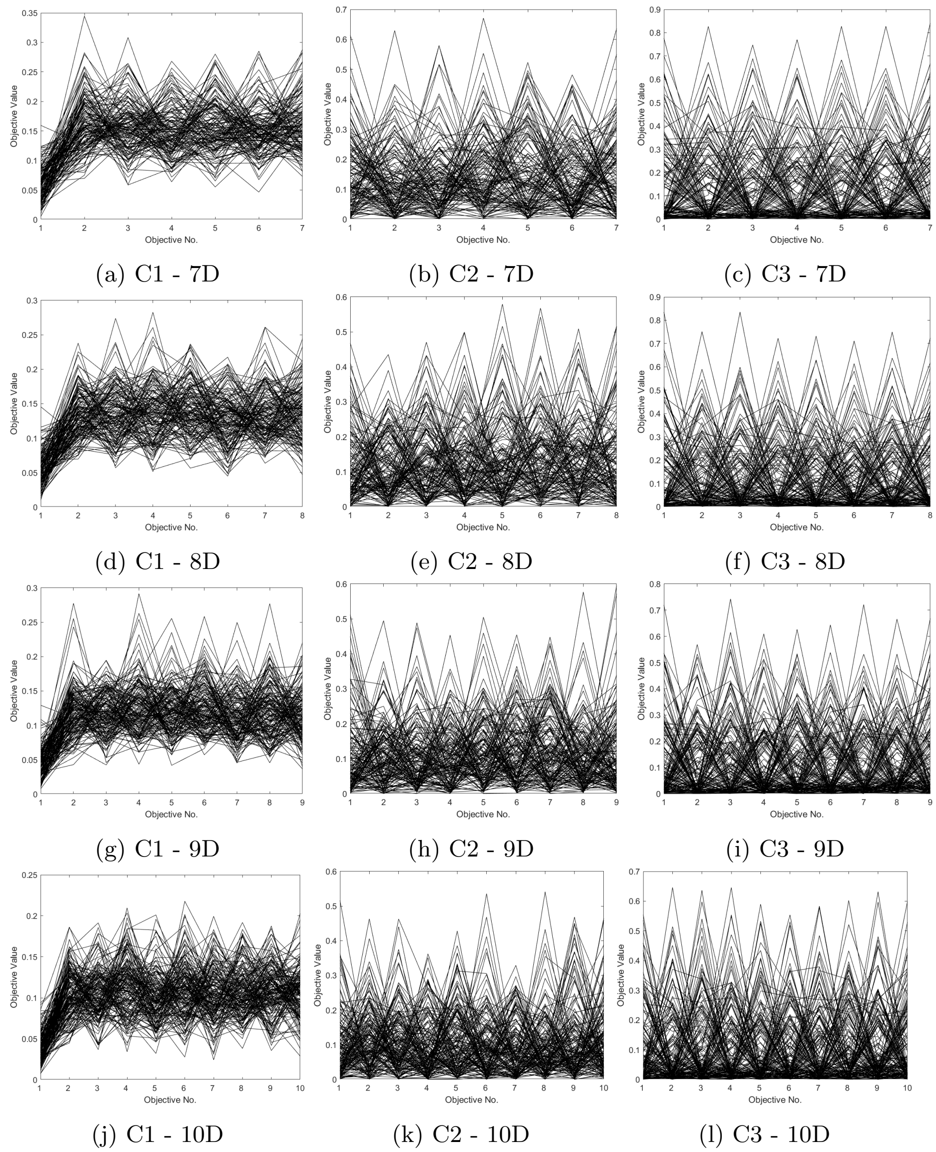
(a) C1 - 3D (b) C2 - 3D (c) C3 - 3D

(d) C1 - 4D (e) C2 - 4D (f) C3 - 4D

(g) C1 - 5D (h) C2 - 5D (i) C3 - 5D

(j) C1 - 6D (k) C2 - 6D (l) C3 - 6D

Fig. 4: Artificial solution sets generated in a unit simplex. Solutions in C1 are concentrated in a corner, in C2 are randomly generated, and in C3 are uniformly distributed.



(a) C1 - 7D (b) C2 - 7D (c) C3 - 7D

(d) C1 - 8D (e) C2 - 8D (f) C3 - 8D

(g) C1 - 9D (h) C2 - 9D (i) C3 - 9D

(j) C1 - 10D (k) C2 - 10D (l) C3 - 10D

Fig. 5: Artificial solution sets generated in a unit simplex (continuation)

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