Using evolutionary computation to infer the decision maker's preference model in presence of imperfect knowledge: a case study in portfolio optimization

Abstract

It is usually very difficult to elicit the parameter values of models representing decision makers' preferences. Consequently, some imprecision, ill-determination and arbitrariness are unavoidable. Moreover, such elicitation cannot be performed by traditional optimization techniques in a reasonable time. Therefore, we present here a novel elicitation method guided by a genetic algorithm whose main contribution is coping with imperfect knowledge. The latter is done by using interval numbers representing all the possible values that the parameters can attain. The assessment of the method showed its high ability to reproduce the decision maker's preferences. Finally, as the method proposed in this paper is the complement of the authors' previous work regarding the optimization of stock portfolios, we provide a case study in such a field. We use differential evolution to obtain the most satisfactory portfolio. The results reported here show that the best portfolio returns are obtained when the elicitation method is exploited, and we conclude that the new overall approach might be an interesting alternative to the already-existing methods.

Keywords: Evolutionary algorithms; multi-criteria decision aiding; outranking approach; preferences elicitation; portfolio optimization.

1. Introduction

Multi-criteria decision aiding (MCDA) provides a wide range of appropriate methods for choosing, ranking and sorting (ordinal classification) problematics. However, the aid provided by MCDA is not effective unless the aggregation model appropriately represents the decision maker's (DM) preferences. Generally, the MCDA models use many parameters to represent the DM's preferences. The values of these parameters can be obtained either with direct or indirect elicitation methods. In the former, the decision maker, often aided by a decision analyst, has to directly assign the parameter values to the preference model. Whereas in indirect elicitation methods, the parameter values are deduced from a battery of easy-to-make decisions made by the DM.

Eliciting the parameter values is very important in developing a multi-criteria decision aiding approach. Some authors (for example [1]) consider the direct elicitation method as less adequate relative to indirect elicitation. Some limitations of the former are the following: i) the preference model's parameters are meaningless as long as the multi-criteria aggregation procedure in which they are used has not been specified; ii) holistic decisions made by the DM using his/her own judgment procedure when comparing pairs of actions and/or assigning actions to categories/classes are more appropriate; iii) the DM may not be accessible (e.g., the manager of an international company) or may be an ill-defined entity (e.g., a heterogeneous group); iv) the DM usually has difficulties to explicitly specify numerical parameters and the time and cognitive effort required to do so may be inhibitory.

The indirect elicitation methods constitute the well-known preference disaggregation analysis (PDA) paradigm. PDA methods analyze decisions made by the DM in order to identify the aggregation model that underlies the outcome of the known decisions. The indirect elicitation methods infer the decision model's parameters from holistic decisions provided by the DM and use regression-like methods to produce a decision model as consistent as possible with the set of reference (training) decisions. The PDA paradigm is of growing interest because it requires less cognitive effort from the DM. The main reason is that DMs frequently prefer making decision judgments than explaining them.

Indirect elicitation approaches have been used for decades to build functional or utility decision models (e.g., [2, 3, 4]). In MCDA, Jacquet-Lagreze and Siskos [5] pioneered the *UTA* method. Regarding the outranking approach, indirect elicitation methods are even more significant, because the DM must establish parameter values that are very unfamiliar to her/him (e.g., veto thresholds). In this frame, some important references are the works of Mousseau and Słowiński [6], Doumpos *et al*. [7], and Fernandez *et al*. [8]. Indirect elicitation approaches have been satisfactorily used by many authors in the context of financial decision making (e.g., [9, 10, 11, 12]) and particularly in the context of portfolio selection (e.g., [13, 14]). All these proposals identify punctual values for the model's parameters, which are supposed to be appropriate to explain or suggest new decisions.

Despite the wide use of indirect elicitation methods, they cannot avoid certain imperfect information in setting the model's parameters; the concept of what is the appropriate value of a decision model parameter is poorly-defined due to several reasons: a) the DM's decision policy may not match with the model's assumptions and its mathematical structure; b) the DM's preferences are ill-defined (e.g., a heterogeneous group); c) the DM is a mythical or inaccessible person (e.g., public opinion); d) often many parameter settings reproduce the known decision examples; and e) imprecise (even missing) information on criterion scores. Thus, there is always imprecision, uncertainty, ill-definition or arbitrariness (imperfect knowledge, according to Roy *et al*. [15]) to be handled by the PDA when it infers the values of the parameters.

Recently, Fernandez *et al*. [16] presented an extension of the outranking approach that is able to deal with imperfect knowledge on the parameters of the model and on criterion scores. Although the DM likely feels more comfortable making a direct elicitation of model parameter values as interval numbers, this approach does not avoid the convenience of indirect setting [16]. It would be more convenient if, instead of punctual values, the indirect elicitation method offers the flexibility to consider the parameters as ranges of numbers, where imperfect knowledge is contained within intervals. Such a method would combine the advantages of the indirect elicitation with the flexibility of the interval outranking approach.

The interval outranking approach was recently applied to solve a many-objective stock portfolio optimization problem in [17]. Such paper proposed an interesting approach to select the best stock portfolio considering imperfect knowledge (in the sense of [15]) that characterizes the DM's implicit model of preferences, performing a pressure toward the DM's most preferred portfolios; it represents the DM's conservatism to risk, and the portfolios' expected return and risk. However, a direct elicitation of the interval-outranking model's parameters representing the DM's preferences was performed there. Therefore, an interesting research question is if the application of evolutionary computation to the indirect elicitation of these parameters implies that i) the interval-outranking finds more preferred solutions, and/or ii) the portfolios found by the overall approach generate greater returns. Our main objective is thus addressing this question. We do it by proposing and assessing a novel method that indirectly elicits the interval-outranking model's parameters using a set of judgments made by the DM.

The rest of the paper is structured as follows. In Section 2 we briefly describe some previous related work. In Section 3, we present our proposal to get an approximation to the DM's model of preferences when the parameters are described as numerical ranges. In Section 4 we describe some experiments to validate the proposal whose results are shown in Section 5. In Section 6, a case study is presented where elicited preference parameter values are used to select the most preferred portfolios. Finally, we conclude this paper in Section 7.

2. Some background

2.1 Previous related work

The outranking approach, introduced by Roy in 1968 and firstly exploited by the ELECTRE family of methods (cf. e.g. [18]), is a well-known methodology used to model the preferences of decision makers. It is based on preference relations defined between pairs of alternatives (or actions) and built on the basis of the assessment of actions on a set of multiple criteria. Some of the most interesting features of this approach are their ability to model intransitive preferences, non-compensatory effects, and poorly known criterion scores. These features have made the outranking approach more suitable for some situations than other methodologies (e.g., value function approaches). Perhaps the most common criticism against the outranking approach is its dependence on many parameters and the common difficulty of directly eliciting these parameters. An interval extension of the outranking approach was applied in [17] to portfolio optimization, but its direct elicitation of the approach's parameters is an important limitation.

Regarding indirect elicitation of preference parameters in the context of the outranking approach, Mousseau and Słowiński [6] pioneered the exploitation of the preference disaggregation analysis to infer the parameters of an ELECTRE method. Under certain strong simplifications, they used a conventional mathematical programming tool to solve a highly complex non-linear and non-convex optimization problem. Using linear programming, Mousseau *et al*. [19] presented a proposal to infer some specific parameters (weights of criteria) from assignment examples. Using linear and mixed-integer linear programming formulations, Zheng *et al*. [20] and Bisdorff *et al*. [21] also inferred the weights of criteria. Through assignment examples, Ngo Te and Mousseau [22] elicited additional parameters (boundary profiles) for the method ELECTRE TRI. Addressing special situations, Mousseau *et al*. [23, 24] proposed to use indirect elicitation methods to infer preference parameters under inconsistent sets of assignment examples.

Most of these works do not elicit all the parameters of the outranking approach simultaneously; particularly, they elude the inference of veto thresholds [25]. This is because it would imply a very complex nonlinear programming problem. To overcome the computational complexity of inferring more general outranking models, metaheuristics have been recently widely used. The tendency to use evolutionary computation in this framework, particularly genetic algorithms, is overwhelming (cf. [26, 27]). This is, for example, because of its treatment of nonlinearity and its ability to perform a global optimization in polynomial time [28]. Assche and De Smet [29] proposed a heuristic based on a genetic algorithm to identify the weights, indifference and preference thresholds but also profiles characterizing classes of a sorting method based on the outranking approach. Doumpos *et al.* [7] and Fernandez *et al.* [9] used evolutionary algorithms to infer the entire set of ELECTRE model's parameters from a set of assignment the decision maker's examples. Fernandez *et al*. [30] used NSGA-II to infer the parameters outranking approach under scarce reference information and effects of reinforced preference. The proposal of Sobrie *et al.* [31] take into account the structure of the problem to perform crossovers and mutations in a genetic algorithm when inferring the parameters of the Majority Rule Sorting procedure, a simplified version of the ELECTRE TRI sorting model. Covantes *et al*. [32] found good results by using a genetic algorithm to infer the parameters of the THESEUS method, an outranking-based approach to assign actions to preferentially ordered classes. Fernandez *et al*. [55] use a single-objective optimization genetic algorithm to indirectly infer the parameters using preference information embedded in assignment examples. Fernandez *et al*. [56] also use genetic algorithms to infer the parameters of an outranking-based approach which is called ELECTRE TRI-nB and represents an interval extension of the well-known ELECTRE TRI-B.

Given the intensity and frequency with which genetic algorithms are used to solve problems similar to the one addressed in this work, as well as some preliminary results and the experience of the authors using this type of tools, we decided to use the classical version of genetic algorithms to address our research problem in Section 3.2.

2.2 Interval numbers

The concept of interval number was originated in the so-called Interval Analysis Theory [33, 34]. Such a number represents a numerical quantity whose exact value is unknown. This lack of knowledge is encompassed in a range of numbers where the exact value is expected to be. Thus, an interval number represents an unspecified quantity whose value is given in a set of values. Let κ be a real value lying between k^+ and k^-. The interval number representing κ is therefore $\boldsymbol{K} = [k^-, k^+]$. Any $r \in [k^-, k^+]$ is a *realization* of $\boldsymbol{K}$. Furthermore, q, a real number, can be represented by an interval number as $[q, q]$. In order to state clearer notations, in the rest of this document interval numbers will be denoted by boldface italic letters.

Now, given two interval numbers $\boldsymbol{K} = [k^-, k^+]$ and $\boldsymbol{L} = [l^-, l^+]$, the basic operations of interval numbers are as follows:

$$\boldsymbol{K} + \boldsymbol{L} = [k^- + l^-, k^+ + l^+].$$

$$\boldsymbol{K} - \boldsymbol{L} = [k^- - l^+, k^+ - l^-].$$

$$\boldsymbol{K} \times \boldsymbol{L} = [\min\{k^- l^-, k^- l^+, k^+ l^-, k^+ l^+\}, \max\{k^- l^-, k^- l^+, k^+ l^-, k^+ l^+\}].$$

$$\boldsymbol{K} \div \boldsymbol{L} = [k^-, k^+] \times \left[\frac{1}{l^-}, \frac{1}{l^+}\right].$$

There is no way to be sure about the order of interval numbers. Nevertheless, [35] proposed how to define a possibility grade of $\boldsymbol{K}$ being greater than or equal to $\boldsymbol{L}$, $p(\boldsymbol{K} \geq \boldsymbol{L})$. This possibility function is given by

$$p(\boldsymbol{K} \geq \boldsymbol{L}) = \begin{cases} 1 & \text{if } p_{\boldsymbol{KL}} > 1, \\ p_{\boldsymbol{KL}} & \text{if } 0 \leq p_{\boldsymbol{KL}} \leq 1, \\ 0 & \text{if } p_{\boldsymbol{KL}} < 0. \end{cases} \tag{1}$$

Where $p_{\boldsymbol{KL}} = \dfrac{k^+ - l^-}{(k^+ - k^-) + (l^+ - l^-)}$.

If $k^+ = k^-$ and $l^+ = l^-$, then

$$p(\boldsymbol{K} \geq \boldsymbol{L}) = \begin{cases} 1 & \text{if } k^- \geq l^-, \\ 0 & \text{otherwise.} \end{cases}$$

Let k and l be two unsettled realizations from $\boldsymbol{K}$ and $\boldsymbol{L}$, respectively; $p(\boldsymbol{K} \geq \boldsymbol{L})$ is the credibility degree of the assertion "given that both realizations are established, k is not lesser than l". Thus, the possibility function denotes robustness of $\boldsymbol{K} \geq \boldsymbol{L}$, even when these quantities are undetermined.

2.3. Interval outranking

Recently, Fernandez *et al.* [16] presented an extension of the outranking approach, called *interval-based outranking*, that handles imperfect knowledge on the preferences of the DM and the criterion scores. Their proposal uses Interval Theory as the basic component to model this imperfect knowledge. We now present a brief description of such extension for completeness of this paper.
Let U be the universe of actions, $A \subseteq U$ a set of actions and each $x \in U$ be evaluated on a family of n criteria $\mathcal{I} = \{\boldsymbol{g_1}, \boldsymbol{g_2}, \cdots, \boldsymbol{g_n}\}$ defined on U. In the interval outranking, each criterion function is an interval number. Let us assume that increasing the score of x on criterion j improves the performance of action x, for any $j = 1,2, \cdots n$. Some parameters used by that approach are the following (note the that these parameters are defined using Interval Theory):

- $\boldsymbol{g_j}(x) = [g_j^-(x), g_j^+(x)]$, the score of action $x \in A$ on criterion g_j;
- $\boldsymbol{w_j} = [w_j^-, w_j^+]$, the weight of criterion g_j;
- $\boldsymbol{v_j} = [v_j^-, v_j^+]$, the veto threshold of criterion g_j; and
- $\boldsymbol{\lambda} = [\lambda^-, \lambda^+]$ reflects a threshold for a sufficient strength of the concordance coalition.

Where $j = 1, \cdots, n$.

Since the imperfect knowledge on the criterion scores is represented through intervals, no preference and indifference thresholds are used in [16].

As in the classical outranking approach, the interval outranking approach estimates a credibility index, $\beta(x, y) \in [0,1]$, between pairs of alternatives about the assertion "x is at least as good as y", xSy. The marginal credibility index of x being at least as good as y on criterion $\boldsymbol{g_j}$ is:

$$\alpha_j(x, y) = p\left(\boldsymbol{g_j}(x) \geq \boldsymbol{g_j}(y)\right).$$

Where $p(\cdot)$ is the possibility function described in Eq. (1).

The concordance coalition, $C(xS_\delta y)$, is composed with all the criteria $\boldsymbol{g_j}$ fulfilling $\alpha_j(x, y) \geq \delta$ for a given credibility threshold δ. This concordance coalition is associated with an index $\delta = \min\{\alpha_j(x, y)\}$ such that $\boldsymbol{g_j} \in C(xS_\delta y)$, where δ is the credibility of the statement "all the criteria within the concordance coalition agree with the outranking relation". All $\boldsymbol{g_j}$ that are not within $C(xS_\delta y)$ constitute the discordance coalition, $D(xS_\delta y)$. The previous definitions are formalized as

$$\boldsymbol{g_j} \in C(xS_\delta y) \text{ iff } \alpha_j(x, y) \geq \delta, \text{and}$$

$$D(xS_\delta y) = \mathcal{I} - C(xS_\delta y).$$

On the other hand, as stated by the classical outranking approach, it is necessary to ensure $\sum \boldsymbol{w_j} = [1,1]$. Any realization not fulfilling this condition is not valid. Therefore, Fernandez et al. [16] set the following constraints to ensure validity of the model:

$$\sum_{j=1}^{n} w_j^- \leq 1 \tag{2}$$

$$\sum_{j=1}^{n} w_j^+ \geq 1 \tag{3}$$

The concordance index of $xS_\delta y$, $\boldsymbol{c}(x, y, \delta) = [c^-(x, y), c^+(x, y)]$, is then defined as follows:

$$c^-(x, y) = \sum_{\boldsymbol{g_j} \in C(xS_\delta y)} w_j^-,$$

if

$$\sum_{\boldsymbol{g_j} \in C(xS_\delta y)} w_j^- + \sum_{\boldsymbol{g_j} \in D(xS_\delta y)} w_j^- \leq 1, \text{and}$$

$$\sum_{\boldsymbol{g_j} \in C(xS_\delta y)} w_j^- + \sum_{\boldsymbol{g_j} \in D(xS_\delta y)} w_j^+ \geq 1.$$

Otherwise, $c^-(x, y)$ is defined as

$$1 - \sum_{g_j \in D(xS_\delta y)} w_j^+.$$

Similarly,

$$c^+(x,y) = \sum_{g_j \in C(xS_\delta y)} w_j^+$$

only if

$$\sum_{g_j \in C(xS_\delta y)} w_j^+ + \sum_{g_j \in D(xS_\delta y)} w_j^- \leq 1\text{, and}$$

$$\sum_{g_j \in C(xS_\delta y)} w_j^+ + \sum_{g_j \in D(xS_\delta y)} w_j^+ \geq 1.$$

Otherwise, $c^+(x,y)$ is

$$1 - \sum_{g_j \in D(xS_\delta y)} w_j^-.$$

Fernandez *et al.* [16] show that

- $c^+(x,y) \geq c^-(x,y)$,
- if $C(xS_\delta y) = \emptyset$ then $\boldsymbol{c}(x,y,\delta) = [0,0]$, and
- if $C(xS_\delta y) = \mathcal{J}$ then $\boldsymbol{c}(x,y,\delta) = [1,1]$.

Let Δ be the set $\left\{\alpha_j \in \mathbb{R}: p\left(\boldsymbol{g_j}(x) \geq \boldsymbol{g_j}(y)\right) = \alpha_j > 0, j = 1, \cdots, n\right\}$. For each $\delta \in \Delta$, Fernandez *et al.* [16] state that *x* outranks *y* with marginal likelihood index β_δ, and majority strength $\lambda = [\lambda^-, \lambda^+]$ ($\lambda^- > 0.5$), if and only if

i. $p(\boldsymbol{c}(x,y,\delta) \geq \boldsymbol{\lambda}) \geq \pi$;
ii. $1 - \max_{g_j \in D(xS_\delta y)} \{p(\boldsymbol{g_j}(y) \geq \boldsymbol{g_j}(x) + \boldsymbol{v_j})\} \geq \pi$; and
iii. $\beta_\delta = \max\{\pi$ fulfilling i and ii$\}$;

Where $\delta \geq \pi \in \mathbb{R}$ and $p(\cdot)$ is defined in (1). Thus, $xS_\delta y$ is fulfilled with likelihood index $\beta(x,y) \in [0,1] = \max\{\beta_\delta\}$ and majority strength $\lambda = [\lambda^-, \lambda^+]$ ($\lambda^- > 0.5$). If Δ is an empty set, then $\beta(x, y)$ is zero. Moreover, the model assumes that the DM uses a credibility threshold $\beta_0 > 0.5$ such that if $\beta(x, y) \geq \beta_0$ then the assertion "x is at least as good as y" is accepted.

Finally, given a set $\mathcal{P} = \{\boldsymbol{w_1}, \cdots, \boldsymbol{w_n}, \boldsymbol{v_1}, \cdots, \boldsymbol{v_n}, \boldsymbol{\lambda}, \beta_0\}$ and for each pair $(x, y) \in A \times A$, the following preference relations may be defined based on the credibility index associated with "x is at least as good as y" and calculated on the basis of $\mathcal{P}$:

- **Strict preference**: $xP_{\mathcal{P}}y \Leftrightarrow \beta(x, y) \geq \beta_0 \; and \; \beta(y, x) < 0.5$,
- **Weak preference**: $xQ_{\mathcal{P}}y \Leftrightarrow \beta(x, y) \geq \beta_0 \; and \; 0.5 \leq \beta(y, x) < \beta_0$,
- **K preference**: $xK_{\mathcal{P}}y \Leftrightarrow \beta(y, x) < 0.5 < \beta(x, y) < \beta_0$,
- **Indifference**: $xI_{\mathcal{P}}y \Leftrightarrow \beta(x, y) \geq \beta_0 \; and \; \beta(y, x) \geq \beta_0$.

3. Our proposal

The imperfect knowledge that characterizes the decision maker's (DM) implicit model of preferences (cf. [15]) gives rise to the idea that vague or ill-determined information should be considered during the modeling of the DM's preferences (see [16]). However, it is often difficult for the DM to express specific values for the parameters of models representing her/his own preferences [6], even when these parameters are defined as ranges of numbers as described in the previous section.

In this paper, we propose a novel way to infer the DM's implicit system of preferences through an interval-based preference disaggregation approach. The main characteristic of the proposal relies in allowing the DM's decision policy to contain imperfect knowledge. An indirect elicitation with this characteristic is particularly important in portfolio optimization where average investors tend to be not able/willing to spend much time expressing the exact parameter values of a model representing their preferences. The only DM's work required by the proposed approach consists in the creation of a reference set containing DM's holistic decisions. The DM's decision policy is reflected by such reference set, and it is from this set where the proposal draws an approximation to the implicit system of preferences of the decision maker.

3.1 Interval-based preference disaggregation analysis

Let us now introduce some assumptions that are basic for presenting our proposal:

1. There is a finite set A of actions described by a set of criteria $\mathcal{J} = \{\boldsymbol{g_1}, \cdots, \boldsymbol{g_n}\}$, where $\boldsymbol{g_j}(x) = [g_j^-(x), g_j^+(x)]$ is the interval number that represents the performance evaluation of action $x \in A$ in attribute $\boldsymbol{g_j}$.
2. The DM is willing to establish a binary relation "at least as good as" on $T \times T$, where T is a proper subset of A. That is, for each ordered pair (x,y) belonging to $T \times T$, one of the following statements is true: i) "x is at least as good as y"; or ii) "x is not at least as good as y".

Our goal is to find a set of parameters $\mathcal{P}^* = \{\boldsymbol{w_1^*}, \cdots, \boldsymbol{w_n^*}, \boldsymbol{v_1^*}, \cdots, \boldsymbol{v_n^*}, \boldsymbol{\lambda}^*, \delta^*, \beta_0^*\}$ (preference model), that permits to construct an interval outranking model as consistent as possible with the relation "is not worse than", denoted by $\succcurlyeq$. To achieve this, let's assume that a binary preference relation is built for each $(x, y) \in T \times T$ for a given set of parameters $\mathcal{P}'$ and let us consider the following sets (cf. Subsection 2.3):

$$H_P = \{(x, y): xP_{\mathcal{P}'}y \text{ with } (x) \nsucc (y)\},$$

$$H_Q = \{(x, y): xQ_{\mathcal{P}'}y \text{ or } xK_{\mathcal{P}'}y \text{ with } (x) \nsucc (y)\},$$

$$H_I = \{(x, y): xI_{\mathcal{P}'}y \text{ with } (x) \nsim (y)\}.$$

Where

$(x) \nsucc (y) \Leftrightarrow not\ (x \succcurlyeq y\ and\ not(\ y \succcurlyeq x))$,

$x{\sim}y \Leftrightarrow x \succcurlyeq y\ and\ y \succcurlyeq x$, and

$x \nsim y \Leftrightarrow not\ (x{\sim}y)$.

Our proposal to find the "best" $\mathcal{P}'$ is to solve the following multi-objective optimization problem:

$$\underset{\mathcal{P}' \in \Gamma}{\text{minimize}} \left(card(H_P), card(H_Q), card(H_I)\right) \tag{4}$$

With preferential priority in lexicographical order favoring $card(H_P)$. In Problem (4), Γ is the set of preference models that fulfill Eqs. (2) and (3) and $card(\omega)$ is the cardinality of set ω.

3.2 A Genetic algorithm for solving Problem (4)

Some recent research reported in the related literature (cf. Subsection 2.1) concluded that Genetic Algorithms present more promising results than other metaheuristics, such as Particle Swarm optimization, Tabu Search and Simulated Annealing, when solving multiobjective optimization problems similar to (4) but with real numbers. Consequently, in this work we use a genetic algorithm capable of dealing with parameters defined as interval numbers in order to search for the solution of Problem (4).

Each chromosome in our genetic algorithm consists of the parameters of the outranking based on Interval Theory; that is, $\boldsymbol{w_1}, \cdots, \boldsymbol{w_n}, \boldsymbol{v_1}, \cdots, \boldsymbol{v_n}, \boldsymbol{\lambda}$, and β_0 as shown in Figure 1. Thus, each individual is composed of $2n + 2$ genes.

Figure 1. Individual representing a solution to Problem (4).

$\boldsymbol{w_1}$	$\boldsymbol{w_2}$	…	$\boldsymbol{w_n}$	$\boldsymbol{v_1}$	$\boldsymbol{v_2}$	…	$\boldsymbol{v_n}$	$\boldsymbol{\lambda}$	β_0

There are $n + 2$ crossing and mutation points. Furthermore, in order to fulfill consistency constraints (2) and (3), the weights are all considered as only one gene. The points to apply the crossover and mutation operators are shown in Figure 2.

Figure 2. $n + 2$ cutoff points for individuals.

$1 \quad 2 \quad n \quad n+1 \quad n+2$

$\boldsymbol{w_1}$	…	$\boldsymbol{w_3}$	$\boldsymbol{v_1}$	…	$\boldsymbol{v_3}$	$\boldsymbol{\lambda}$	β_0

Algorithm 1 describes the procedure. The algorithm first randomly creates an initial population of L individuals, P_0. After that, and for each generation, from the current population P_g the algorithm creates an offspring H_g, also of size L, using the following operators of Selection, Crossover and Mutation.

The selection of parents at each generation of the genetic algorithm is done through a binary tournament; the different winning individuals in two independent tournaments are crossed in a single-point crossover to generate an offspring individual according to Table 2.

Mutation consists of the random generation of a gene. If gene 1 is selected to mutate, then the set of weights would be randomly generated satisfying the consistency constraints (2) and (3). If gene 2 is chosen to mutate, then a random value would be generated for the veto of the first criterion, $\boldsymbol{v_1}$. The probability with which an individual is selected to mutate is 1%.

Since the set of weights is considered as a unique gene, there is no need to verify feasibility of the offspring individuals.

The next step is the combination of parents and offspring in a pool from which the algorithm extracts the individuals with the best fitness. Such fitness is evaluated from the objectives in Problem (4), fulfilling constraints (2) and (3), and based on a reference set χ_m of cardinality m. The individuals with the best fitness within the pool form the next generation of parents, P_{g+1},. This procedure is repeated for G generations. Later on, the algorithm returns the individual that represents the feasible solutions with the best fitness values in the last population. This vector is obtained as the centroid (average of the parameters) of individuals with the best fitness values. It can be demonstrated that if the centroid is obtained from a set of feasible solutions, then such a centroid is also feasible. In order to discard randomness in the procedure, we generate L centroids. And, in order to take advantage of these centroids, we use them as a "seed population" for the last run of the algorithm. The centroid generated in this final run is recommended as the best solution to Problem (4).

The values of parameters L (population size), G (number of generations), and χ_m (set of reference actions) should be provided by the decision maker according to the problem's context.

Algorithm 1. Genetic Algorithm proposed to solve Problem (4).
Require: L, the size of the population; G, the number of generations; χ_m, a reference set of cardinality m.
Ensure: ρ_{final}, individual representing the population with the best fitness value

$i \leftarrow 1$
for $i \leq L$ **do**
    $g \leftarrow 0$
    $P_g \leftarrow$ createInitialPopulation()
    for $g \leq G$ **do**
        $H_g \leftarrow$ createOffspring(P_g, selection, crossover, mutation)
        $P_{g+1} \leftarrow$ bestAptitude($P_g \cup H_g$)

    $g \leftarrow g + 1$
  end for
  $\rho_i \leftarrow$ findCentroid(P_g)
$i \leftarrow i + 1$
end for
$g \leftarrow 0$
$P_g \leftarrow \{\rho_1, \rho_2, \cdots, \rho_L\}$
for $g \leq G$ **do**
  $H_g \leftarrow$ createOffspring(P_g, selection, crossover, mutation)
  $P_{g+1} \leftarrow$ bestAptitude($P_g \cup H_g$)
  $g \leftarrow g + 1$
end for
$\rho_{final} \leftarrow$ findCentroid(P_G)

It is possible to incorporate information into the genetic algorithm that will help to reduce the search space. Some ways to add this type of information are the following:

- The DM can assign values to some of the parameter boundaries. Because the parameters are expressed as interval numbers, it is relatively easy for the DM to assign the boundaries of some of these parameters. For example, the DM can provide a value μ_j such that if the maximum difference between the score of actions y and x in criterion g_j is equal to or greater than μ_j (that is, if $g_j^-(y) - g_j^+(x) \geq \mu_j$), then there is no doubt that xSy must be vetoed. Therefore, it is possible to limit the search space of the algorithm by doing $v_j^+ = \mu_j$.
- The DM can express that criterion g_j is more important than criterion g_i. In this case, the algorithm must ensure $w_j^- > w_i^+$.
- It must be satisfied that $\lambda^- > 0.5$ and $\lambda^+ < 1$.
- It must also be satisfied that $\beta_0 > 0.5$.

4. Experiments

This section details the experiments carried out to test the performance of the proposed approach and shows the validation of the parameters generated, mainly in its ability to reproduce the DM's preferences. The actions used here are artificially created.

4.1 Creating experimental instances

To assess the proposed approach, it is necessary to simulate the DM's decision policy and to generate sets of instances as test cases for evaluating the approach's robustness. For this purpose, we simulate the DM's preferences through the random generation of the parameter vector $\mathcal{P}$. We assume a context where the DM is not able/disposed to spend a lot of time expressing his/her preferences (e.g., establishing a binary preference relation between each pair of actions within a reference set); so, we allow him/her to express his/her preferences in form of an ordinal classification of the actions. Each instance of the experiments consists then of a reference set T containing a finite number of actions assigned to classes ordered according to the preferences in $\mathcal{P}$. Each action $x \in T$ is assigned to a class C_j belonging to the following set: $C_3 = Good$, $C_2 = Doubt$ and $C_1 = Bad$. We denote such assignment as $C_{\mathcal{P}}(x) = j$. We assume for the experimental case that the assignments are consistent with the following conditions:

$$C_{\mathcal{P}}(x) - C_{\mathcal{P}}(y) \geq 2 \Rightarrow xP_{\mathcal{P}}y, \tag{5}$$

$$C_{\mathcal{P}}(x) - C_{\mathcal{P}}(y) = 1 \Rightarrow xP_{\mathcal{P}}y \vee xQ_{\mathcal{P}}y \vee xK_{\mathcal{P}}y, \tag{6}$$

$$C_{\mathcal{P}}(x) - C_{\mathcal{P}}(y) = 0 \Rightarrow xI_{\mathcal{P}}y \vee xQ_{\mathcal{P}}y \vee yQ_{\mathcal{P}}x \vee xK_{\mathcal{P}}y \vee yK_{\mathcal{P}}x, \tag{7}$$

$$xP_{\mathcal{P}}y \Rightarrow C_{\mathcal{P}}(x) > C_{\mathcal{P}}(y), \tag{8}$$

$$xQ_{\mathcal{P}}y \vee xK_{\mathcal{P}}y \Rightarrow C_P\ (x) \geq C_P(y), \tag{9}$$

$$xI_{\mathcal{P}}y \Rightarrow C_{\mathcal{P}}(x) = C_{\mathcal{P}}(y). \tag{10}$$

If an action cannot be assigned to one of the classes in consistency with Equations (5-10), then the current action is rejected and a new action is generated. This procedure continues until the cardinality of the reference set is satisfied.

Sometimes the DM is not able/willing to engage in an arduous elicitation procedure, thus, allowing her to express her preferences as assignments of actions instead of asking her to specify a preference relation between each pair of actions is convenient. Such scenario is typical in portfolio optimization.

The first step to create a reference set is to determine a central profile for each class. The central profiles of the classes are assigned in the following way. First, we randomly create a sufficiently large set of actions described by the criteria in $\mathcal{J}$. (Sets with 2000 actions are used in the experiments described below.) Later on, these actions are ranked through the outranking net flow score[1] using the simulated parameter vector $\mathcal{P}$. Finally, the central profile of a given class is defined as the action with the most representative position within the whole rank. For example, the central profile of the lowest class (C_1) is close to the position [2000/6]. As stated above, Equations (5-10) must always be fulfilled when assigning an action to a class.

To assign the rest of actions within the reference set to the classes, we follow the next procedure: i) randomly create a new action described by its impact on the criteria; ii) determine if it can be assigned to a class (fulfilling Equations (5-10)); iii) if it cannot be assigned to any class, go to step i; iv) if it can be assigned to just one class, assign the solution to that class; v) if it can be assigned to more than one class, assign the action to the central class among those classes where the solution fulfills Equations (5-10).

The bounds of $\boldsymbol{g_i}(x) = [g_i^-(x), g_i^+(x)]$ are generated as $g_i^-(x) = \min\{d_1, d_2\}$, $g_i^+(x) = \max\{d_1, d_2\}$ where $d_j = \max\{10, \min\{1, \hat{\imath}(1 - \epsilon_j)\}\}$, $\hat{\imath} \in [1,10]$, $\epsilon_j \in [-0.3,0.3]$, $j = 1,2$. The parameters of $\mathcal{P}$ are generated as follows: First, β_0 is randomly generated in (0.5, 0.6) while the i-th veto is defined as $v_i^- = 0.7\widehat{v_i}$ and $v_i^+ = 1.3\widehat{v_i}$, where $\widehat{v_i}$ is randomly generated in $[3, 5]$. We calculate the core values of the weights as $\widehat{w} = \frac{1}{n}$ and the weight of criterion $\boldsymbol{g_i}$ as $w_i^- = (1 - \omega_i)\,\widehat{w}, w_i^+ = (1 + \omega_i)\,\widehat{w}$, where ω_i is randomly generated in [0,0.3]. The lower bound of λ, λ^-, is randomly generated in $[0.51, 0.76]$ and its upper bound is set as $\lambda^+ = 1.3\lambda^-$. For all cases $i = 1, \cdots, n$.

To create the individuals in the initial population of the genetic algorithm, these parameters are randomly generated as described above, but with wider ranges of search: β_0 and the bounds of $\boldsymbol{\lambda}$ are randomly generated in (0.5, 1), ω_i is randomly generated in [0.1,0.5], $v_i^- = 0.5\widehat{v_i}$ and $v_i^+ = 1.5\widehat{v_i}$.

[1] The net flow score is commonly used to rank a set of actions (cf. [36]). If $\beta(x, y)$ is a fuzzy preference relation on a set A', the net flow score associated to $a \in A'$ is defined as $F_n(a) = \sum_{c \in A' - \{a\}}[\beta(a, c) - \beta(c, a)]$ (see [37]).

4.2 Validation procedure

We consider that it is important to validate the effectiveness of the approach when it deals with

- different decision models (e.g., when dealing with different DMs);
- different number of criteria;
- different number of actions in the reference set;
- out-of-sample situations: testing the model's capacity of generalization when approaching new decisions on actions out of the reference set.

We are particularly interested in testing the proposed approach's ability to create a decision model that reproduces the same preference relations as assumed from the assignments made by the DM. We use the following validation procedure:

1. Use two sets of criteria with different cardinality to describe actions; namely, six and twelve criteria.
2. Simulate the decision model of a DM through the random generation of the parameter vector $\mathcal{P}$.
3. Create five reference sets, $\chi_{10}, \chi_{20}, \chi_{30}, \chi_{40},$ and χ_{50}, with cardinalities of 10, 20, 30, 40 and 50, respectively, using three classes, $C_3 = Good$, $C_2 = Doubt$ and $C_1 = Bad$. We denote the assignment by the decision model $\mathcal{P}$ of action x to class j in the m-th reference set as $C_{\mathcal{P}_m}(x) = j$.
4. Obtain, through the proposal in Section 3, a set of parameters $\mathcal{P}^*$ as consistent as possible with the assignments made by the simulated DM (whose real decision model is $\mathcal{P}$) in each reference set. The maximum consistency is identified with the best compromise of Problem (4) and the optimization is performed using Algorithm 1.
5. Obtain the *in-sample* effectiveness of the proposal as follows: first, determine the preference relation for each pair of actions $(x, y) \in \chi_m \times \chi_m, m \in \{10, 20, 30, 40, 50\}$ through $\mathcal{P}^*$, and call it $x\mathcal{R}_m^* y$, $\mathcal{R} \in \{P, Q, K, I\} \cup \{O\}$. $\{P, Q, K, I\}$ is the set of preference relations described in bullet point 4 of Section 3, while O indicates that the relation between x and y is not in this set. It is necessary to note here that the definition of the preference relations does not guarantee that one of

the four relations will occur between x and y for $\mathcal{P}^*$. However, the way that the reference sets are created (see Subsection 4.1) allows one of these relations to be always hold between each pair of actions for $\mathcal{P}$. Thus, the situation where we cannot set one of these relations between each $(x, y) \in \chi_m \times \chi_m$ for $\mathcal{P}^*$ shall be counted as an inconsistency between $\mathcal{P}^*$ and $\mathcal{P}$. Finally, contrast the preference relation obtained through $\mathcal{P}'$ with the one inferred from the assignments made by the DM of x and y to their respective classes in χ_m and calculate an error indicator for the method in each instance of reference set χ_m as

$$\xi_m = \sum_{(x,y)\in\chi_m\times\chi_m} \left[\xi_{P_m}(x, y) + \xi_{Q_m}(x, y) + \xi_{K_m}(x, y) + \xi_{I_m}(x, y) + \xi_{O_m}(x, y)\right]$$

where

$\xi_{P_m}(x, y) = 1$ if $xP_m^*y \Rightarrow C_{\mathcal{P}_m}(x) > C_{\mathcal{P}_m}(y)$ is false and 0 otherwise,

$\xi_{Q_m}(x, y) = 1$ if $xQ_m^*y \Rightarrow C_{\mathcal{P}_m}(x) \geq C_{\mathcal{P}_m}(y)$ is false and 0 otherwise,

$\xi_{K_m}(x, y) = 1$ if $xK_m^*y \Rightarrow C_{\mathcal{P}_m}(x) \geq C_{\mathcal{P}_m}(y)$ is false and 0 otherwise,

$\xi_{I_m}(x, y) = 1$ if $xI_m^*y \Rightarrow C_{\mathcal{P}_m}(x) = C_{\mathcal{P}_m}(y)$ is false and 0 otherwise, and

$\xi_{O_m}(x, y) = 1$ if xOy.

Hence, the effectiveness of the method when evaluated in the context of the preference relations in χ_m is defined as

$$1 - \frac{\xi_m}{\eta} \tag{11}$$

(i.e., the strict negation of the proportion of errors with respect to the number of preference relations); where $\eta = \frac{m}{2}(m-1),\ m \in \{10, 20, 30, 40, 50\}$.

6. Obtain the *out-of-sample* effectiveness of the proposal as follows: first, use $\mathcal{P}$ to assign new actions (different to the ones in the reference sets) as described in Subsection 4.1 (we generate sets of 100 actions). Then, use $\mathcal{P}^*$ to find the binary preference relations between pairs of these new actions. Finally, determine the out-

of-sample effectiveness of the proposal using an equivalent validation method as the one used in step 5.

One instance of the experiment consists in a simulated DM, a given cardinality of the criteria set, and a cardinality of the reference set. We consider 40 instances to be sufficient to perform a satisfactory validation of the proposed approach using the validation process described in steps 4-6.

5. Results

Here, we analyze the effectiveness of the proposed approach to state the same binary relation as the ones inferred from the assignments made by the simulated DM. Such analysis is performed both in-sample and out-of-sample with respect to the actions in the reference sets. We evaluate the results obtained when the actions are described by six and twelve criteria.

5.1 Actions described by six criteria

5.1.1 In-sample effectiveness

Table 1 shows the average effectiveness of our approach, as calculated by Eq. (11) and its standard deviation for each reference set. We calculate these results from the effectiveness of our proposal in the 40 instances of the experiment. The Wilcoxon Signed-Ranks test for two paired samples indicated that the difference of each pair of average performances is considered to be statistically significant with a 0.95 confidence level. This means that the increments in the cardinality of the reference sets allowed the model to increase its performance and shows the number of actions that the DM should classify in order to obtain an expected performance.

Table 1. Average in-sample effectiveness of the proposal relative to preference relations for each reference set using six criteria

Reference set	Average effectiveness	Standard deviation
χ_{10}	0.9822	0.0023
χ_{20}	0.9992	1.36E^{-05}

χ_{30}	0.9993	$6.02E^{-06}$
χ_{40}	0.9996	$2.21E^{-06}$
χ_{50}	0.9997	$8.48E^{-07}$

Evidently, an effectiveness lower than 100% is due to the error indicator, ξ_m, being greater than zero. Each of the error types, $\xi_{P_m}, \xi_{Q_m}, \xi_{I_m}$, and ξ_{O_m}, has a different level of proportion in the total error (see step 5 of Subsection 4.2). Table 2 shows the average ratio in ξ_m of each of the error types.

Table 2. Average proportion of each type of error in ξ_m

Reference set	$\xi_{P_m}(x,y)$	$\xi_{Q_m}(x,y)$	$\xi_{K_m}(x,y)$	$\xi_{I_m}(x,y)$	$\xi_{O_m}(x,y)$
χ_{10}	0.43	0	0	0.14	0.43
χ_{20}	0.66	0	0.03	0.03	0.28
χ_{30}	0.64	0	0	0.16	0.20
χ_{40}	0.61	0.01	0.03	0.07	0.28
χ_{50}	0.59	0	0.01	0.12	0.28

Table 2 indicates that inconsistencies respect to the strict preference provide most of the error encompassed in the global error. Of course, not all preference relations occur with the same frequency in the experiment. Actually, it is the strict preference the one with the highest frequency, so it is not surprising that ξ_{P_m} has the largest proportion in the error indicator. Table 3 shows the effectiveness with respect to the frequency of each preference relation. This effectiveness is obtained as the average number of times that the preference relation inferred from the simulated DM's assignments and the preference relation found through $\mathcal{P}^*$ coincide. For example, with cardinality equal to 50, there are 819 pairs on average in the 40 instances where strict preference exists, according to the $\mathcal{P}^*$ model. Nevertheless, the ξ_{P_m} indicator was only 2 on average for strict preference (that is, there were only two times where $xP^*_{50}y \Rightarrow C_{\mathcal{P}_{50}}(x) > C_{\mathcal{P}_m}(y)$ was false, out of the 819 opportunities where it could happen). Thus, for the case of strict preference, the model has an effectiveness of 99.76% in χ_{50}.

Table 3. Average effectiveness by preference relation

Reference set	***P***	***Q***	***K***	***I***

χ_{10}	0.9973	1.0000	0.9999	1.0000
χ_{20}	0.9994	1.0000	0.9999	0.9998
χ_{30}	0.9997	1.0000	1.0000	1.0000
χ_{40}	0.9999	1.0000	1.000	1.0000
χ_{50}	0.9976	1.0000	1.0000	0.9997

Tables 4-7 show the comparison of the models $\mathcal{P}$ and $\mathcal{P}^*$ in reference sets χ_{30}. The parameter values are rather similar, although there are some significant deviations. But the really important feature of the indirect elicited parameters is to allow making decisions consistent with the reference sets, even when this set of parameters is not alike the ones of the simulated DM.

Table 4. Comparison of the cutting level β_0 of the simulated DM and the one found by the proposed approach in χ_{30}

Instance	β_0	
	DM	**PDA**
1	0.515	0.529
2	0.541	0.547
3	0.516	0.576
4	0.549	0.56
5	0.541	0.547
6	0.544	0.682
7	0.549	0.562
8	0.513	0.518
9	0.536	0.555
10	0.519	0.526

Table 5. Comparison of the cutting level $\boldsymbol{\lambda}$ of the simulated DM and the one found by the proposed approach in χ_{30}

Instance	$\boldsymbol{\lambda}$	
	DM	**PDA**
1	[0.529,0.541]	[0.559,0.67]
2	[0.512,0.523]	[0.56,0.667]
3	[0.522,0.54]	[0.567,0.67]
4	[0.515,0.523]	[0.561,0.668]
5	[0.52,0.527]	[0.553,0.659]
6	[0.521,0.534]	[0.557,0.666]
7	[0.516,0.529]	[0.557,0.661]

8	[0.522,0.536]	[0.561,0.666]
9	[0.529,0.541]	[0.571,0.676]
10	[0.518,0.54]	[0.566,0.641]

Table 6. Comparison of the vector of weights of the simulated DM and the one found by the proposed approach in χ_{30}

Instance	Weights	w_1	w_2	w_3	w_4	w_5	w_6
1	DM	[0.102,0.231]	[0.125,0.208]	[0.104,0.23]	[0.131,0.202]	[0.103,0.231]	[0.117,0.216]
	PDA	[0.115,0.218]	[0.118,0.215]	[0.114,0.219]	[0.117,0.216]	[0.117,0.216]	[0.119,0.215]
2	DM	[0.086,0.247]	[0.11,0.224]	[0.118,0.215]	[0.112,0.221]	[0.098,0.236]	[0.127,0.207]
	PDA	[0.122,0.212]	[0.116,0.217]	[0.116,0.217]	[0.118,0.215]	[0.121,0.212]	[0.112,0.221]
3	DM	[0.133,0.201]	[0.115,0.219]	[0.11,0.223]	[0.14,0.193]	[0.144,0.189]	[0.091,0.242]
	PDA	[0.118,0.215]	[0.118,0.215]	[0.116,0.218]	[0.115,0.219]	[0.118,0.215]	[0.115,0.218]
4	DM	[0.093,0.241]	[0.126,0.207]	[0.135,0.198]	[0.112,0.221]	[0.098,0.235]	[0.12,0.213]
	PDA	[0.118,0.215]	[0.118,0.215]	[0.112,0.221]	[0.118,0.215]	[0.124,0.21]	[0.114,0.219]
5	DM	[0.114,0.219]	[0.088,0.245]	[0.133,0.2]	[0.119,0.214]	[0.11,0.223]	[0.136,0.197]
	PDA	[0.12,0.214]	[0.119,0.215]	[0.113,0.22]	[0.115,0.218]	[0.117,0.217]	[0.119,0.214]
6	DM	[0.091,0.242]	[0.144,0.19]	[0.121,0.212]	[0.116,0.217]	[0.149,0.184]	[0.131,0.202]
	PDA	[0.117,0.216]	[0.114,0.219]	[0.115,0.219]	[0.117,0.216]	[0.116,0.217]	[0.116,0.217]
7	DM	[0.106,0.227]	[0.135,0.198]	[0.107,0.226]	[0.119,0.215]	[0.137,0.197]	[0.092,0.242]
	PDA	[0.115,0.218]	[0.116,0.217]	[0.117,0.217]	[0.12,0.214]	[0.119,0.214]	[0.114,0.219]
8	DM	[0.129,0.204]	[0.095,0.239]	[0.104,0.229]	[0.122,0.211]	[0.091,0.242]	[0.084,0.25]
	PDA	[0.121,0.212]	[0.119,0.214]	[0.116,0.217]	[0.119,0.215]	[0.112,0.221]	[0.117,0.217]
9	DM	[0.112,0.221]	[0.103,0.23]	[0.116,0.217]	[0.107,0.227]	[0.12,0.213]	[0.131,0.202]
	PDA	[0.115,0.218]	[0.118,0.216]	[0.119,0.214]	[0.12,0.213]	[0.12,0.213]	[0.118,0.215]
10	DM	[0.138,0.196]	[0.125,0.209]	[0.095,0.238]	[0.122,0.211]	[0.123,0.21]	[0.101,0.232]
	PDA	[0.118,0.216]	[0.117,0.216]	[0.115,0.218]	[0.12,0.213]	[0.113,0.221]	[0.118,0.216]

Table 7. Comparison of the vectors of vetoes of the simulated DM and the one generated by the proposed approach in χ_{30}

Instance	Veto thresholds	v_1	v_2	v_3	v_4	v_5	v_6
1	DM	[0.652,0.807]	[0.663,0.842]	[0.657,0.82]	[0.604,0.703]	[0.753,0.871]	[0.757,0.864]
	PDA	[0.663,0.795]	[0.635,0.745]	[0.733,0.904]	[0.719,0.863]	[0.733,0.902]	[0.733,0.915]
2	DM	[0.612,0.697]	[0.616,0.786]	[0.64,0.747]	[0.698,0.812]	[0.746,0.895]	[0.755,0.835]
	PDA	[0.665,0.797]	[0.718,0.875]	[0.651,0.78]	[0.703,0.849]	[0.629,0.74]	[0.735,0.9]
3	DM	[0.685,0.82]	[0.686,0.793]	[0.699,0.875]	[0.649,0.838]	[0.635,0.762]	[0.616,0.761]
	PDA	[0.736,0.911]	[0.676,0.807]	[0.677,0.807]	[0.682,0.816]	[0.629,0.746]	[0.653,0.788]
4	DM	[0.663,0.841]	[0.678,0.783]	[0.706,0.821]	[0.701,0.9]	[0.715,0.873]	[0.732,0.873]
	PDA	[0.7,0.849]	[0.713,0.854]	[0.697,0.836]	[0.714,0.873]	[0.698,0.841]	[0.735,0.897]
5	DM	[0.688,0.833]	[0.647,0.808]	[0.668,0.834]	[0.662,0.794]	[0.649,0.83]	[0.656,0.781]
	PDA	[0.684,0.849]	[0.692,0.828]	[0.661,0.802]	[0.734,0.905]	[0.648,0.772]	[0.707,0.863]
6	DM	[0.75,0.936]	[0.661,0.842]	[0.696,0.794]	[0.618,0.787]	[0.713,0.787]	[0.705,0.818]
	PDA	[0.669,0.796]	[0.723,0.889]	[0.706,0.843]	[0.667,0.799]	[0.707,0.854]	[0.633,0.746]
7	DM	[0.655,0.758]	[0.709,0.911]	[0.747,0.854]	[0.639,0.782]	[0.713,0.877]	[0.76,0.838]
	PDA	[0.652,0.774]	[0.706,0.868]	[0.644,0.757]	[0.705,0.859]	[0.737,0.904]	[0.727,0.893]

8	DM	[0.674,0.775]	[0.731,0.866]	[0.61,0.771]	[0.746,0.832]	[0.714,0.829]	[0.689,0.858]
	PDA	[0.689,0.815]	[0.73,0.907]	[0.715,0.866]	[0.733,0.899]	[0.715,0.874]	[0.654,0.784]
9	DM	[0.617,0.736]	[0.626,0.778]	[0.685,0.771]	[0.673,0.812]	[0.725,0.817]	[0.707,0.853]
	PDA	[0.637,0.757]	[0.685,0.82]	[0.728,0.862]	[0.693,0.829]	[0.646,0.771]	[0.648,0.77]
10	DM	[0.622,0.724]	[0.725,0.867]	[0.64,0.813]	[0.713,0.897]	[0.654,0.75]	[0.616,0.689]
	PDA	[0.622,0.742]	[0.674,0.811]	[0.696,0.833]	[0.679,0.816]	[0.719,0.877]	[0.644,0.763]

5.1.2 Out-of-sample effectiveness

The final goal of the PDA paradigm is to create a decision model consistent with the DM's preferences; so, the decision model should suggest decisions that may be considered appropriate by the DM. In this subsection we assess out-of-sample the effectiveness of our proposal to find such model; that is, we use the decision models built by the proposed approach to determine binary preference relations between actions different to the ones within the reference sets. We analyze the proposed approach's capability to "predict" the preference relation between pairs of actions described by six criteria such as the DM would have done it, considering a set of 100 assignments other than those in the original reference sets. These assignments are performed by the same simulated DMs for whom the preference models $\mathcal{P}^*$ were created. We measure this effectiveness based on the solution generated by our approach in each instance per reference set using the same effectiveness measure from the previous section.

Table 8 shows the average effectiveness of the proposed approach and its standard deviation for each cardinality of the original reference sets used to elicit $\mathcal{P}^*$. Unlike the results obtained in the in-sample test (Table 1), here some pairs of average performances are not statistically different; namely, those of reference sets χ_{20} and χ_{30}, and χ_{40} and χ_{50}. Furthermore, when performing the same statistical test (Wilcoxon Signed-Ranks test for two paired samples with 0.95 confidence level) between the respective reference sets of the in-sample and out-of-sample effectiveness, we saw that the high performance of the model was not maintained.

Table 8. Average out-of-sample effectiveness of the proposal relative to preference relations for each reference set using six criteria

Reference set	**Average effectiveness**	**Standard deviation**
χ_{10}	0.9270	0.1150

χ_{20}	0.9878	0.0137
χ_{30}	0.9891	0.0151
χ_{40}	0.9946	0.0061
χ_{50}	0.9947	0.008

5.2 Twelve criteria

5.2.1 In-sample effectiveness

Here, we show the results obtained when carrying out experiments with actions described by twelve criteria. Table 9 shows the first results. In this table we can see that the effectiveness of the proposed approach when it worked with actions described by twelve criteria is actually not worse than with actions described by six criteria. The procedure to obtain these results is the same as the one stated in Section 5.1.1.

Table 9. Average in-sample effectiveness of the proposal relative to preference relations for each reference set using twelve criteria

Reference set	Average effectiveness	Standard deviation
χ_{10}	0.9794	0.0018
χ_{20}	0.9992	5.04E-06
χ_{30}	0.9995	1.95E-06
χ_{40}	0.9998	2.15E-07
χ_{50}	0.9996	7.84E-07

5.2.2 Out-of-sample effectiveness

We show now the proposed approach's capability to predict the preference relation between pairs of actions when these are different to the reference sets used by the approach to elicit $\mathcal{P}^*$. Table 10 shows the average effectiveness of the indirect elicitation proposal and its standard deviation for each reference set. The actions are described by twelve criteria and the procedure to obtain the results is stated in Section 5.1.2.

Table 10. Average out-of-sample effectiveness of the proposal relative to preference relations for each reference set using twelve criteria

Reference set	Average effectiveness	Standard deviation
χ_{10}	0.8634	0.1401
χ_{20}	0.982	0.0184
χ_{30}	0.9882	0.0158
χ_{40}	0.9911	0.0125
χ_{50}	0.9937	0.0106

6. Case study: eliciting preferences in portfolio optimization

An application in the context of portfolio optimization was performed in [17], where four underlying criteria (defined as interval numbers) are used to maximize stock portfolios' returns. There, a direct elicitation of the model's parameters is performed. Here, we show the performance of the approach proposed in Section 3 to indirectly elicit the same application's preference parameter values by comparing the results of both elicitation procedures.

6.1 Background to the case study

Portfolio optimization is a problem that requires to select the allocation of resources that maximizes the impact(s) on the DM's objective(s). Many objectives can be considered during portfolio optimization; from them, maximization of the portfolio return is definitely the most common [38, 17]. However, even when this is the only objective being optimized and, because of the uncertainty involved in the estimation of its impact, many underlying criteria are often used to underlie its estimation. We present below some perspectives on how to characterize portfolios; such perspectives can be used as criteria underlying the maximization of the portfolio return.

6.1.1 Confidence intervals

In [38] both the estimation of the portfolio's return and its involved risk are incorporated into the optimization model in the form of confidence intervals. That work allows to consider the DM's attitude in presence of risk by letting the DM to define what he/she considers the appropriate probability of the confidence intervals. It is also able to represent multiple points of the probability distribution in a single criterion; thus, it reduces the necessity of many criteria to satisfactorily describe such distribution.

The main idea of that work is that the portfolios with the best chance to get high results are the ones whose confidence interval are most to the right of their density function. Thus, it is necessary to maximize, in the sense of Equation (1), such confidence intervals. Below, we present a formalization of this idea.

Let $x = [x_1, x_2, x_3 \cdots]^\intercal$ be a portfolio characterized by the proportion of resources assigned to a set of assets (e.g., stocks, funds, …). In the financial literature each x_j could be negative (that is, borrowing resources), but it is common to constrain its values to $[0,1]$. Also let $R(x)$ be the return of portfolio x and $P(\omega)$ the probability of event ω. Then, the confidence interval around such return is defined as $\boldsymbol{\theta}_{\boldsymbol{\gamma}}(x) = [\alpha, \beta]$, such that $P(\alpha \leq R(x) \leq \beta) = \gamma$. The proposal of Solares *et al.* [38] is to select the portfolio that solves the following multicriteria optimization problem:

$$\underset{x \in \Delta}{\text{maximize}} \left(\boldsymbol{\theta}_{\boldsymbol{\gamma_1}}(x), \cdots, \boldsymbol{\theta}_{\boldsymbol{\gamma_k}}(x) \right), \tag{12}$$

where Δ is the set of feasible portfolios fulfilling and k is the number of confidence intervals.

Even when this proposal is interesting, it is highly important for the stock optimization model to allow considering additional information to the statistical one.

6.1.2 Financial fundamental study

One of the most well-known techniques used by stock investors is the so-called financial fundamental analysis. This analysis is performed on the basis of fundamental indicators defined as numerical proportions derived from the companies' financial statements. Some relevant fundamental indicators are shown in Table 11 (cf. [39]), see [17].

Table 11. Some relevant fundamental indicators.

Name	**Description**
Return on assets	Earnings before interest and taxes divided by total assets.
Return on equity	Net income divided by shareholders equity.
Earnings Per Share	Net income minus dividends on preferred stocks all divided by average outstanding shares.
Dividend yield	Annual dividends per share divided by price per share.

Price on earnings	Market value per share divided by earnings per share
Price on book	Stock price divided by all total assets minus intangible assets and liabilities.
Price on sales	Share price divided by revenue per share.
Price on cash Flow	Share price divided by cash flow per share

6.1.3 Financial technical study

The financial technical study considers the behavior of the stock markets using price and volume of transactions to create decision rules about the convenience of buying/selling stocks [40]. For example, if the decision rule related to the jth indicator presents evidence about the price of the ith stock rising, then $it_j^i = 1$ and the DM should support the stock to eventually increase the return of the portfolio; otherwise, $it_j^i = 0$ and there would be no evidence, according to this indicator, showing that the stock's price will rise.

Some of the most outstanding technical indicators with their corresponding rules are presented in Table 12 (e.g., [40, 41, 42]). See [17] for the algebraic specification of the rules associated with the indicators and [42] for graphical illustrations.

Table 12. Technical indicators commonly mentioned in the related literature and their associated inference rules.

Indicator	Name	Associated rule
it_1	Exponential Moving Average (EMA)	Price line crosses above the EMA line.
it_2	Double Crossover (DC)	EMA(11) crosses above the EMA(20) line.
it_3	Rate of change (ROC)	ROC line crosses above 0.
it_4	Relative Strength Index (RSI)	RSI line crosses above 30.
it_5	Moving average convergence/divergence (MACD)	MACD(12,26) crosses above MM(9); where MM is a moving average.
it_6	On Balance Volume (OBV)	OBV is rising simultaneously with price indicating an uptrend.
it_7	Bollinger Band (BB)	Price is simultaneously above LB(20) and below MM(20); where LB is MM(20) minus 2 standard deviations.
it_8	True Strength Index (TSI)	TSI crosses below trigger on oversold region (-25)

In every case, the value within parentheses represents the number of historical periods considered to calculate the value of the corresponding measure.

6.1.4 Portfolio optimization with two surrogated criteria

Using Fuzzy Logic and the interval outranking method, Fernandez *et al*. [17] presented an approach where an optimization problem with virtually any number of criteria can be indirectly addressed by tackling an optimization problem with only two surrogated criteria. Specifically, the interval outranking model is used in that work to model the preferences of the DM and incorporate them into the search process, while Fuzzy Logic is used to create a *non-outranked truth degree* of x in a set of alternatives A. Let us now present the main characteristics of that approach.

For each pair $(x, y) \in A \times A$, it is possible to obtain through the procedure described in Subsection 2.3 a credibility degree of the assertions i) "x outranks y", denoted by $\beta(xSy)$; ii) "y outranks x", denoted by $\beta(ySx)$; and iii) "y dominates x", denoted by $yD(\alpha)x$. Now, let us make $\hat{A} = A - \{x\}$ and $s = card(\hat{A})$, then the *non-outranked truth degree* of x in A is specified using the compensatory fuzzy logic based on the geometric mean[2] as (cf. [17]):

$$NS_A(x) = \sqrt[s]{NS(x, \hat{A})}.$$

Where

$$NS(x, \hat{A}) = \prod_{y \in \hat{A}} \sqrt{(1 - yD(\alpha)x)\left(1 - \sqrt{\left(1 - \left(1 - \beta(ySx)\right)\right)\left(1 - \beta(xSy)\right)}\right)}.$$

According to Fernandez *et al*. [17], a high valued $NS_A(x)$ represents that there is no evidence of other solutions in A being better than x. Therefore, a high valued $NS_A(x)$ is a necessary condition for x to be the best compromise among the criteria. However, it is not a sufficient condition, given that x may be incomparable with many of the solutions in A. Thus, further information is required to ensure superiority of x over the other solutions.

[2] See Espin *et al*., 2014 for the definition of compensatory fuzzy logic based on the geometric mean.

To achieve this, the alternative used in [17] is the net flow score (see [36, 37]), defined as follows. Let $\beta(xSy)$ be an outranking credibility on $x, y \in A \times A$, the net flow score of x is $F_n(x) = \sum_{y \in A-\{x\}}\big(\beta(xSy) - \beta(ySx)\big)$. Since $F_n(x) > F_n(y)$ indicates preference of x over y, the net flow score may be used to select a solution between x and y when both have the same non-outranked truth degree. Therefore, a best compromise solution is represented in [17] as the non-dominated set of solutions obtained from addressing the following problem:

$$\underset{x \in \Omega}{\text{maximize}}\big(NS_A(x), F_n(x)\big), \tag{13}$$

with preemptive priority favoring $NS_A(x)$, where Ω is the set of feasible portfolios.

6.2 Portfolio optimization problem

Fernandez *et al.* [17] presented an original assessment of portfolios using financial analyses (as opposed to only evaluating individual stocks, as it is commonly done). They implemented Fuzzy Logic to build a truth degree for each stock being "good" with respect to the analyses; later on, the degrees of all the stocks within the portfolio are aggregated to produce interval numbers representing the quality indexes of portfolio x being good from the fundamental and technical viewpoints, $\boldsymbol{F}(x)$ and $\boldsymbol{T}(x)$, respectively (cf. [17] to see the construction of both quality indexes).

On the other hand, Solares *et al.*, [38] use two confidence intervals to simulate a risk-averse DM. One of the confidence intervals contains the portfolio return with a 70% of probability, $\boldsymbol{\theta}_{\boldsymbol{\gamma}_{70}}(x)$; whereas the other, contains it with the 99% of probability, $\boldsymbol{\theta}_{\boldsymbol{\gamma}_{99}}(x)$. Furthermore, three constraints are used there to represent a more realistic investment scenario: budget constraint (how much will be invested), non-negativity constraint (no short-sales) and bounds on individual stocks constraints (a maximum limit of what can be invested in each stock).

Here, as in [17] and similarly to [38], we use the following multi-criteria problem to assess our proposed approach described in Section 3:

$$\underset{x \in \Omega}{\text{maximize}}\ \big(\boldsymbol{\theta}_{\boldsymbol{\gamma}_{70}}(x), \boldsymbol{\theta}_{\boldsymbol{\gamma}_{99}}(x), \boldsymbol{F}(x), \boldsymbol{T}(x)\big). \tag{14}$$

Subject to

$\sum x_j = 1 \longrightarrow$ Budget constraint;

$x_j \geq 0 \longrightarrow$ Non-negativity conditions;

$x_j \leq 0.4 \longrightarrow$ Bounds on individual stocks.

Where

x_j is the proportion of resources allocated to the jth stock,

$\boldsymbol{\theta_{70}}(x) = \{[\alpha_{70}, \beta_{70}]: P(\alpha_{70} \leq R(x) \leq \beta_{70}) = 0.70\}$,

$\boldsymbol{\theta_{99}}(x) = \{[\alpha_{99}, \beta_{99}]: P(\alpha_{99} \leq R(x) \leq \beta_{99}) = 0.99\}$,

$R(x)$ is the return of portfolio x,

$\boldsymbol{F}(x)$ is the assessment of portfolio x made by the fundamental analysis,

$\boldsymbol{T}(x)$ is the assessment of portfolio x made by the technical analysis.

6.3 Experimental design

We are interested in testing the performance of the proposed approach to indirectly elicit the parameter values that represent the DM's decision policy. Thus, we compare the ability of its solutions to reproduce the DM's decisions with that obtained using a direct elicitation. Specifically, we use the DM's actual decision policy to compare our solutions with those obtained by Fernandez *et al*. [17] when solving Problem (14).

6.3.1 Instance generation

We first simulate the DM's decision policy and generate sets of instances. For this purpose, we create 20 instances simulating the DM's preferences through the random generation of the parameter vector $\mathcal{P}^i$, $i = 1, \cdots, 20$.

6.3.1.1 Creating reference sets

Each instance uses a reference set T containing 20 portfolios assigned to classes consistently with Constraints (5) to (10). The assignments are also consistent with the corresponding decision maker's $\mathcal{P}^i$, as we could expect if the investors were real. Each portfolio is assigned to one of three classes: $C_3 = Good$, $C_2 = Doubt$ and $C_1 = Bad$. The assignments of portfolios to classes are made guaranteeing (as much as possible) a uniform number of portfolios among the classes according to the procedure described in Subsection 4.1.

6.3.1.2 Simulating investors

Aiming to create a "fair" comparison between the direct elicitation published by Fernandez *et al*. [17] and the approach presented here, we use the same parameter values of the investors simulated in that work. For completeness purposes, we show here the procedure followed in [17] to create these parameter values.

The parameters of $\mathcal{P}$ are generated as follows. First, β_0 (a real parameter) and the bounds of $\boldsymbol{\lambda}$ (an interval number) are randomly generated in the interval (0.5, 0.6). Then, given that the simulated DMs are highly risk-averse (Subsection 6.2), it is plausible to assume that they require the weights related to the confidence intervals criteria to be greater than the weights related to the rest of criteria (cf. [17]); thus, we assign each bound of $\boldsymbol{w_1}, \cdots, \boldsymbol{w_4}$ in [0,1] guaranteeing that i) $\boldsymbol{w_1} + \boldsymbol{w_2} \geq \boldsymbol{w_3} + \boldsymbol{w_4}$, and ii) the weights fulfill constraints (2) and (3). We calculate the core values of the weights as $\widehat{w} = \frac{1}{4}$ and the weight of criterion $\boldsymbol{g_i}$ as $w_i^- = (1 - \omega_i)\,\widehat{w}, w_i^+ = (1 + \omega_i)\,\widehat{w}$, where ω_i is randomly generated in the range [0,0.3]. Finally, the bounds of the i-th veto, $\boldsymbol{v_i} = [v_i^-, v_i^+]$, are assigned as $v_i^- = r_i \breve{v}_i$ and $v_i^+ = r_i \breve{v}_i \widehat{v}_i$, where $r_i = \max_{x \in A}\{\boldsymbol{g_i}(x)\} - \min_{x \in A}\{\boldsymbol{g_i}(x)\}$, $\breve{v}_i$ is randomly selected from the range [0.3,0.38] and $\widehat{v}_i$ is randomly selected from the range [1.1,1.3].

6.3.2 Dataset

Our dataset is composed of the actual monthly returns of the stocks within the very well-known Dow Jones Industrial Average (DJIA) index. We use the period April 2011 to March 2016 to perform the assessment; such period was chosen given its high number of uptrends, downtrends and horizontal market's movements, providing enough diversity of information for assessment purposes. The 20 instances described in the previous subsection are generated in each month of the evaluation period.

We implement a back-testing strategy [43] of 36 months/1 month similarly to [38, 42, 44]: we select thirty-six months for training and one month for validation. The assessment is performed again every month for the rest of evaluation periods selecting a new portfolio each time and accumulating gains/losses.

As in [17], the historical prices are obtained from [45], while the data used to compute the fundamental ratios is obtained from [46]. All this data as well as the specific results described below are available under request.

6.3.3 Portfolio optimization procedure

For each instance of the experiments, our approach uses the following procedure to select the best compromise portfolio. First, Montecarlo simulation is used to estimate the confidence intervals by defining the returns' probability distributions. Such simulation implements 200 statistical points (through the pseudo-random-numbers generator defined by Matsumoto and Nishimura [47]). After this, the assessment of the portfolio using the financial analyses is performed through the process presented in [17]. All this allows to estimate the portfolios' fitness of a set of candidate solutions regarding Problem (14), these candidate solutions are presented to the i-th decision maker in order to create a reference set of holistic decisions. The procedure of Section 3 is then followed using such reference set to create an approximation to the DM's preferences, $\mathcal{P}^{i^*}$. Then, the $\mathcal{P}^{i^*}$ vector is used to aggregate each portfolio's fitness through the procedure described in Subsection 6.1.4. Finally, the nondominated solutions to the subrogated Problem (13) are presented to the DM as the best compromise solutions to Problem (14).

Different metaheuristics have proved to be able to successfully tackle different models of portfolio optimization problems; particularly, swarm intelligence [48, 49] has been found to be robust addressing this problem. However, given that Problem (13) is a lexicographical optimization problem, we use differential evolution as the method addressing it. Several authors have found good behaviors of this metaheuristic to deal with non-linear single-objective optimization problems (e.g., [50, 51]). The population size of differential evolution was set to $ps = 100$; we use the achievement of $gn = 100$ generations as the stopping criterion, while each chromosome is a real-valued vector specifying the proportions of resources to be assigned to the stocks. These parameters were adopted based on some preliminary experiments made by the authors in this and other similar works. Other additional control parameters of differential evolution are (cf. [38, 52]): the crossover probability, CR; the mutation rate, p_m; the differential weight, F; and the distribution index, η. Similarly to the cited works, we defined these parameters as $CR = 1$, $p_m = 1/ps$, $F = 0.5$, and $\eta = 20$. A run of the algorithm consists in obtaining a set of the best portfolios (according to the objectives of Problem (13)) after gn generations. In each run, the initial population involves the random generation of the individuals, and at least one individual is obtained as the best compromise solution to Problem (13). We create a "seed population" of size ps with the solutions found in (up to ps) runs, and a final run is achieved using this seed population as the initial population.

6.4 Results

6.4.1 Assessing our approach's performance in portfolio optimization

6.4.1.1 Assessing the impacts on the criteria

This Section shows the comparison of the portfolios created by our approach with the portfolios created by the approach proposed in [17]. The goal of the comparison is to define how "satisfied" the DM would be when applying the indirect elicitation defined in Section 3 with respect to the direct elicitation performed in [17]. Thus, we calculate the proportion of times that one approach's solutions are preferred to the other approach's solutions. As stated in Sections 6.3.1 and 6.3.2, 20 instances are created in each experimental period and 24 periods are considered in the whole experimentation; thus, 480 experimental points are used to perform the comparison. Given that both approaches use the interval outranking method as a model to characterize the decision maker's preferences, such method is used to evaluate the DM's satisfaction. Furthermore, the DM's real system of preferences simulated in the ith instance, $\mathcal{P}^i$, to compare the solutions.

As an example of the comparison, Table 13 shows the values in $\mathcal{P}^4$ (shown in the simulated DM row) and the corresponding values found by the elicitation procedures for the period April 2014, Table 14 presents the portfolios created by the approaches for the elicited parameters, and Table 15 provides the fitness values of these portfolios in the sense of Problem (14). It may not be obvious that the sets in this table do not necessarily have to coincide. Recall that each DM's decision model in the experiments is simulated, in reality the parameter values may not be defined. Therefore, the main objective of the elicitation of these parameters is not to find their exact values, but to be able to reproduce the already-expressed judgment decisions and allow new ones to be made as the DM would have.

Table 13. Model parameters values of the simulated DM and parameter values found by the elicitation procedures in the period April 2014.

Model parameters	β_0	$\boldsymbol{\lambda}$	$\boldsymbol{w_1}$	$\boldsymbol{w_2}$	$\boldsymbol{w_3}$
Simulated DM	0.54	[0.52,0.57]	[0.57,0.84]	[0.31,0.38]	[0.00,0.06]
Direct elicitation	0.51	[0.65,0.66]	[0.50,1.00]	[0.25,0.46]	[0.00,0.04]
Our approach	0.51	[0.56,0.62]	[0.62,0.85]	[0.10,0.80]	[0.07,0.31]
Model parameters	$\boldsymbol{w_4}$	$\boldsymbol{v_1}$	$\boldsymbol{v_2}$	$\boldsymbol{v_3}$	$\boldsymbol{v_4}$
Simulated DM	[0.02,0.12]	[0.04,0.05]	[0.09,0.10]	[0.33,0.42]	[0.11,0.13]
Direct elicitation	[0.02,0.10]	[0.03,0.04]	[0.08,0.11]	[0.26,0.47]	[0.09,0.09]
Our approach	[0.07,0.57]	[0.03,0.04]	[0.08,0.10]	[0.39,0.53]	[0.12,0.16]

Table 14. Proportion of resources allocated to stocks for April 2014 using the parameter values of Table 13.

Stock	**Direct elicitation**	**Our Approach**
American Express Company (AXP)	0	0
Boeing Co. (BA)	0	0.018
Caterpillar Inc. (CAT)	0	0
Cisco Systems, Inc. (CSCO)	0	0
Chevron Corporation (CVX)	0	0
EI du Pont de Nemours and Co (DWDP)	0.003	0
Walt Disney Company (DIS)	0	0
General Electric Company (GE)	0	0
Goldman Sachs Group Inc. (GS)	0	0
Home Depot, Inc. (HD)	0.05	0

International Business Machines Corporation (IBM)	0.28	0.347
Intel Corporation (INTC)	0	0
Johnson and Johnson (JNJ)	0.007	0
JPMorgan Chase and Co. (JPM)	0	0
Coca-Cola Company (KO)	0	0
McDonald's Corporation (MCD)	0.315	0.376
3M Co. (MMM)	0	0
Merck and Co., Inc. (MRK)	0	0
Microsoft Corporation (MSFT)	0	0
Nike Inc. (NKE)	0	0
Pfizer Inc. (PFE)	0	0.036
Procter and Gamble Co. (PG)	0	0
ATandT Inc. (T)	0.345	0.17
Travelers Companies Inc. (TRV)	0	0
UnitedHealth Group Inc. (UNH)	0	0
United Technologies Corporation (UTX)	0	0
Visa Inc. (V)	0	0.053
Verizon Communications Inc. (VZ)	0	0
Wal-Mart Stores Inc. (WMT)	0	0
Exxon Mobil Corporation (XOM)	0	0

Table 15. Portfolios' fitness built by the approaches in Problem (14).

Criterion	**Direct elicitation**	**Our Approach**
70 percent confidence interval	[-0.0131, 0.0247]	[-0.0087, 0.0270]
99 percent confidence interval	[-0.0417, 0.0414]	[-0.0449, 0.0452]

Fundamental analysis' quality index	[0.3719,0.3720]	[0.3760,0.3761]
Technical analysis' quality index	[0.3533,0.3534]	[0.3393,0.3394]

Let y and x be the portfolios of Table 14, built through the direct elicitation proposed in [17] and our approach, respectively; the credibility indexes of the outranking relations between these portfolios are shown in Table 16.

Table 16. Credibility indexes of the outranking relations between the portfolios shown in Table 15. y is a portfolio created using direct elicitation, x is a portfolio created using our approach.

	Value	**Strict preference**
$\beta(x,y)$	0.55	Yes
$\beta(y,x)$	0.45	No

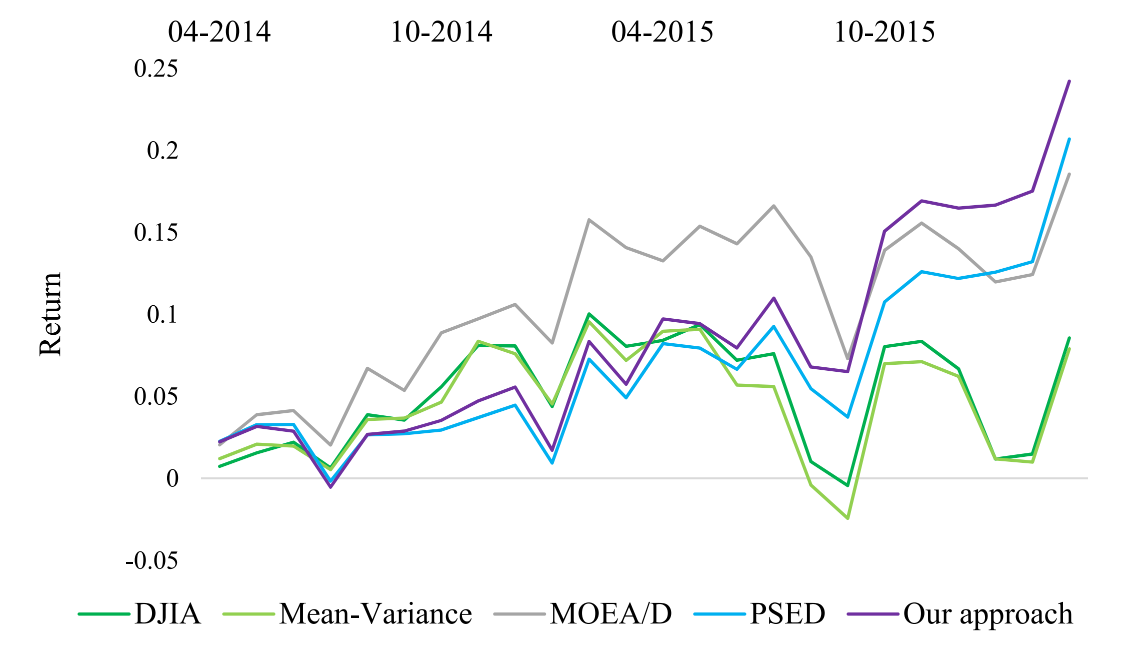
Of course, there is not dominance between the portfolios but, according to Table 16, the solution found by our approach is strictly preferred to the benchmark solution.

Applying the same analysis to the 480 points of comparison, we obtained that at least one of our solutions is strictly-preferred to 10% of the solutions found by the approach presented in [17], while at least one of their solutions is strictly-preferred to 5% of our solutions. When performing Student's t-test on difference of means with the null hypothesis that these proportions are the same, the two-tailed P value equals 0.0032. Hence, this indicates that the difference is statistically significant with a 99% confidence level. This allows us to conclude that the indirect elicitation based on the preference disaggregation analysis proposed in Section 3 produced solutions that better satisfy the DM's preferences.

6.4.1.2 Assessing the portfolio returns obtained

We now use four benchmarks to compare our results in the context of actual portfolio returns: the results presented in Ref. [17], the DJIA index, the Modern Portfolio Theory [53], and the Pareto Front found by MOEA/D [54]. Problem (14) is used as the multicriteria problem to be solved by the optimization methods (i.e., all the approaches but the DJIA index). We present the results in Figure 3.

Figure 3. Accumulative monthly returns of the portfolios built by the benchmark approaches and the portfolios built by our approach.



In Figure 3, the portfolio returns are shown accumulatively in a monthly basis. The accumulative returns regarding the Mean-Variance approach and MOEA/D are the average portfolio returns in their respective Pareto front, whereas the monthly returns published in [45] are used to represent the DJIA index. For the approach in [17] and the one proposed here, we use the average returns of the portfolio returns achieved for the 20 decision makers using first the direct elicitation proposed in [17], denoted as PSED, and then using our indirect elicitation procedure of Section 3, respectively. Undoubtedly, our approach outperformed all the benchmarks; however, it is important to note that even when high accumulative returns are obtained by our approach, which is desired, this is a result of the overall methodology and not only of the elicitation procedure, which is the main proposal of this work. Therefore, the comparison with these benchmarks in this paper is rather complementary. Nevertheless, we highlight that the high returns obtained are caused because the overall methodology i) requires the (simulated) DMs to establish which are the best portfolios, and ii) optimizes portfolios based on the inferred preferences; thus, high average of accumulative returns can be expected. Furthermore, since our approach's results are the average returns of all the simulated DMs, it would be interesting to find out what are the

DMs' systems of preferences that generate the greater returns; however, this is out of the scope of this paper and we defer it as future work.

On the other hand, our approach's outperformance is mainly obtained on the large downtrend of mid-2015; although, in the first periods, it was MOEA/D the one with the best performance. The good performance of MOEA/D is caused by its representation of all the investors' attitudes from the solutions within the Pareto front. That is, the results of MOEA/D in Figure 3 are representing an average accumulative return of averse, neutral and acceptant behaviors in presence of risk. Thus, it can be expected that MOEA/D will have good performance in good market conditions (first periods of the timeframe used) and bad performance in bad market conditions (last periods), as Figure 3 is actually demonstrating. This is not the case for our approach since it is representing only risk-averse investors. Thus, it can be expected that our approach will not have great results in good market conditions, but it will protect itself in bad market conditions, as it is actually happening. This clearly indicates a good representation of the investor behavior in presence of risk.

6.4.2 Validating the ability to reproduce the DM's decisions in portfolio optimization

Here, we evaluate the approach's ability to define a set of preference parameters through which i) it is possible to establish the same preference relations as the ones inferred from the assignments made by the simulated DMs (see the assumptions made in Section 3); and ii) it is possible to suggest the same assignments as the ones made by the simulated DMs.

6.4.2.1 Using preference relations as a benchmark

The average effectiveness measured with the validation procedure of Subsection 4.2 and Equation (11) for the direct elicitation performed in [17] in the 480 experimental points is 84%. While the average effectiveness of the approach proposed here is 96%. The difference is statistically significant according to student's t test. This indicates that the approach proposed here was able to better reproduce the DM's preferences in the context of binary preference relations.

6.4.2.2 Using ordinal classification as a benchmark

Now, the solutions found by the approaches are used to assign portfolios in preferentially ordered classes. So, the quality of the solutions is revised by contrasting the assignments made using the decision models created by the approaches and the decision models of the actual (simulated) DMs. The comparison is based on the new sets of decisions defined in the previous subsection. All the assignments are performed as described in Subsection 4.1. The effectiveness for each approach is defined as the proportion of portfolios that were assigned to the same class as the DMs' assignments in the 480 experimental points.

The effectiveness in the context of ordinal classification of the model presented in [17] is 70%. The effectiveness of our approach is 80%. This difference is statistically significant according to student's t test, indicating that the indirect elicitation procedure proposed in Section 3 is more effective in reproducing the DM's assignments.

7. Conclusions

Our purpose in this paper is to advance the state of the art in the elicitation of the decision maker's system of preferences. Our main contribution is to address the case where the decision maker's preferences are imperfectly known. We assume that imperfect knowledge about such preferences can be coped with by interval numbers. Thus, we proposed a Preference Disaggregation Analysis model based on Interval Theory to indirectly elicit the decision maker's preference parameters.

We extensively assessed the proposed method in several scenarios where simulated decision makers assigned sets of reference actions to preferentially ordered classes. Such reference sets were composed of 10, 20, 30, 40 and 50 actions, respectively. The results shown in Tables 1, 8, 9 and 10 support that the effectiveness of our proposal is high -in most cases superior to 99%- when the actions are described by six and twelve criteria. This effectiveness is measured as the average proportion of coincidences of the binary preference relation established by the simulated DM and the one established by the elicited decision model, for 40 different decision makers' systems of preferences.

Additionally, we presented a case study where the decision maker's preferences are indirectly elicited in order to perform a search towards the most preferred solutions in portfolio optimization. We used real historical data to perform a back-testing assessment where our approach's performance to construct a reliable preference model is verified. The assessment procedure was extensive; we compared our approach's performance with that of four benchmarks in several scenarios. We conclude from Tables 15 and 16 and the discussion of Subsection 6.4.1.1 that our approach demonstrated, in the context of the experiments, that the portfolios it allowed to find are more preferred by investors than other methodologies, including a recent direct elicitation method. The latter conclusion supports findings of important works (e.g., [1, 6, 7, 8]) remarking superiority of indirect elicitation procedures over direct ones. Furthermore, Figure 3 shows how the high preference on the portfolios found implicated a correct representation of the investors' attitude in presence of risk and high accumulative returns. Such returns were superior to that of all the benchmarks.

One interesting future research line is to use new ways to generate the reference sets; e.g., using different sorting methods, different cardinalities of the reference sets and the criteria sets, and/or a diverse number of classes. Assessing the approach's performance on all these new scenarios would allow one to perform a sensitivity analysis and could identify opportunity areas for improving our proposed approach.

Another interesting research line consists in modifying Constraints (5-10) to introduce new preference relations and/or increase the granularity of the system of relations. By doing this, the improved capacity of the method might allow it to better represent the decision maker's system of preferences. However, this would increase the method's complexity and the possibility of new sources of errors. Thus, further assessments would be required to establish plausibility.

The parameters of the genetic algorithm used in Section 3.2 to elicit the preferences of the decision maker as well as the differential evolution used in Subsection 6.3.3 to optimize portfolios were selected according to the authors' previous experiences, some preliminary experimentations and the related literature. It is interesting to know, however, what would be the approach's performance if these parameters would have been defined according to systems experts in finding such parameter values.

In view of the results obtained when assessing the portfolio returns generated by our approach (Subsection 6.4.1.2), it is interesting to study the effect that the investors' preferences implicated on the returns. Recall that our approach's results shown in Figure 3 are the average returns obtained by all the simulated DMs, so, it is interesting to know what are the DM's decision policies that had the best impacts on the maximization of the portfolios returns. Such analysis would require classes of known DM's decision policies and a sufficiently large sample of simulated DMs for each class.

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