The $\widehat{g}$-dominance Relation for Preference-Based Evolutionary Multi-Objective Optimization

Wenjian Luo*, Luming Shi*, Xin Lin*, Carlos A. Coello Coello†
*Anhui Province Key Laboratory of Software Engineering in Computing and Communication,
School of Computer Science and Technology, University of Science and Technology of China, Hefei 230027, Anhui, China
Email: wjluo@ustc.edu.cn, {shi52535, iskcal}@mail.ustc.edu.cn
†Departmento de Sistemas, UAM-Azcapotzalco, Mexico City, MEXICO
Email: ccoello@cs.cinvestav.mx

***Abstract*—In evolutionary multi-objective optimization, the results generated by an evolutionary algorithm usually contain an approximation, as good as possible, of the entire Pareto-optimal front. However, sometimes the number of Pareto-optimal solutions may be so large that the decision maker (DM) is incapable of manipulating or understanding them. Methods for considering only the Pareto-optimal solutions that the DM prefers indeed constitute a hot research topic in the evolutionary computation field. In this paper, we introduce a new dominance relation called $\widehat{g}$-dominance, which is an improved version of the g-dominance relation and can be easily implemented in traditional multi-objective evolutionary algorithms. In this work, the proposed $\widehat{g}$-dominance is implemented in NSGA-II. Our experimental results show the effectiveness of $\widehat{g}$-NSGA-II with respect to the original g-NSGA-II.**

***Index Terms*—multi-objective optimization, evolutionary computation, preference, g-dominance**

I. Introduction

Multi-objective optimization has been studied in depth for several years, as many real-world problems involve optimizing several (often conflicting) objectives. These are the so-called multi-objective optimization problems (MOPs). When solving a MOP, the goal is to generate solutions that represent the best possible trade-offs among the objectives. Such solutions constitute the so-called *Pareto optimal set*. During the past three decades, many evolutionary algorithms have been proposed, including the Nondominated Sorting Genetic Algorithm-II (NSGA-II) [1], the Strength Pareto Evolutionary Algorithm 2 (SPEA2) [2], and the Multi-Objective Evolutionary Algorithm based on Decomposition (MOEA/D) [3].

Since decision makers (DMs) normally have a preference when selecting the final solution, considerable attention has been paid to evolutionary algorithms (EAs) for preference-based multi-objective optimization in recent years. Specifically, in order to handle the DMs' preferences, many techniques have been proposed and have been embedded in a variety of evolutionary algorithms to solve MOPs and search only the region of interest (ROI) that the DM prefers.

Reference point techniques are one of the most commonly used methods to represent the DM's preferences. A novel concept called g-dominance was proposed in [4] to define the preferred Pareto-optimal solutions based on a reference point. An additional novel concept called r-dominance, was proposed in [5]. Before and after the appearance of these two concepts, many techniques were proposed. For example, in [6], NSGA-II was modified to search the partial Pareto-optimal solutions that are close to the reference point. Based on the use of a weighted achievement scalarizing function, an evolutionary algorithm for solving MOPs according to the DM's preference was proposed in [7]. The preference information of the reference point was decomposed into a number of scalar optimization subproblems in the study presented in [8], where the algorithmic framework was based on MOEA/D. A technique involving reference points consisting of the desired aspiration levels for the objective functions was examined in [9]. A hybrid multi-objective immune algorithm, where the number of reference points is flexible, was proposed in [10]. Additionally, the NSGA-III was proposed to deal many-objective problems (i.e., MOPs having more than 3 objectives) as well as to handle DMs' preferences using reference points [11].

Reference direction techniques are an additional approach for expressing the DM's preferences. For example, based on NSGA-II, a novel reference direction-based NSGA-II called RDNSGA-II, was proposed in [12]. In [13], a modified SPEA was proposed, in which the reference direction is used to guide the search for multi-objective and many-objective optimization.

In addition to the reference point and the reference direction techniques, other approaches have also been used to express the DM's preference in MOEAs. For example, the preferred sub-problems selected by the DMs were used to guide the search process of MOEA/D in [14]. Based on the light beam search presented in [15], a modified NSGA-II designed to search the Pareto-optimal solutions preferred by the DM was proposed in [16]. The Gaussian functions on a hyperplane was used to represent the DM's preferences in [17].

From among the above techniques, g-dominance [4] is a highly versatile, and easy to implement dominance relation. However, when the reference point is close to (or on) the Pareto-optimal front (PF), the performance of g-dominance relation is not ideal. In particular, when the PF is discontinu-

This study is partially supported by the National Natural Science Foundation of China (No. 61573327). The fourth author is on sabbatical leave from CINVESTAV-IPN. He gratefully acknowledges support from CONACyT grant no. 2016-01-1920 (*Investigación en Fronteras de la Ciencia 2016*).

ous, there can be no Pareto-optimal solutions in the region that is selected as the preferred one by the g-dominance relation.

In this paper, the idea of bilevel optimization is used to describe a MOP with preferences, and an improved version of g-dominance called $\widehat{g}$-dominance is proposed. $\widehat{g}$-dominance is implemented in NSGA-II, and the corresponding algorithm is called $\widehat{g}$-NSGA-II. Our experimental results show that $\widehat{g}$-NSGA-II has obvious advantages over g-NSGA-II in terms of diversity and convergence, where g-NSGA-II is the NSGA-II using g-dominance instead of $\widehat{g}$-dominance.

The remainder of this paper is organized as follows. Section II describes previous related work, including the basic concepts of multi-objective optimization, the g-dominance relation, and NSGA-II. Section III is devoted to describe our proposed $\widehat{g}$-dominance relation and its implementation. Our experimental design and the results obtained are shown in Section IV. Finally, Section V summarizes this paper.

II. Related Work

In this section, first the definition of a MOP is provided, and then, the g-dominance relation is introduced. Finally, the well-known NSGA-II is introduced, since it is adopted as our baseline algorithm in this study.

A. *Definition of Multi-objective Optimization Problem*

A MOP is defined as follows (assuming minimization of all the objectives) [18]:

$$Minimize\ F(x) = (f_1(x), \cdots, f_M(x))^T,$$
$$Subject\ to \begin{cases} h_i(x) = 0, & i = 1, \cdots, n_p \\ g_j(x) \le 0, & j = 1, \cdots, n_q \\ \quad x \in \Omega \end{cases}, \quad (1)$$

where $x = (x_1, \cdots, x_n)$ is the decision variable vector, n is the number of decision variables, $\Omega \in R^n$ is the decision space, $f_k(x) : R^n \to R$ $(k = 1, \cdots, M)$ represents the objective functions, M is the number of objectives, h_i and g_j are the equality and inequality constraints of the problem, respectively, and n_p and n_q are the corresponding number of equality and inequality constraints.

On the basis of the above definition, we explain some terms related to MOPs as follows [1].

First, the **Pareto dominance** relationship is defined as follows. For two given solutions $x, y \in \Omega$, x Pareto dominates y if and only if $f_i(x) \le f_i(y)$ for each $i \in \{1, \cdots, M\}$ and there exists at least one $j \in \{1, \cdots, M\}$, $f_j(x) < f_j(y)$.

The second term is **Pareto-optimal solution**. That is, for a given solution $x \in \Omega$, if x is a Pareto-optimal solution, then there is no other solution $y \in \Omega$ that dominates x.

Finally, the **Pareto-optimal set** (**PS**) is defined as the set of all the Pareto-optimal solutions, i.e., $PS = \{x \in \Omega | x\ is\ Pareto\ optimal\}$, while the **PF** is defined as $PF = \{f(x) \in R^M | x \in PS\}$.

B. *g-dominance*

The g-dominance relation proposed by *Molina et al.* [4] is a new dominance relation that combines the DM's preferences with the Pareto dominance principle. This relation divides the objective space into two parts according to a reference point. One part is the region on which the DM does not concentrate and the second is the DM's preference.

The approach in which the g-dominance relation divides the objective space is described as follows. Given reference point g, for any point w in the objective space, the association of point w depends on $Flag_g(w)$:

$$Flag_g(w) = \begin{cases} 1, & if\ w_i \le g_i, \forall i = 1, \ldots, M \\ 1, & if\ w_i \ge g_i, \forall i = 1, \ldots, M \\ 0, & otherwise \end{cases}, \quad (2)$$

where M is the dimension of the objective space.

In objective space, point w with $Flag_g(w) = 1$ satisfies the DM's preferencea. The g-dominance relation is defined as follows.

Given a reference point g, a solution x g-dominates another solution y when one of the following conditions is satisfied:

(1) $Flag_g(x) > Flag_g(y)$;
(2) $Flag_g(x) = Flag_g(y)$, x Pareto dominates y.

The convenience of the g-dominance relation is such that it can be easily implemented in a variety of MOEAs. MOEAs with the g-dominance relation will converge to solutions in the ROI because of the two following reasons:

(1) When the reference point is feasible, the ROI is formed by the solutions on the PF that dominate the reference point.
(2) When the reference point is infeasible, the ROI consists of the solutions on the PF that are dominated by the reference point.

C. *NSGA-II*

NSGA-II, which was proposed by *Deb et al.* [1], is one of the most popular multi-objective evolutionary algorithms (MOEAs) currently available. The pseudocode for NSGA-II is presented in Algorithm 1.

In Algorithm 1, the function $NonDominatedSort(R(t))$ divides the immediate population $R(t) = P(t) \cup Q(t)$ into multiple levels. The first level, F_1, includes all the non-dominated individuals in $R(t)$; the second level, F_2, includes all non-dominated individuals in $R(t) - F_1$, and so on. The individuals in F_i are better than those in F_j when $i < j$.

The function $CrowdingDistance(F_i)$ is used to calculate the crowding distance for all individuals in F_i. An individual with a larger crowding distance is considered to be better. The function $NewPopulation(P(t+1))$ is utilized to generate the next population using recombination and mutation.

In this study, NSGA-II was employed as the basic framework for the $\widehat{g}$-dominance relationship. We refer the reader to [1] for the detailed formula for calculating the crowding distance, as well as for the details of the recombination and mutation operators.

Algorithm 1 NSGA-II

Input: N (the size of population)
Output: approximated PF and PS

$t = 1$;
Create an initial population $P(t)$ randomly;
$Q(t) = \emptyset$;
while termination condition is not satisfied **do**
  F =NonDominatedSort($P(t) \cup Q(t)$);
  $P(t+1) = \emptyset$;
  $i = 1$;
  while $|P(t+1)| + |F_i| \leq N$ **do**
    $P(t+1) = P(t+1) \cup F_i$;
    $i = i + 1$;
  end while
  CrowdingDistance(F_i);
  Add best $N - |P(t+1)|$ individuals in F_i to $P(t+1)$;
  $Q(t+1)$ = NewPopulation($P(t+1)$);
  $t = t + 1$;
end while

III. Proposed $\widehat{g}$-dominance

In this section, first the basic idea of the proposed $\widehat{g}$-dominance relation is introduced. Second, the detailed definition of the $\widehat{g}$-dominance relation is provided. Then, the similarities and differences between $\widehat{g}$-dominance and g-dominance are illustrated using some examples. Finally, the algorithm $\widehat{g}$-NSGA-II is explained.

A. Basic Idea

For preference-based MOPs, the purpose of MOEAs is to search the ROI, instead of the entire PF. Intuitively, preference-based MOPs can be regarded as a kind of special bilevel optimization problems, as follows:

(1) Lower level requirement: the solutions generated by the MOEA should be Pareto optimal;
(2) Upper level requirement: the solutions generated by the MOEA should be within the ROI.

In other words, the lower level optimization problem is the original MOP, and the upper level problem is a new MOP constructed to attain the DM's preferences. However, thus far, insufficient attention has been paid to preference-based MOPs from the viewpoint of special bilevel optimization problems [19], [20].

In this paper, the upper level of the problem is defined as

$$Minimize\ P(F(x)) = (|f_1(x) - g_1|, \cdots, |f_M(x) - g_M|)^T, \quad (3)$$

where $g = (g_1, \cdots, g_M)$ is the given reference point, $F(x) = (f_1(x), \cdots, f_M(x))$ is a point in the objective space, and M is the dimension of the objective space. Obviously, its goal is to minimize the distance from the individuals to the reference point at each dimension.

Then, the minimized bilevel MOP with preferences is defined as

$$\begin{aligned} &Minimize\ P(F(x)) = (|f_1(x) - g_1|, \cdots, |f_M(x) - g_M|)^T, \\ &Subject\ to\ Minimize\ F(x) = (f_1(x), \cdots, f_M(x))^T, \\ &Subject\ to \begin{cases} h_i(x) = 0, & i = 1, \cdots, n_p \\ g_j(x) \leq 0, & j = 1, \cdots, n_q \\ x \in \Omega \end{cases}, \end{aligned} \quad (4)$$

where $F(x) = (f_1(x), \cdots, f_M(x))$ is the decision variable vector for $P(F(x))$, $p_k(F(x)) = |f_k(x) - g_k| : R \to R$ $(k = 1, \cdots, M)$ represents the objective functions for satisfying the preferences, and the remaining definitions are consistent with equations (1) and (3).

The problems defined by equation (4) are a type of MOP with preferences. The ROI expressed by equation (4) constitutes the Pareto-optimal solutions with the minimum distances from the individuals to the reference point, g, at each dimension. In the following, we define such a preference as the $\widehat{g}$-dominance relation and compare the $\widehat{g}$-dominance and the g-dominance relation.

B. $\widehat{g}$-dominance

Before defining the $\widehat{g}$-dominance relation, some concepts are presented as follows. For two given individuals x, y in the decision space of the above bilevel problem,

(1) x f-dominates y ($x \prec_f y$): for the lower level optimization problem $F(x)$, x Pareto dominates y, that is, if and only if $f_i(x) \leq f_i(y)$ for each $i \in \{1, \cdots, M\}$ and there exists at least one $j \in \{1, \cdots, M\}$, $f_j(x) < f_j(y)$;
(2) x p-dominates y ($x \prec_p y$): for the upper level optimization problem $P((F(x))$, $F(x)$ Pareto dominates $F(y)$, that is, if and only if $|f_i(x) - g_i| \leq |f_i(y) - g_i|$ for each $i \in \{1, \cdots, M\}$ and there exists at least one $j \in \{1, \cdots, M\}$, $|f_j(x) - g_j| < |f_j(y) - g_j|$.

The $\widehat{g}$-dominance relation is now introduced. A solution x $\widehat{g}$-dominates a solution y when one of the following conditions is satisfied:

(1) x f-dominates y;
(2) x and y is non-f-dominated each other, and x p-dominates y.

Moreover, the $\widehat{g}$-dominance optimal solution is defined as follows.

x is $\widehat{g}$-dominance optimal, if and only if x is Pareto optimal and no Pareto-optimal individual y satisfies $y \prec_p x$.

C. Comparisons of g-dominance and $\widehat{g}$-dominance

In this part, g-dominance and $\widehat{g}$-dominance are compared. We discuss first the common ground of the two relations and then their differences. For convenience, the problem ZDT3 from [21] is taken as an example, and the three different reference points used are (0.5, 0.5), (0.2, -0.2), and (0.3,0.2419), respectively.

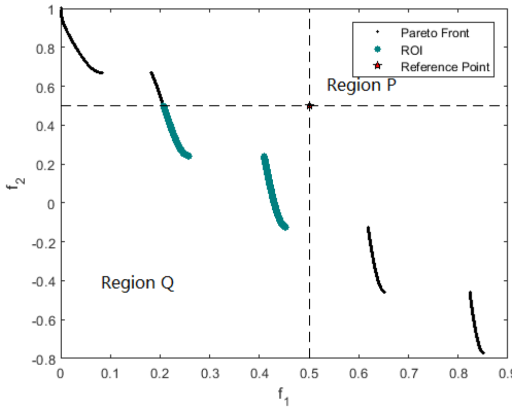

Fig. 1. Real regions of interest defined by the g-dominance and the $\widehat{g}$-dominance relation when the reference point is located in the feasible region.

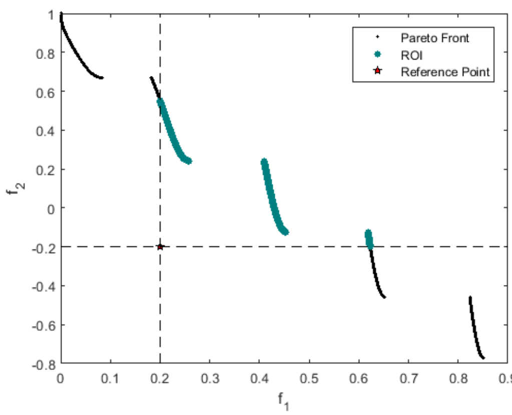

Fig. 2. Real regions of interest defined by the g-dominance and the $\widehat{g}$-dominance relation when the reference point is located in the infeasible region.

1) Common Ground: Figs. 1 and 2 show two examples, when the reference points are in the feasible and infeasible regions, respectively.

In Fig. 1, the reference point g is set as $(0.5, 0.5)$, and the black curve represents the PF of ZDT3.

(1) For g-dominance, according to Equation (2), regions P and Q are preference areas with a $Flag_g$ of 1; the real ROI of the g-dominance relation is composed of non-dominated solutions in the region Q, i.e., the green portion of the PF in Fig. 1.

(2) For $\widehat{g}$-dominance, the real ROI consists of non-f-dominated solutions that are also non-p-dominated. The non-f-dominated solutions in region Q are p-dominance equivalent, because, given any two non-f-dominated solutions x, y in the region Q, there must be:
$|f_1(x)-g_1| < |f_1(y)-g_1|$ & $|f_2(x)-g_2| > |f_2(y)-g_2|$
or
$|f_1(x)-g_1| > |f_1(y)-g_1|$ & $|f_2(x)-g_2| < |f_2(y)-g_2|$.
Meanwhile, the non-f-dominated solutions outside region Q are p-dominated by the solutions in the region Q, because, for any non-f-dominated solution y outside the region Q, there must exist at least one non-f-dominated solution x in the region Q satisfying:
$|f_1(x)-g_1| < |f_1(y)-g_1|$ & $|f_2(x)-g_2| \leq |f_2(y)-g_2|$
or
$|f_1(x)-g_1| \leq |f_1(y)-g_1|$ & $|f_2(x)-g_2| < |f_2(y)-g_2|$.
In summary, the ROI of the $\widehat{g}$-dominance consists of the non-dominated solutions in the region Q. Thus, the real ROIs defined by the g-dominance and the $\widehat{g}$-dominance relation are consistent.

Fig. 2 reveals the same situation when the reference point is set as $(0.2, -0.2)$ in the infeasible region. That is, the real ROIs defined by the g-dominance and the $\widehat{g}$-dominance are consistent.

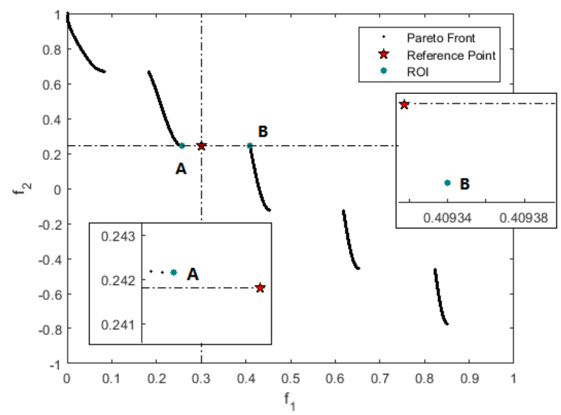

Fig. 3. Real region of interest (ROI) defined by the $\widehat{g}$-dominance when the reference point is located in the special region. The ROI of g-dominance does not exist.

2) Differences: In Fig. 3, the reference point is set as $(0.3, 0.2419)$. Meanwhile, in this figure, the positions of points A and B on the PF are $(0.2577, 0.2422)$ and $(0.4093, 0.2418)$, respectively. The two subgraphs in the figure show the relative position relationship between Point A (or B) and the reference point. It can be clearly observed that individuals on the PF do not dominate the reference point, and they are also not dominated by the reference point.

In this case, the real ROI defined by g-dominance is non-existent rather than in the Pareto-optimal region.

In contrast, the ROI defined by the $\widehat{g}$-dominance is located in the Pareto-optimal region, which consists of points A and B. Note that $P(A) = (0.0423, 0.0003)$, $P(B) = (0.1093, 0.0001)$, and A and B are p-dominance equivalent.

Furthermore, the cases where the ROI defined by the g-dominance does not exist may be rare in a two-dimensional space; however, this would be a more common situation in a three- or higher-dimensional space. Overall, the $\widehat{g}$-dominance can be regarded as an improved version of the g-dominance.

D. $\widehat{g}$-NSGA-II

The main idea of $\widehat{g}$-NSGA-II is to achieve the bilevel dominating relationship. We need only to modify the non-dominated sorting procedure in the original NSGA-II.

That is, Algorithm 2 is adopted to replace the function $NonDominatedSort(.)$ in Algorithm 1. At each time step t, the input parameter $TempP$ of Algorithm 2 is $P(t) \cup Q(t)$.

Algorithm 2 $\widehat{g}$-Non-Dominated Sort

Input: $TempP$ (population), g (reference point)
Output: F (the result of non-$\widehat{g}$-dominance sorting)

1: $maxLayer = 1$;
2: **while** length($TempP$) > 0 **do**
3: $\bar{P}$ = all $\widehat{g}$-dominance optimal solutions in $TempP$;
4: $TempP$ = $TempP - \bar{P}$;
5: $F(\bar{P}) = maxLayer$;
6: $maxLayer = maxLayer + 1$;
7: **end while**

In Algorithm 2, first, at step 3, the non-f-dominated solutions in population $TempP$ are selected (denoted as $NonFDS$ for convenience), and then, the non-p-dominated solutions in $NonFDS$ are selected as $\bar{P}$. Second, $TempP = TempP - \bar{P}$, and $F(\bar{P}) = maxLayer$ is set, where $maxLayer$ is initialized to 1. Third, $maxLayer = maxLayer + 1$, and the above process is repeated until $TempP$ is empty.

IV. Experiments

In this section, the experimental settings and results are presented. Both $\widehat{g}$-NSGA-II and g-NSGA-II were implemented on the MATLAB platform PlatEMO [22].

A. Benchmark

The ZDT and DTLZ instances [21], [23], [24] are widely used in multi-objective optimization and can illustrate relatively well the ability of MOEAs to generate solutions in various situations. In this study, ZDT1∼4, ZDT6, and DTLZ1∼6 were adopted.

In our experiments, the ZDT test suite consiste of two-objective problems and the DTLZ test suite consisted of three-objective problems. The dimension of all the ZDT problems was set to 30, except for ZDT4 and ZDT6 whose dimensionality was set to 10. The dimension of the DTLZ problems was set to 12. The maximum allowable number of fitness evaluations for the ZDT test problems was set to 20000 and it was set to 30000 for the DTLZ test problems.

B. Metric

The inverted generational distance (IGD) [25] is a widely employed indicator for measuring the distance between the PF and the solutions generated by MOEAs, which considers both diversity and convergence. It is defined as:

$$IGD(P^*, P) = \frac{\sum_{v \in P^*} d(v, P)}{|P^*|}, \tag{5}$$

where P^* denotes the real PF consisting of a set of uniformly distributed points, P is the approximate PF composed of a comparable set of points generated by the MOEA, and $d(v, P)$ represents the Euclidean distance from a point v in P^* to its nearest point in P.

The IGD indicator considers the entire PF rather than the region preferred by the DM. To concentrate on the DM's preferred solutions, we modified the IGD indicator as follows:

$$IGD(ROI^*, ROI) = \frac{\sum_{v \in ROI^*} d(v, ROI)}{|ROI^*|}, \tag{6}$$

where ROI^* contains the DM's preferred solutions on the real PF and ROI represents the results produced by the algorithms.

The modified IGD can better compare the distance between the true ROI and the solutions generated by the algorithm. For the sake of convenience, this modified version is called IGD-ROI hereafter.

C. Reference Points

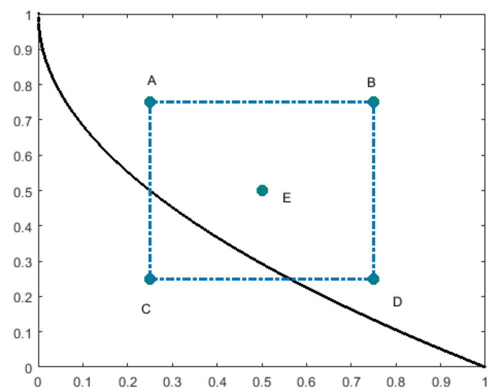

Fig. 4. Five reference points in a two-dimensional problem.

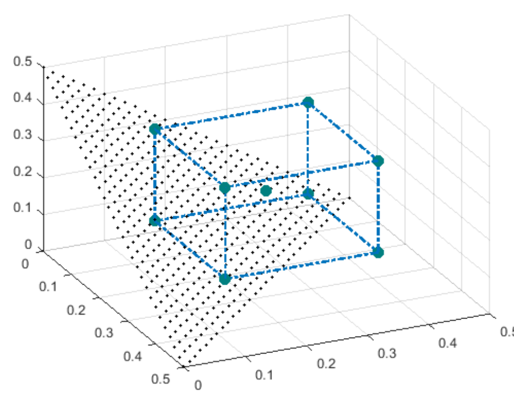

Fig. 5. Nine reference points in a three-dimensional problem.

Each problem was tested using different reference points to allow extensive comparisons. The reference points $Ref_{i,j}$ were set as follows:

(1) For two-dimensional problems:

$$Ref_{i,j} = (c_1 + (-1)^i * s_1, c_2 + (-1)^j * s_2), \quad i, j = 1, 2; \tag{7}$$

(2) For three-dimensional problems:

$$Ref_{i,j,k} = (c_1 + (-1)^i * s_1, c_2 + (-1)^j * s_2, \\ c_3 + (-1)^k * s_3), \qquad i,j,k = 1,2; \tag{8}$$

where

$$\begin{aligned} c &= (c_1, \cdots, c_D) \\ &= ((ub_1 - lb_1)/2, \cdots, (ub_D - lb_D)/2), \\ s &= (s_1, \cdots, s_D) \\ &= ((ub_1 - lb_1)/4, \cdots, (ub_D - lb_D)/4), \end{aligned}$$

and lb_d and ub_d are the lower and upper boundaries of the true PF in the d^{th} dimensional objective space, c_d is the average of lb_d and ub_d, and s represents a vector consisting of offsets in various dimensions, $1 \leq d \leq D$. $Ref_{i,j}$ and $Ref_{i,j,k}$ denote the reference points in two- and three-dimensional objective space, respectively. Moreover, the center point c was also tested as a reference point Ref_c in each problem.

For example, the true PF of ZDT1 [21], [23], [24] and the positions of the five reference points are illustrated in Fig. 4. The points A, B, C, D, and E correspond to $Ref_{1,2} = (0.5 + (-1)^1 * 0.25, 0.5 + (-1)^2 * 0.25) = (0.25, 0.75)$, $Ref_{2,2} = (0.75, 0.75)$, $Ref_{1,1} = (0.25, 0.25)$, $Ref_{2,1} = (0.75, 0.25)$, and $Ref_c = (0.5, 0.5)$, respectively. It can be seen that these reference points include multiple types of locations, such as feasible areas, infeasible areas, and regions near the PF.

For an additional example, Fig. 5 shows the nine reference points on DTLZ1. As shown in the figure, in the three-dimensional problem, the above reference point selection method can better cover most of the representative regions in the objective space.

D. Parameters Settings

The size of the population of all test cases was set to 100. The simulated binary crossover and polynomial-based mutation operators were adopted in $\widehat{g}$-NSGA-II. For the crossover operator, the crossover probability was $p_c = 1$ and the distribution index was set to $d_c = 20$. For the mutation operator, the mutation probability was set to $p_m = 1/n$ (n is the dimensionality of the solution space) and the distribution index was set to $d_m = 20$.

The parameters settings for g-NSGA-II were the same as those for $\widehat{g}$-NSGA-II to allow a fair comparison.

E. Experimental Results

The average values of IGD-ROI and their standard deviations are shown in Tables I, II, and III. Each experiment was independently run 20 times. A t-test was applied to determine whether the performance of $\widehat{g}$-NSGA-II is significantly better at a 0.05 significance level.

1) Results of ZDT problems: It can be seen in Table I that, for all cases of the ZDT instances, except for one (ZDT6), $\widehat{g}$-NSGA-II performs better than g-NSGA-II.

On ZDT6, when the position of the reference point was set as $(0.5394, 0.2303)$, g-NSGA-II performed better. However, the performance of $\widehat{g}$-NSGA-II was close to that of g-NSGA-II. In fact, g-NSGA-II obtained 0.0028 and $\widehat{g}$-NSGA-II obtained 0.0030.

The IGD-ROI is a good indicator of the difference between the results obtained by the algorithms and the target ROI. That is, the above experimental results show that the solutions obtained by $\widehat{g}$-NSGA-II are more in line with the needs of the DM than those generated by g-NSGA-II on the ZDT test problems. Meanwhile, the results of the t-test show that the improvement achieved is statistically significant.

TABLE I
MEAN AND STANDARD DEVIATION VALUES OF IGD-ROI (ZDT)

Prob.	Reference point	g-NSGA-II	$\widehat{g}$-NSGA-II
ZDT1	(0.25,0.75)	0.0029(8.68E-04)(++)	**0.0016(2.93E-04)**
	(0.25,0.25)	0.0030(1.02E-03)(++)	**0.0019(2.51E-04)**
	(0.75,0.25)	0.0028(1.40E-03)(++)	**0.0011(1.73E-04)**
	(0.75,0.75)	0.0059(1.91E-03)(++)	**0.0039(5.54E-04)**
	(0.5,0.5)	0.0026(6.90E-04)(++)	**0.0016(2.43E-04)**
ZDT2	(0.25,0.75)	0.0031(8.09E-04)(++)	**0.0017(2.35E-04)**
	(0.25,0.25)	0.0056(1.28E-03)(++)	**0.0039(4.68E-04)**
	(0.75,0.25)	0.0033(1.80E-03)(++)	**0.0013(2.99E-04)**
	(0.75,0.75)	0.0070(9.43E-03)(++)	**0.0020(3.79E-04)**
	(0.5,0.5)	0.0038(1.28E-03)(++)	**0.0017(2.71E-04)**
ZDT3	(0.2129,1.33)	0.0029(9.18E-04)(++)	**0.0022(1.55E-04)**
	(0.2129,0.4433)	0.0008(4.46E-04)(++)	**0.0001(6.17E-05)**
	(0.6388,0.4433)	0.0582(1.19E-01)(+)	**0.0061(1.47E-02)**
	(0.6388,1.33)	0.0064(1.89E-03)(++)	**0.0048(4.21E-04)**
	(0.4259,0.8867)	0.0052(3.08E-03)(++)	**0.0032(2.96E-04)**
ZDT4	(0.25,0.75)	0.0753(9.17E-02)(++)	**0.0090(1.41E-02)**
	(0.25,0.25)	0.0272(4.78E-02)(+)	**0.0065(3.96E-03)**
	(0.75,0.25)	0.0597(6.71E-02)(++)	**0.0166(3.38E-02)**
	(0.75,0.75)	0.0211(1.93E-02)(++)	**0.0080(4.41E-03)**
	(0.5,0.5)	0.0764(8.88E-02)(++)	**0.0056(2.61E-03)**
ZDT6	(0.1798,0.6909)	0.0024(7.03E-04)(+)	**0.0021(6.67E-04)**
	(0.1798,0.2303)	0.0042(1.08E-03)(+)	**0.0040(7.59E-04)**
	(0.5394,0.2303)	**0.0028(7.77E-04)**	0.0030(9.44E-04)
	(0.5394,0.6909)	0.0020(2.10E-03)(++)	**0.0005(3.82E-04)**
	(0.3596,0.4606)	0.0031(1.34E-03)(++)	**0.0025(5.08E-04)**

++ and + indicate $\widehat{g}$-NSGA-II performs significantly better and better than g-NSGA-II, respectively.

2) Results on the DTLZ test problems: In Tables II and III, it can be seen that, as compared with g-NSGA-II, $\widehat{g}$-NSGA-II obtained outstanding results on four instances: DTLZ1, DTLZ3, DTLZ5, and DTLZ6. The t-test values show that $\widehat{g}$-NSGA-II performed significantly better than g-NSGA-II on these four problems. The performance of g-NSGA-II was better than that of $\widehat{g}$-NSGA-II on DTLZ2 for four reference points and on DTLZ4 for six reference points.

Overall, our experimental results show that $\widehat{g}$-NSGA-II can better handle the DM's preferences.

F. Two Special Cases

In this study, two special cases were used to compare the performance of $\widehat{g}$-NSGA-II and g-NSGA-II. Fig. 6 and Fig. 7 show the results of a specific run of the two cases. In fact, in most cases, the results of the two cases are similar to those in the figures.

TABLE II

MEAN AND STANDARD DEVIATION VALUES OF IGD-ROI (DTLZ, PART 1)

Prob.	Reference point	g-NSGA-II	$\widehat{g}$-NSGA-II
DTLZ1	(0.125,0.375,0.375)	11.3340(4.61E+00)(++)	**0.0875(1.49E-01)**
	(0.125,0.125,0.375)	10.4750(2.98E+00)(++)	**0.0753(1.20E-01)**
	(0.125,0.375,0.125)	12.2340(3.61E+00)(++)	**0.1226(2.05E-01)**
	(0.125,0.125,0.125)	10.0440(3.23E+00)(++)	**0.0943(1.30E-01)**
	(0.25,0.25,0.25)	9.8832(2.84E+00)(++)	**0.0381(6.11E-02)**
DTLZ2	(0.25,0.75,0.75)	**0.0111(4.54E-04)**	0.0114(4.61E-04)
	(0.25,0.25,0.75)	0.0241(3.60E-03)(++)	**0.0212(1.70E-03)**
	(0.25,0.75,0.25)	0.0213(1.55E-03)(++)	**0.0194(6.27E-04)**
	(0.25,0.25,0.25)	0.0499(3.27E-03)(+)	**0.0477(2.83E-03)**
	(0.5,0.5,0.5)	0.0146(1.44E-03)(++)	**0.0130(8.21E-04)**
DTLZ3	(0.25,0.75,0.75)	25.1670(6.75E+00)(++)	**0.1570(2.98E-01)**
	(0.25,0.25,0.75)	22.3890(6.50E+00)(++)	**0.2640(4.07E-01)**
	(0.25,0.75,0.25)	22.5890(7.74E+00)(++)	**0.3879(5.64E-01)**
	(0.25,0.25,0.25)	24.0880(5.55E+00)(++)	**0.2254(3.31E-01)**
	(0.5,0.5,0.5)	24.5360(7.57E+00)(++)	**0.2639(4.24E-01)**
DTLZ4	(0.25,0.75,0.75)	**0.0110(3.40E-04)**	0.0114(3.87E-04)
	(0.25,0.25,0.75)	0.0216(1.53E-03)(+)	**0.0214(1.23E-03)**
	(0.25,0.75,0.25)	0.0202(1.33E-03)(++)	**0.0193(5.73E-04)**
	(0.25,0.25,0.25)	**0.0466(3.07E-03)**	0.0477(2.05E-03)
	(0.5,0.5,0.5)	0.0143(1.20E-03)(++)	**0.0132(5.66E-04)**
DTLZ5	(0.1768,0.5303,0.75)	0.2080(4.15E-04)(++)	**0.0020(6.58E-05)**
	(0.1768,0.1768,0.75)	0.0017(8.56E-05)(++)	**0.0016(7.24E-05)**
	(0.1768,0.5303,0.25)	0.0052(7.53E-04)(++)	**0.0043(4.12E-04)**
	(0.1768,0.1768,0.25)	0.0042(2.09E-04)(++)	**0.0038(1.58E-04)**
	(0.3536,0.3536,0.5)	0.0022(4.90E-04)(++)	**0.0019(7.66E-05)**
DTLZ6	(0.1768,0.5303,0.75)	0.2460(3.99E-02)(++)	**0.0022(1.28E-04)**
	(0.1768,0.1768,0.75)	0.0282(4.01E-02)(++)	**0.0017(5.81E-05)**
	(0.1768,0.5303,0.25)	0.0251(1.84E-02)(++)	**0.0046(4.38E-04)**
	(0.1768,0.1768,0.25)	0.0218(2.05E-02)(++)	**0.0039(1.56E-04)**
	(0.3536,0.3536,0.5)	0.0288(2.56E-02)(++)	**0.0019(7.03E-05)**

++ and + indicate respectively that $\widehat{g}$-NSGA-II performs significantly better and better than g-NSGA-II.

1) Case 1: In the first case, the reference point was close to the Pareto optimal front.

Here, we take ZDT3 and the reference point $g = (0.25, 0.25)$ as an example. As shown in Fig. 6, g-NSGA-II cannot properly converge to the ROI. But $\widehat{g}$-NSGA-II can.

When the reference point is very close to the PF, since the solutions in the preferred region of g-NSGA-II are selected first for being evolved, these individuals cannot easily be concentrated on one point at the PF as a result of the gap between them and the PF. Nevertheless, $\widehat{g}$-NSGA-II always selects first the nearest Pareto-optimal solution for evolution. Consequently, $\widehat{g}$-NSGA-II can properly locate the ROI.

2) Case 2: In the second case, the reference point was located in the region where the real ROI of g-dominance is nonexistent.

Here, we take the problem ZDT3 and the reference point $g = (0.3, 0.2419)$ as an example. When the reference point is located at $(0.3, 0.2419)$, the real ROI defined by g-dominance does not exist. The results obtained by the two algorithms are shown in Fig. 7. The results of g-NSGA-II are concentrated on the non Pareto-optimal region, whereas $\widehat{g}$-NSGA-II is effective.

TABLE III

MEAN AND STANDARD DEVIATION VALUES OF IGD-ROI (DTLZ, PART 2)

Prob.	Reference point	g-NSGA-II	$\widehat{g}$-NSGA-II
DTLZ1	(0.375,0.375,0.375)	11.1740(3.77E+00)(++)	**0.0959(1.19E-01)**
	(0.375,0.125,0.375)	11.9580(4.35E+00)(++)	**0.0864(1.24E-01)**
	(0.375,0.375,0.125)	10.3500(3.24E+00)(++)	**0.1711(1.98E-01)**
	(0.375,0.125,0.125)	9.9710(3.55E+00)(++)	**0.0772(1.20E-01)**
DTLZ2	(0.75,0.75,0.75)	**0.0359(1.56E-03)**	0.0365(1.82E-03)
	(0.75,0.25,0.75)	**0.0110(4.28E-04)**	0.0115(3.67E-04)
	(0.75,0.75,0.25)	**0.0107(3.78E-04)**	0.0110(4.75E-04)
	(0.75,0.25,0.25)	0.0233(5.62E-03)(++)	**0.0192(4.64E-04)**
DTLZ3	(0.75,0.75,0.75)	27.5560(7.41E+00)(++)	**0.1199(2.24E-01)**
	(0.75,0.25,0.75)	27.4220(6.15E+00)(++)	**0.0434(3.24E-02)**
	(0.75,0.75,0.25)	26.2620(6.99E+00)(++)	**0.1998(3.88E-01)**
	(0.75,0.25,0.25)	24.0030(8.13E+00)(++)	**0.1572(3.08E-01)**
DTLZ4	(0.75,0.75,0.75)	**0.0361(1.39E-03)**	0.0363(1.43E-03)
	(0.75,0.25,0.75)	**0.0109(4.72E-04)**	0.0113(3.49E-04)
	(0.75,0.75,0.25)	**0.0108(3.68E-04)**	0.0111(3.57E-04)
	(0.75,0.25,0.25)	**0.0197(9.55E-04)**	0.0197(6.36E-04)
DTLZ5	(0.5303,0.5303,0.75)	0.0004(1.97E-05)	0.0004(1.90E-05)
	(0.5303,0.1768,0.75)	0.2079(3.00E-04)(++)	**0.0020(7.51E-05)**
	(0.5303,0.5303,0.25)	0.0021(1.68E-04)(++)	**0.0018(8.26E-05)**
	(0.5303,0.1768,0.25)	0.0053(9.14E-04)(++)	**0.0046(5.47E-04)**
DTLZ6	(0.5303,0.5303,0.75)	0.1349(2.80E-01)(++)	**0.0005(1.28E-05)**
	(0.5303,0.1768,0.75)	0.2490(4.21E-02)(++)	**0.0022(1.19E-04)**
	(0.5303,0.5303,0.25)	0.0648(1.69E-01)(+)	**0.0018(8.45E-05)**
	(0.5303,0.1768,0.25)	0.0295(2.07E-02)(++)	**0.0044(3.27E-04)**

++ and + indicate respectively that $\widehat{g}$-NSGA-II performs significantly better and better than g-NSGA-II.

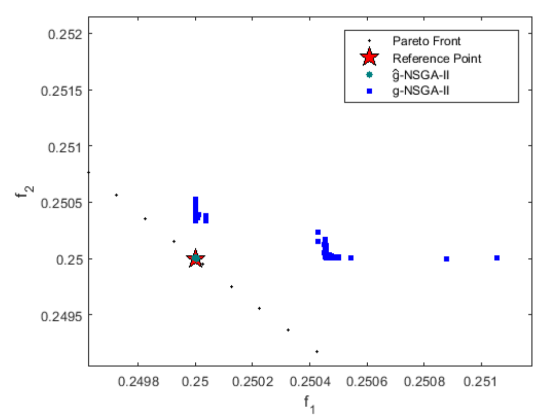

Fig. 6. Results obtained by g-NSGA-II and $\widehat{g}$-NSGA-II when the reference point is close to the Pareto optimal front (One Run).

In this case, according to equation (2), all individuals with $Flag_g(.) = 1$ are not Pareto optimal. Therefore, g-NSGA-II cannot find the PF in this case. However, $\widehat{g}$-NSGA-II is effective, because it selects the nearest individuals from the Pareto-optimal solutions.

V. CONCLUSION AND FUTURE WORK

In this paper, a new dominance relation called $\widehat{g}$-dominance was proposed. This approach hybridizes the reference point method and the Pareto dominance principle. $\widehat{g}$-dominance can

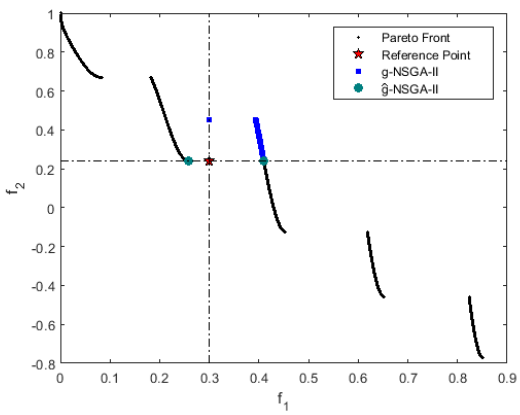

Fig. 7. Results obtained by the g-NSGA-II and $\widehat{g}$-NSGA-II when the reference point is located in the region where the real region of interest of g-dominance is nonexistent (One Run).

be regarded as an improved version of g-dominance. The new $\widehat{g}$-dominance relation was implemented in NSGA-II, and our experimental results showed that $\widehat{g}$-NSGA-II performs better in terms of processing user preferences than the original g-NSGA-II.

Some studies have shown that the more complex the problem environment, the greater the pressure on the DM to make decisions [26], [27]. However, the incorporation of preferences will undoubtedly reduce decision costs. In the future, it would be worthwhile to implement and test the $\widehat{g}$-dominance relation in dynamic MOPs, as well as in many-objective optimization problems. Moreover, the use of multiple reference points still needs further study. Finally, when the user continually changes the reference point, the above strategy is undoubtedly inefficient and wastes resources. Evidently, we need better means of handling time-evolving reference points.

References

[1] K. Deb, A. Pratap, S. Agarwal, and T. Meyarivan, "A fast and elitist multiobjective genetic algorithm: NSGA-II," *IEEE transactions on evolutionary computation*, vol. 6, no. 2, pp. 182–197, 2002.

[2] E. Zitzler, M. Laumanns, and L. Thiele, "SPEA2: Improving the Strength Pareto Evolutionary Algorithm," in *EUROGEN 2001. Evolutionary Methods for Design, Optimization and Control with Applications to Industrial Problems*, K. Giannakoglou, D. Tsahalis, J. Periaux, P. Papailou, and T. Fogarty, Eds., Athens, Greece, 2001, pp. 95–100.

[3] Q. Zhang and H. Li, "MOEA/D: A multiobjective evolutionary algorithm based on decomposition," *IEEE Transactions on evolutionary computation*, vol. 11, no. 6, pp. 712–731, 2007.

[4] J. Molina, L. V. Santana, A. G. Hernández-Díaz, C. A. C. Coello, and R. Caballero, "g-dominance: Reference point based dominance for multiobjective metaheuristics," *European Journal of Operational Research*, vol. 197, no. 2, pp. 685–692, 2009.

[5] L. B. Said, S. Bechikh, and K. Ghédira, "The r-dominance: a new dominance relation for interactive evolutionary multicriteria decision making," *IEEE Transactions on Evolutionary Computation*, vol. 14, no. 5, pp. 801–818, 2010.

[6] K. Deb and J. Sundar, "Reference point based multi-objective optimization using evolutionary algorithms," in *Proceedings of the 8th annual conference on Genetic and evolutionary computation*. ACM, 2006, pp. 635–642.

[7] A. B. Ruiz, R. Saborido, and M. Luque, "A preference-based evolutionary algorithm for multiobjective optimization: the weighting achievement scalarizing function genetic algorithm," *Journal of Global Optimization*, vol. 62, no. 1, pp. 101–129, 2015.

[8] G. Yu, J. Zheng, R. Shen, and M. Li, "Decomposing the user-preference in multiobjective optimization," *Soft Computing*, vol. 20, no. 10, pp. 4005–4021, 2016.

[9] L. Thiele, K. Miettinen, P. J. Korhonen, and J. Molina, "A preference-based evolutionary algorithm for multi-objective optimization," *Evolutionary computation*, vol. 17, no. 3, pp. 411–436, 2009.

[10] L. Jiao, W. Zhang, R. Liu, and F. Liu, "A hybrid multiobjective immune algorithm with region preference for decision makers," in *Evolutionary Computation (CEC), 2010 IEEE Congress on*. IEEE, 2010, pp. 1–8.

[11] K. Deb and H. Jain, "An evolutionary many-objective optimization algorithm using reference-point-based nondominated sorting approach, part I: Solving problems with box constraints." *IEEE Trans. Evolutionary Computation*, vol. 18, no. 4, pp. 577–601, 2014.

[12] K. Deb and A. Kumar, "Interactive evolutionary multi-objective optimization and decision-making using reference direction method," in *Proceedings of the 9th annual conference on Genetic and evolutionary computation*. ACM, 2007, pp. 781–788.

[13] S. Jiang and S. Yang, "A strength Pareto evolutionary algorithm based on reference direction for multiobjective and many-objective optimization," *IEEE Transactions on Evolutionary Computation*, vol. 21, no. 3, pp. 329–346, 2017.

[14] M. Gong, F. Liu, W. Zhang, L. Jiao, and Q. Zhang, "Interactive MOEA/D for multi-objective decision making," in *Proceedings of the 13th annual conference on Genetic and evolutionary computation*. ACM, 2011, pp. 721–728.

[15] A. Jaszkiewicz and R. Słowiński, "The 'Light Beam Search' approach – an overview of methodology applications," *European Journal of Operational Research*, vol. 113, no. 2, pp. 300–314, 1999.

[16] K. Deb and A. Kumar, "Light beam search based multi-objective optimization using evolutionary algorithms," in *Evolutionary Computation, 2007. CEC 2007. IEEE Congress on*. IEEE, 2007, pp. 2125–2132.

[17] K. Narukawa, Y. Setoguchi, Y. Tanigaki, M. Olhofer, B. Sendhoff, and H. Ishibuchi, "Preference representation using gaussian functions on a hyperplane in evolutionary multi-objective optimization," *Soft Computing*, vol. 20, no. 7, pp. 2733–2757, 2016.

[18] H. Geng, M. Zhang, L. Huang, and X. Wang, "Infeasible elitists and stochastic ranking selection in constrained evolutionary multi-objective optimization," in *Asia-Pacific Conference on Simulated Evolution and Learning*. Springer, 2006, pp. 336–344.

[19] A. Sinha, P. Malo, and K. Deb, "A review on bilevel optimization: from classical to evolutionary approaches and applications," *IEEE Transactions on Evolutionary Computation*, vol. 22, no. 2, pp. 276–295, 2018.

[20] A. Sinha, P. Malo, K. Deb, P. Korhonen, and J. Wallenius, "Solving bilevel multicriterion optimization problems with lower level decision uncertainty," *IEEE Transactions on Evolutionary Computation*, vol. 20, no. 2, pp. 199–217, 2016.

[21] E. Zitzler, K. Deb, and L. Thiele, "Comparison of multiobjective evolutionary algorithms: Empirical results," *Evolutionary computation*, vol. 8, no. 2, pp. 173–195, 2000.

[22] Y. Tian, R. Cheng, X. Zhang, and Y. Jin, "Platemo: A matlab platform for evolutionary multi-objective optimization [educational forum]," *IEEE Computational Intelligence Magazine*, vol. 12, no. 4, pp. 73–87, 2017.

[23] K. Deb, L. Thiele, M. Laumanns, and E. Zitzler, "Scalable Test Problems for Evolutionary Multiobjective Optimization," in *Evolutionary Multiobjective Optimization. Theoretical Advances and Applications*, A. Abraham, L. Jain, and R. Goldberg, Eds. USA: Springer, 2005, pp. 105–145.

[24] Q. Zhang, A. Zhou, S. Zhao, P. N. Suganthan, W. Liu, and S. Tiwari, "Multiobjective Optimization Test Instances for the CEC 2009 Special Session and Competition," University of Essex, Colchester, UK and Nanyang Technological University, Singapore, Tech. Rep. 264, 2008.

[25] C. A. C. Coello and N. C. Cortés, "Solving multiobjective optimization problems using an artificial immune system," *Genetic Programming and Evolvable Machines*, vol. 6, no. 2, pp. 163–190, 2005.

[26] B. Schwartz, *The Paradox of Choice: Why More is Less*. ECCO Press, 2016, ISBN 978-0062449924.

[27] E. Reutskaja and R. M. Hogarth, "Satisfaction in choice as a function of the number of alternatives: When goods satiate," *Psychology & Marketing*, vol. 26, no. 3, pp. 197–203, 2009.