Handling uncertainty through confidence intervals in portfolio optimization

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Abstract

The approach proposed here uses evolutionary algorithms combined with interval analysis to optimize the allocation of resources in portfolio optimization. The proposal uses probabilistic confidence intervals to characterize the solutions. Such characterization allows the investor to consider not only the expected impact of the portfolios but also the risk of not obtaining that expected impact. This approach identifies the behavior of the investor in the face of risk and gives her/him support depending on her/his own preferences. Portfolio optimization is performed through one of the most outstanding evolutionary multi-objective approaches, the so-called Multiobjective Evolutionary Algorithm Based on Decomposition (MOEA/D). To the best of our knowledge, this algorithm has not been used in the context of interval analysis. In this work, MOEA/D has been enhanced so that it can deal with chromosomes and fitness values described as interval numbers.

In order to evaluate the proposed approach, an illustrative application in stock portfolio selection is included. We use as our dataset 13 years of historical monthly prices of stocks in the Dow Jones Industrial Average index (DJIA), including those of the 2008 crisis. Besides, we have carried out an extensive evaluation comparing the performance of the proposed approach with respect to the DJIA index, the Markowitz's mean-variance approach, and other more recent approaches. The results show that the proposed approach outperforms the other ones and allow us to conclude that, within the context of our experiments, i) the proposal was effective in the allocation of resources in most of the periods considered (156 scenarios), ii) the approach is appropriate to find portfolios by explicitly considering the DM's attitude facing risk, and iii) interval analysis was a robust measure of risk even for the 2008 crisis.

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1. Introduction

From the 1970s we saw an accelerated evolution in several fields of science, such as finance, computing and decision making. This evolution has resulted in the selection of investment objects modelled as optimization problems which have attracted a lot of interest within operations research. The selection of objects optimizing the impact in the investor's objectives is a multifaceted problem that raises a number of interesting algorithmic and modeling challenges (e.g., risk modeling and multi-criteria optimization). This task is relevant in a variety of contexts, including resources allocation to non-financial market assets, as well as in the context of financial objects such as the selection of stock portfolios.

Even when formulations of stock portfolio selection with multiple objectives are mentioned in the literature (see e.g., Steuer *et al*., 2007; Qi *et al*., 2017; and Saborido, 2016), the conventional stock portfolio selection has as the investor's (decision maker, DM) only objective the maximization of return (profit). A formulation with multiple criteria is usually used in order to maximize this single objective (e.g., Agarwal, 2017). The need for multiple criteria originates from the DM being unwilling to accept that the uncertainty of the return can be fully encompassed in a single real criterion. Not even through a common way of estimating the return such as the expected value. Therefore, the decision maker would want to select the best alternative based on the evaluation of additional information such as the volatility around the expected return (Markowitz, 1952).

The most commonly used measures of this type of uncertainty in the allocation of resources belongs to probability theory (e.g., Markowitz, 1968; Leavens, 1945; Artzner *et al*., 1999; and Rachev *et al*., 2008). The main idea is that uncertainty can be represented by volatility. However, even with the same expected return and volatility, decision makers with different attitudes facing risk can select different alternatives. This implies that, in order to select alternatives that are satisfactory for the DM, his/her attitude facing risk must be explicitly considered. Many of the measures of uncertainty in probability theory do not allow this in a straightforward manner.

The goal of this paper is to propose and validate a portfolio selection approach that overcomes this limitation and offers satisfactory solutions from the DM's perspective. To achieve this, we propose a novel way of modeling both risk and subjectivity of the DM in terms of significant confidence intervals. In this proposal, intervals around the expected return characterize portfolios during the optimization. The optimization is performed through a widely accepted decomposition-based evolutionary algorithm, MOEA/D. This algorithm is modified so that it can deal with chromosomes and fitness values described as interval numbers. Furthermore, we use some ideas from the literature to enhance the algorithm's solutions diversity. The performance of the approach is evaluated in the context of stock portfolio optimization. The dataset used in the validation consists in the return of stocks in the Dow Jones Industrial Average index during the period 1998-2016. Results show clear advantages of the new method with respect to the market index and other approaches from the literature.

The paper is structured as follows: Section 2 offers a background theory that sustains the developed work, namely, portfolio selection theory, interval theory and the description of MOEA/D. Also, this section briefly mentions different methodologies that address the portfolio selection decision. Section 3 describes the approach proposed in this paper. Its validation is carried out in Section 4 through an illustrative application in stock portfolio optimization. Section 5 concludes this work.

2. Some background

2.1. Portfolio selection

The portfolio selection may be divided into two stages. "The first stage begins with observation and experience, and ends with beliefs about the future results of investment objects. While the second stage begins with beliefs about future outcomes and ends with the choice of objects and the proportion of resources allocated to each of them" (Markowitz, 1952). This work deals with the second stage.

A portfolio is a vector $x = [x_1, x_2, \dots, x_n]^T$ in the decision space that specifies the proportions of money to invest in n investment proposals, such that x_i is the proportion to invest in the i-th investment proposal. The image of a portfolio in the criteria space is a vector that represents the impact on k criteria established by the investor (Decision Maker). The portfolio problem is to select the feasible portfolio that maximizes the impact on the criteria, formally:

$$\underset{x\in\Omega}{\text{maximize}}\left(I(x)=\{I_{1(x)},I_{2(x)},\cdots,I_{k(x)}\}\right) \tag{1}$$

where $I_j(x)$ is the impact of portfolio x on criterion j and Ω is the set of feasible portfolios (set of portfolios that fulfill the constraints).

Some common constraints of (1) are the following.

$\sum_{j=1}^{n} x_j = 1 \longrightarrow$ Budget constraint.

$x_j \geq 0\ (j = 1,\cdots,n) \longrightarrow$ Non-negativity/No short sales constraint.

$l_j y_j \leq x_j \leq u_j y_j\ (j = 1,\cdots,n) \longrightarrow$ Bounds on individual proposal constraint.

$\sum_{j=1}^{n} y_j \leq N \longrightarrow$ Cardinality constraint.

$y_j \in \{0,1\}(j = 1,\cdots,n) \longrightarrow$ Auxiliary variables.

where $y_j = 1$ if $x_j > 0$ and $y_j = 0$ otherwise, and l_j, u_j are the minimum and maximum proportion that should be assigned to investment proposal j.

Because of imprecision, vagueness, and/or ill definition, the "true" value of $I_j(x)$ (if we accept the premise that it exists) cannot be accurately known. So, we actually optimize estimations of $I_j(x)$. Nevertheless, when estimating $I_j(x)$ we incorporate uncertainty into the selection process given that it is possible that portfolio x does not generate the expected impact. The literature offers several ways to consider this uncertainty.

Addressing the problem of maximization of return, in 1952 the now called modern portfolio theory was founded by Markowitz in (Markowitz, 1952). The major contribution of that paper was the formalization of the portfolio problem as a multi-criteria problem and the argument that for any expected return, the decision maker should prefer the portfolio with the lowest uncertainty[1]. Particularly, the idea stated in (Markowitz, 1952) to maximize the return of the portfolio is to use two underlying criteria: maximize the expected return and

[1] Although some authors refer to uncertainty and risk as synonyms, in this work we will treat the concept of risk as the uncertainty that negatively affects the decision maker.

minimize the variance of the portfolio. Actually, many forms of the mean-variance optimization exist; for example, the *classic formulation* is the bi-criteria problem optimization:

$$\max_{x \in \Omega} \left(E\big(R(x)\big), -\sigma^2 \left(E\big(R(x)\big) \right) \right).$$

Whereas the so-called *risk aversion formulation* is defined as a single-objective- optimization problem:

$$\max_{x \in \Omega} \left(E\big(R(x)\big) - \frac{\gamma}{2} \sigma^2 \left(E\big(R(x)\big) \right) \right).$$

Where $E\big(R(x)\big)$ is the expected return of portfolio x, $\sigma^2 \left(E\big(R(x)\big) \right)$ is its variance, and γ is a parameter for risk aversion, balancing investor trade-offs in mean-variance space. For most portfolio allocation decisions in investment management applications, the risk aversion is somewhere between 2 and 4 (Fabozzi *et al*., 2007; and Das *et al*., 2010).

The mean-variance approach has been the main idea in most of the theoretical research on the portfolio problem. However, its application in practice has been rather scarce (Kolm *et al*., 2014). The lack of applicability of the Markowitz model in practice is due mainly to the following limitations (see Artzner *et al*., 1999; Greco *et al*., 2013; and Black and Litterman, 1992):

- The model requires the impacts on the objectives to be distributed in a normal way, or the DM to present a utility function that can be described by a quadratic function (Harvey and Siddique, 2000). Yet, empirical evidence suggests that the distributions of the returns typically have heavier tails than those that are implied by the normal distribution, and are often not symmetric with respect to the mean (cf. Kon, 1984; Mills, 1995; Peiro, 1999; and Tay and Premaratne, 2002).
- The risk measure used by the model has some undesirable characteristics. One of them is lack of monotonicity. The mean-variance approach can select as the best alternative a portfolio that is dominated in the return's states of the nature; that is, portfolio y can be selected over portfolio x even when in all possible scenarios the return of x is not worse than the return of y (Greco *et al*., 2013).

- The model presents a high sensitivity to errors in the estimations (Fabozzi *et al.*, 2007; Tüntücü and Konig, 2004). Estimation error has always been acknowledged as a substantial problem in portfolio construction (Scherer, 2007). There are several alternative methodologies to approach the problem. These methodologies range from Bayesian techniques (Black and Litterman, 1992; Frost and Savarino, 1986; Jorion, 1991) to portfolio resampling (Medaglia *et al.*, 2007; Greco *et al.*, 2013; Jorion, 2007).
- The model makes a poor modeling of the DM's attitude facing risk. Given the possibility of not obtaining the expected return, the model must incorporate the DM's subjectivity in the selection process.

Several authors have proposed different alternatives to overcome the limitations of the mean-variance model. For example, Markowitz (1968) proposed to substitute the variance for the semi-variance. This way, some drawbacks related to the variance as the measure of risk are discarded from the model. Nonetheless, other limitations continue present, such as the lack of robustness with respect to the values of the parameters.

Robust optimization (see Ben-Tal *et al.*, 2009; and Soyster, 1973) has been applied by some authors (e.g., Ghaoui *et al.*, 2003; Quaranta and Zaffaroni, 2008; see Kolm *et al.*, 2014) with the intention of solving some problems caused by imperfect knowledge on the values of the criteria. Robust optimization implicitly considers that these values have been estimated with errors and uses an interesting concept called uncertainty set to protect the results of the model against the worst scenarios. This way, the resultant portfolio tends to be more stable and less sensitive to changes of the model's parameters (cf. Fabozzi *et al.*, 2007). Nevertheless, such protection could be considered by the DM as too pessimistic and can result in under performance of the portfolios when the values of the parameters tend to the "true" parameters and/or in situations where the returns of the stocks in the portfolio tend to grow.

With respect to the risk of not achieving the expected returns of the portfolios, some authors have proposed to incorporate higher statistical moments, such as skewness and kurtosis, in order to better describe the probability distribution of the portfolio's return (e.g., Saranya and Prasanna, 2014; Scott and Horvath, 1980; Dittmar, 2002; and Harvey and Siddique, 2000). However, the incorporation to the model of the DM's risk attitude using such statistical specific-knowledge tools is too complicated. To surpass this situation, some authors have used probabilistic quantiles to provide valuable information to the DM (see e.g., Jorion, 2007; Artzner *et al*., 1999; Greco *et al*., 2013). These kinds of approaches can deal with virtually any probability distribution, consider higher statistical moments and use many quantiles in order to better describe the probability distribution of the portfolio's return. However, in the pursuit of a better description of this probability distribution, the quantity of criteria could be so high that it exceeds the cognitive limitations of the DM (cf. Miller, 1956).

In this work, we intend to overcome these limitations by characterizing the portfolios' performance as confidence intervals. This way, the DM can express her/his attitude facing risk in a straightforward and understandable manner, the uncertainty of not achieving the expected return is considered, and there is no need of using many underlying criteria to describe the probability distribution of the returns. Let us now briefly describe the so-called interval theory.

2.2. Interval-based decision aid

The so-called interval analysis theory was originated independently by Sunaga (1958) and Moore (1962). Interval theory's principal concept is the interval number. Such a number represents a numerical quantity whose exact value is unknown. Given this imperfect knowledge about the quantity, a range of numbers is used to encompass all the possible values that the quantity could obtain. In this way, an interval number stands for an indeterminate number that takes its possible value within a set of numbers. Let us consider the quantity ι whose real value lies between bounds i^- and i^+. The interval number for such quantity is set then as $I = [i^-, i^+]$. Any $r \in [i^-, i^+]$ is called a *realization* of I. We can also translate a real number, q, into an interval number as $[q, q]$.

In what follows, let us look at the basic operations of interval numbers. Given the interval numbers $I = [i^-, i^+]$ and $J = [j^-, j^+]$, the following equations represent the addition, subtraction, multiplication and division, of I and J, respectively.

$$I + J = [i^- + j^-, i^+ + j^+],$$

$$I - J = [i^- - j^+, i^+ - j^-],$$

$$I \times J = [\min\{i^- j^-, i^- j^+, i^+ j^-, i^+ j^+\}, \max\{i^- j^-, i^- j^+, i^+ j^-, i^+ j^+\}],$$

$$I \div J = [i^-, i^+] \times \left[\frac{1}{j^-}, \frac{1}{j^+}\right].$$

More recently, Shi *et al*. (2005) proposed a way to determine the order of interval numbers. For instance, suppose we want to determine the order of $I = [i^-, i^+]$ and $J = [j^-, j^+]$. First, we need to find the possibility of I being greater than or equal to J. The possibility function proposed in (Shi *et al*., 2005) is given by

$$p(I \geq J) = \begin{cases} 1 & \text{if } p_{\{IJ\}} > 1, \\ p_{\{IJ\}} & \text{if } 0 \leq p_{\{IJ\}} \leq 1, \\ 0 & \text{if } p_{\{IJ\}} < 0. \end{cases} \tag{2}$$

Where $p_{\{IJ\}} = \frac{i^+ - j^-}{(i^+ - i^-) + (j^+ - j^-)}$.

Furthermore, if $i^+ = i^-$ and $j^+ = j^-$, then

$$p(I \geq J) = \begin{cases} 1 & \text{if } I \geq J, \\ 0 & \text{otherwise.} \end{cases}$$

Let i and j be two currently undetermined realizations from I and J, respectively; $p(I \geq J)$ can be interpreted as a degree of credibility of the statement "once both realizations are determined, i will be greater than or equal to j". This helps the DM to ensure the robustness of $I \geq J$, that is, to have a strong belief on I being not less than J when they are instanced as real numbers (Fernandez *et al*., 2018).

It is easily proved (see Fernandez *et al*., 2018) that Eq. (2) (2)fulfills some interesting properties: let $I = [i^-, i^+]$, $J = [j^-, j^+]$, and $K = [k^-, k^+]$, then:

i. $p(I \geq J) = 0$ if $j^- > i^+$ and $p(I \geq J) = 1$ if $j^+ \leq i^-$.

ii. If $j^- = i^-$ and $j^+ = i^+$, it is said that I is equal to J, denoted as $I = J$. Then $p(I \geq J) = 0.5$.

iii. If $i^- > j^+$, it is said that I is strictly greater than J, denoted as $I > J$. Then $p(I \geq J) = 1$.

iv. If $i^+ < j^-$, it is said that I is strictly lower than J, denoted as $I < J$. Then $p(I \geq J) = 0$.

v. When $p(I \geq J) > 0.5$, it is said that I is strictly greater than J, denoted as $I > J$. When $p(I \geq J) < 0.5$, it is said that I is strictly lower than J, denoted as $I < J$.

vi. If $p(I \geq J) = \alpha_1 \geq 0.5$ and $p(J \geq K) = \alpha_2 \geq 0.5$ then $p(I \geq K) \geq \min\{\alpha_1, \alpha_2\}$.

vii. If $\hat{\imath}$ and $\hat{\jmath}$ are respectively the middle points of the confidence intervals I and J, we have $I > J$ if and only if $\hat{\imath} > \hat{\jmath}$ and $I = J$ if and only if $\hat{\imath} = \hat{\jmath}$.

viii. If $p(I \geq J) = \alpha > 0.5$ then $p(J \geq I) = 1 - \alpha < 0.5$.

Finally, we define the concept of a maximum among a set of interval numbers as follows. Let $\mathcal{B}$ be a set of interval numbers, $b^* \in \mathcal{B}$ is the maximum of $\mathcal{B}$, denoted by $max\{\mathcal{B}\}$, if and only if $p(b^* \geq b) \geq 0.5$ for all $b \in \mathcal{B}$.

2.3. MOEA/D

The conflictive character of criteria in (1) leads to that no point in Ω optimizes all criteria simultaneously. One common way to deal with this situation is through the so-called multi-objective evolutionary algorithms (MOEAs). Most MOEAs are based on the concept of Pareto Dominance (Srinivas and Deb, 1994; Li and Zhang, 2009). Let $u, v \in \mathbb{R}^k$, we say that u dominates v if and only if $u_i \geq v_i$, $(i = 1, \cdots, k)$, and $u_i > v_i$ for at least one i. A portfolio $x^* \in \Omega$ is Pareto Optimal to (1) if there is no portfolio $x \in \Omega$ such that $I(x)$ dominates $I(x^*)$. $I(x^*)$ is then called a Pareto Optimal (criteria) vector. The set of all the Pareto Optimal portfolios is called the Pareto Set (PS) and the set of all the Pareto Optimal criteria vectors is the Pareto Front (PF).

Recently, Fernandez *et al.* (2018) proposed an approach in which Eq. (2) is used to state if there is dominance between two solutions when the values of the criteria are imprecise. In their approach, criteria are described by interval numbers instead of real numbers. Thus, dominance is not crisp, but there is a "degree of credibility", α, of the dominance. Let x and y be two solutions and α a real number; y is α-dominated by x if and only if $\min_{1 \leq j \leq k} p\left(I_j(x) \geq I_j(y)\right) = \alpha \geq 0.5$.

With the goal of aiding in the selection process, MOEAs find only a subset of the PS. Later, the subset is presented to the DM who is in charge of selecting the best alternative according to her own preferences. In order to present a representative subset of alternatives, MOEAs look for a manageable number of Pareto Optimal vectors which are evenly distributed along the PF, and thus are good representatives of the entire PF. The fitness measure in MOEAs based on the Pareto Dominance concept is determined by the individual's Pareto Dominance relations with respect to other individuals. Using this fitness measure alone discourages the diversity of the search (Li and Zhang, 2009). Several other efforts have been made in order to discover complementary fitness measures.

One of the lines in this context is the aggregation of criteria. The idea is that a solution to the original problem could be an optimal solution of a single criterion optimization problem in which the criterion is an aggregation function of all the original criteria. Therefore, the approximation to the PF can be decomposed into a number of single objective optimization subproblems. MOEA/D (Zhang and Li, 2007) is a MOEA that implements this idea. The objective in each of the subproblems that MOEA/D optimizes is an aggregation of all the criteria. Neighborhood relations among these subproblems are defined based on the distances between their aggregation coefficient vectors. Each subproblem (i.e., scalar aggregation function) is optimized in MOEA/D by using information mainly from its neighboring subproblems.

MOEA/D requires a decomposition technique for converting the approximation of the PF of Problem (1) into a number of single objective optimization problems. In principle, any decomposition approach can serve for this purpose. A common approach used in the MOEA/D context is the Tchebycheff method (Miettinen, 1999). A single objective optimization subproblem in this approach is

$$\underset{x\in\Omega}{\text{minimize}}\left(g\left(x\middle|\lambda^j, z^*\right) = \max_{1\le i\le k}\left\{\lambda_i^j \left|I_i(x) - z_i^*\right|\right\}\right) \tag{3}$$

where $\lambda^j = \left(\lambda_1^j, \cdots, \lambda_k^j\right)^T$ is a weight vector of the criteria that satisfies $\lambda_i^j \geq 0$ for all $i = 1, \cdots, k$ and $\sum_{i=1}^{k} \lambda_i^j = 1$. $z^* = (z_1^*, \cdots, z_k^*)$ is the reference point, that is, $z_i^* = \max\{I_i(x) | x \in \Omega\}$ for each $i = 1, \cdots, k$. And $I_i(x)$ is the impact in the i-th objective, as specified in Problem (1).

Following (Li and Zhang, 2009), it is well known that, under mild conditions, for each Pareto optimal portfolio there exists a weight vector such that it is the optimal solution of (3) and each optimal solution of (3) is a Pareto optimal solution of problem (1).

If $\lambda^1, \cdots, \lambda^N$ is a set of weight vectors, then we have N single objective optimization subproblems. If N is reasonably large and $\lambda^1, \cdots, \lambda^N$ are properly selected, then the optimal solutions to these subproblems will provide a good approximation to the PF or PS of Problem (1) (Zhang and Li, 2007).

In the experiments performed in Section 4 we use normalized values of $I_i(x)$. We now show a simple pseudocode of MOEA/D.

Input. N, T

Output. A final population, $\{x^1, \cdots, x^N\}$.

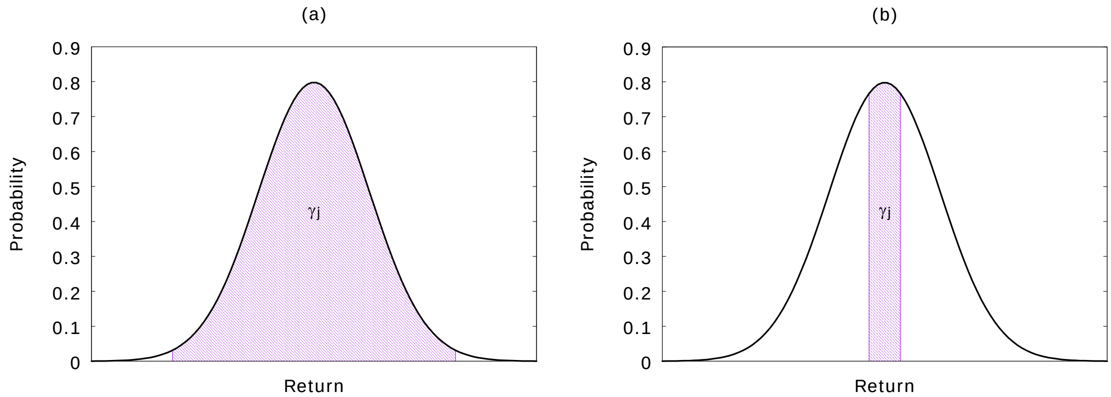
1. For each $i = 1, \cdots, N$, set the indexes of the T closest weight vectors to λ^i (computed through the Euclidean distance) as $B(i) = \{i_1, \cdots, i_T\}$; where $\lambda^{i_1}, \cdots, \lambda^{i_T}$ is known as the neighborhood of λ^i.
2. Generate an initial population $x^1, \cdots, x^N$.
3. Initialize $z^* = (z_1^*, \cdots, z_k^*)$, or a corresponding approximation.
4. For $i = 1, \cdots, N$, do
   a. Randomly select two indexes k, l from $B(i)$, then generate a new solution $\hat{y}$ from x^k and x^l by using genetic operators, and apply a mutation operator to $\hat{y}$.
   b. Apply a problem-specific repair/improvement heuristic on $\hat{y}$ to produce y.
   c. For each $j = 1, \cdots, k$, if $z_j^* < I_j(y)$, then set $z_j^* = I_j(y)$.
   d. For each index $j \in B(i)$, if $g(y|\lambda^j, z^*) < g(x^j|\lambda^j, z^*)$, then set $x^j = y$.
5. If the stopping criterion is satisfied, then stop and output $\{x^1, \cdots, x^N\}$. Otherwise, go to Step **4**.

3. Our proposal

3.1 Problem formalization

Let $E(R(x))$ be a random variable that represents the expected return of portfolio x and $P(\omega)$ the likelihood that event ω will occur. Then, $\theta_\gamma(x) = [\alpha, \beta]: P(\alpha \leq E(R(x)) \leq \beta) = \gamma$ is called the confidence interval around the expected return. Now, let us suppose that it is possible to consider multiple confidence intervals $\theta_{\gamma_j}(x)$. In this proposal, each γ_j is selected by the DM according to her own preferences. This allows us to incorporate her attitude facing risk in the following manner. First, suppose a highly risk-averse DM[2]. This DM would feel more satisfied of making a decision based on intervals with a high probability of containing the real return. That is, she considers more valuable the information about the worst scenarios that could happen when selecting x. So, she would select high values for γ_j looking for protection against those scenarios, see Figure 1 (a). On the other hand, if the DM is lowly risk-averse, she would prefer to make a decision based on intervals that tend to the expected return, see Figure 1 (b).

Figure 1. **a)** Representation of an interval with a high value of γ_j; information required by a highly risk-averse DM. **b)** Representation of an interval with a small value of γ_j; information required by a lowly risk-averse DM.



Thus, our proposal is to select the feasible portfolio that maximizes a set of confidence intervals around the expected return:

$$\underset{x\in\Omega}{\text{maximize}}\left(\theta(x) = \left\{\theta_{\gamma_1}(x), \cdots, \theta_{\gamma_k}(x)\right\}\right). \tag{4}$$

where $\theta_{\gamma_j}(x) = \left\{[\alpha_j, \beta_j]: P\left(\alpha_j \leq E(R(x)) \leq \beta_j\right) = \gamma_j\right\}$, and Ω is the set of feasible portfolios. $P(\omega)$ is the likelihood that event ω will occur and can be approximated through the frequentist approach.

[2] Without loss of generality, we say that a DM is risk-averse when she is reluctant to support an alternative that has uncertain expected return instead of supporting an alternative with less uncertainty but with possible expected minor consequences.

It is important to note that the maximization referred to in Problem (4) is not necessarily related to the wideness of the intervals, but it is based on the possibility function defined in (2). That is, portfolios with the rightmost confidence intervals are preferred.

Each $\theta_{\gamma_j}(x)$ is easily understandable even for an investor (DM) without a sophisticated technical preparation, since it represents the probability that the return of portfolio x actually lies within interval $[\alpha_j, \beta_j]$. This is not the case if one considers the technical criteria used in the mean-variance approach (Markowitz, 1952) or higher statistical moments (Saranya and Prasanna, 2014; Scott and Horvath, 1980; Dittmar, 2002; Harvey and Siddique, 2000).

Moreover, the investor has the capability of defining as many criteria per objective as (s)he wishes; thus, the information describing the distribution is enough to satisfy his/her requirements. Nevertheless, we believe that no more than one or two criteria are sufficient to satisfy his/her requirements for information. This is because of the definition of each $\theta_{\gamma_j}(x)$, which allows the approach to encompass multiple points of the probability distribution in a single criterion. That is, in a single criterion we know with a given probability that the portfolio's return can be *any* of the values within the corresponding interval. This is not possible in point estimators, where the statistical information relies on only one point. In some approaches (see e.g., Greco *et al*., 2013; Markowitz, 1952; Markowitz, 1968) each criterion represents a single point of the probability distribution, so a better description of the distribution requires a higher number of criteria.

3.2 Interval-based portfolio optimization system

3.2.1 Algorithm

Considering several confidence intervals as evaluation criteria leads to a multi-criteria optimization problem that can be solved using a multi-criteria decision aiding method. We use MOEA/D in our system to find acceptable solutions to Problem (4). Nevertheless, even when MOEA/D is widely recognized as the most prominent MOEA based on aggregation of criteria, it has a poor diversity when dealing with instances having complicated PFs (Li and Zhang, 2009). To overcome this shortcoming, we use some improvements introduced by Li and Zhang in (Li and Zhang, 2009); namely, the setting of a maximal number of solutions replaced by each child solution, a selection of parents involving not necessarily only the neighborhood of the candidate solution, and a crossover that involves more than two parents. By using these new mechanisms, the exploration ability of the search can be improved. Moreover, an enhancement of the original algorithm needs to be performed in order to deal with parameters described as interval numbers. We present now the characterization of the system implemented in this work, which is inspired on (Zhang and Li, 2007) and (Li and Zhang, 2009).

Input:

- Problem (4), see Section 4 for an illustrative application;
- 100 generations as the stopping criterion;
- $n_r = 2$: the maximal number of solutions replaced by each child solution;
- $\delta = 0.9$: probability of selecting parents only from the neighborhood (instead of the whole population);
- $N = 100$: the number of the subproblems;
- $T = 20$: the number of weight vectors in the neighborhood of each weight vector.

Output:

- Approximation to the PS: $\{x^1, x^2, \cdots, x^N\}$;
- Approximation to the PF: $\{\theta(x^1), \theta(x^2), \cdots, \theta(x^N)\}$.

Step 1 Initialization

Step 1.1. Work out the T closest weight vectors to each weight vector. (Recall that a weight vector is a vector $\lambda^i = \left(\lambda_1^i, \cdots, \lambda_k^i\right)^T$ that allows to weigh the k criteria in the i-th subproblem and satisfies $\lambda_j^i \geq 0$ for all $j = 1, \cdots, k$ and $\sum_{j=1}^{k} \lambda_j^i = 1$.) For each $i = 1, \cdots, N$, set $B(i) = \{i_1, \cdots, i_T\}$ where $\lambda^{i_1}, \cdots, \lambda^{i_T}$ are the closest weight vectors to λ^i.

Step 1.2 Generate an initial population $x^1, x^2, \cdots, x^N$ by uniformly randomly sampling from Ω. Set $FV^i = \theta(x^i)$ for $i = 1, \cdots, N$.

Step 1.3 Initialize $z^{\text{nad}} = \left(z_1^{\text{nad}}, \cdots . z_k^{\text{nad}}\right)$ by setting $z_j^{\text{nad}} = \min\limits_{1 \leq i \leq N} \alpha_j^i$, and $z = (z_1, \cdots . z_k)$ by setting $z_j = \max\limits_{1 \leq i \leq N} \beta_j^i$. Where α_j^i and β_j^i are the lowest and highest attainable return of solution i in the j-th criterion; that is, $\theta_j(x^i) = [\alpha_j^i, \beta_j^i]$. z^{nad} and z are used in the update step in order to normalize the fitness values of the criteria.

Step 2 Update

For $i = 1, \cdots, N$, do

Step 2.1 Selection of Mating/Update Range:

Define the population P, from where the offspring will be produced, as the neighborhood of λ^i (with a probability of δ) or as the whole population (with a probability of $1-\delta$): uniformly randomly generate a number $rand$ from [0, 1], then set

$$P = \begin{cases} B(i) & \text{if } rand < \delta \\ \{1, \cdots, N\} & \text{otherwise} \end{cases}$$

Step 2.2 Reproduction: Set $r_1 = i$ and randomly select two indexes r_2 and r_3 from P, then generate a solution $\hat{y}$ from x^{r_1}, x^{r_2} and x^{r_3} using genetic operators, finally perform a mutation operation on $\hat{y}$ with probability $p_m = 0.01$ to produce a new solution y.

Step 2.3 Repair: If an element of y is out of the boundary of Ω, go to **step 2.1**.

Step 2.4 Update of z: For each $j = 1, \cdots, k$, if $z_j^{\mathrm{nad}} > \alpha_j$, then set $z_j^{\mathrm{nad}} = \alpha_j$; and if $z_j < \beta_j$, then set $z_j = \beta_j$, where $\theta_j(y) = [\alpha_j, \beta_j]$.

Step 2.5 Update of Solutions: Set $c = 0$ and do the following:

1) If $c = n_r$ or P is empty, go to **Step 3**. Otherwise, randomly pick an index i from P.
2) Normalize $\theta_j(x^i) = [\alpha_j^i, \beta_j^i]$ for $j = 1, \cdots, k$, such that $\theta_j^{\mathrm{norm}}(x^i) = [\alpha_j^{i,\mathrm{norm}}, \beta_j^{i,\mathrm{norm}}]$: Make $\alpha_j^{i,\mathrm{norm}} = \frac{\alpha_j^i - z_j^{\mathrm{nad}}}{z_j - z_j^{\mathrm{nad}}}$ and $\beta_j^{i,\mathrm{norm}} = \frac{\beta_j^i - z_j^{\mathrm{nad}}}{z_j - z_j^{\mathrm{nad}}}$.
3) Calculate $g(x^i|\lambda^i)^{\mathrm{norm}} = \max_{1 \le j \le k}\{[1 - \beta_j^{i,\mathrm{norm}}, 1 - \alpha_j^{i,\mathrm{norm}}]\lambda^i\}$.

4) If $p\left(g\left(y|\lambda^i\right)^{\text{norm}} \geq g\left(x^i|\lambda^i\right)^{\text{norm}}\right) \geq 0.5$ then set $x^i = y$, $FV^i = \theta(y)$ and $c = c + 1$.
5) Remove i from P and go to 1).

Step 3 Stopping Criterion If the stopping criterion is satisfied, namely the number of iterations is 100, then stop and output $\{x^1, x^2, \cdots, x^N\}$ and $\{F(x^1), F(x^2), \cdots, F(x^N)\}$. Otherwise go to **Step 2**.

Since it is often computationally expensive to find the exact ideal point z^*, we use z, which is initialized in **Step 1.3** and updated in **Step 2.4** of the algorithm, as a substitute for z^* in (3). Furthermore, we use a *nadir* point, z^{nad}, to perform the normalization. Given that our implementation of MOEA/D has to deal with interval numbers instead of real numbers, we consider the lower and upper bound of the intervals to define z^{nad} and z. We update this reference through the lowest and highest possible value attainable by the confidence interval. We use 100 generations given previous experience of some of the authors in similar optimization problems. Finally, the individuals present in the population at the last generation are considered as the result of one run. The individuals generated in 20 runs are introduced in a pool, from where the non α-dominated solutions are selected as the final approximation to the PF (see Section 2.2 to see the definition of α-dominance).

3.2.2 Chromosome representation

In this work, the chromosomes or individuals (alternatives of solution) in the population are represented by a string of n real numbers; that is, each gene in the chromosome is a real number. This is a common way of representing portfolios given its practicality in the representation of the resources assigned to stocks. The i-th gene in the chromosome specifies the proportion of resources assigned to stock i. Individuals are represented by a string composed of n positions as shown in Figure 2.

Figure 2. Individual encoding

x_1	x_2	$\cdots$	x_{n-1}	x_n

3.2.3 Selection

Since g, as defined in (3), is continuous on λ, the optimal solutions of the neighboring subproblems should be close in the decision space. MOEA/D exploits the neighborhood relationship among the subproblems for making its search effective and efficient (Zhang and Li, 2007). Nevertheless, as stated above, MOEA/D shows poor diversity in its solutions when facing complicated PFs (Li and Zhang, 2009). One reason of this problem is that the maximal number of solutions replaced by a child solution could be as large as T, the neighborhood size. This implies that a single solution may replace most of the current solutions to its neighboring subproblems. As a result, diversity in the population could be significantly reduced. In this work we intend to overcome this limitation by, as was done in (Li and Zhang, 2009), letting the child solution replace no more than n_r solutions of the current population. Furthermore, the solutions replaced may not necessarily be in the neighborhood of the child solutions. But the proposed approach allows three different parent solutions to be randomly selected from the whole population with a probability of $1 - \delta$. The values T, n_r and δ are 20, 2 and 0.9, respectively, as in (Li and Zhang, 2009).

3.2.4 Crossover

We create one child solution from the information contained in the three parents selected. The crossover procedure works as follows. Let qG_1, qG_2, qG_3 be the quantity of genes satisfying $x_i > 0$ in parent 1, parent 2 and parent 3, respectively. The idea is that the parents provide similar proportions of gene material to the offspring. So, the number of genes satisfying $x_i > 0$ in the child solution is up to $qG_C = \frac{qG_1+qG_2+qG_3}{3}$ and each parent gives $\frac{qG_C}{3}$ randomly chosen genes to the child solution.

3.2.5 Mutation

The mutation operation simply consists in swapping two randomly chosen genes of the child solution. With the intention of a further improvement in the search exploration phase, the probability of mutation is $p_m = 0.01$.

3.2.6 Repairing process

In Section 2.1, we mentioned that the DM can consider several types of constraints during the optimization process, depending on her own preferences. The illustrative application shown in the next section considers only three of these constraints; namely, the budget constraint, the non-negativity constraint and the bounds on individual stocks constraint. We can ensure fulfillment of the last two constraints easily in the chromosomes construction (e.g., gene j is randomly selected in $[l_j, u_j]$ if $l_j \geq 0$). Nevertheless, the fulfillment of the budget constraint is not straightforward. The following techniques have been revised here with this goal, see (Consigli *et al.*, 2017; Gill *et al.*, 1981).

- Discard infeasible solutions. Given that our approach has only few constraints that are not often hit, the simplest approach is to 'throw away' new infeasible solutions. That is, if a solution violates a constraint, we just select another one.
- Normalization. We can introduce mechanisms to correct solutions that violate constraints. For example, dividing every element in x by the sum of the elements of x ensures that all weights sum to unity.
- Ordering. Add the value of the ordered elements of x until the sum, σ, is greater or equal to one. Assign to the last element considered in the previous sum the value $\sigma - 1$. Finally, assign zero to the elements not considered in the sum.
- Penalization. Whenever a constraint is violated, we add a penalty term to the objective function and we consequently degrade the quality of the solution.

In preliminary experiments, we have found that simply discarding the infeasible solutions is the most suitable method in terms of time performance and quality of the solutions. Hence, this is the method we use to satisfy the constraints in the experiments shown below.

3.2.7 Fitness evaluation

As stated above, the Tchebycheff method is used to aggregate the criteria (see Section 2.3), its computation is given in (3). In **Step 2.5** of the algorithm we use this aggregation as the fitness of the solutions. In order to estimate the value of each criterion before the aggregation, a Montecarlo simulation is performed. This simulation allows us to find an approximation to the probability distribution of a given portfolio's return.

A simulation point consists in the random generation of the portfolio's return. This return is calculated as the weighted sum of the return of the stocks in the portfolio. Whereas the return of a stock is generated also in a random process where the historical returns of the stock are sampled. The "actual" return of the stock is randomly generated from a sample of the historical returns of the stock, where the probability of obtaining the actual return is given by the sample. The distribution of one thousand simulation points is assumed to be the real probability distribution of the return of the portfolio, and the confidence intervals are taken from this distribution according to the preferences expressed by the DM (see Section 3.1).

A pseudo-random numbers generator known as *Mersenne Twister* (MT) is used (Matsumoto and Nishimura, 1998) in the simulation. The algorithm of the generator has the following characteristics (Matsumoto and Nishimura, 2004):

- A period of $2^{19,936}$.
- An equidistribution property of 623 dimensions.
- Quick generation. (Although dependent of the system architecture, the authors report that MT is sometimes faster than the ANSI-C standard library.)

3.2.8 Final selection

The number of solutions in the generated approximation to the PF may still be high enough to make the decision difficult. Thus, a final selection procedure needs to be performed.

Let portfolios x and y be two points in the PF of problem (4). Furthermore, let's assume[3]

$$\theta_{70}(x) = [0.0171, 0.0479], \theta_{99}(x) = [-0.1067, 0.0553], \text{ and}$$

$$\theta_{70}(y) = [-0.0004, 0.0220], \theta_{99}(y) = [-0.0428, 0.0311].$$

Hence, $p\big(\theta_{70}(x) \geq \theta_{70}(y)\big) = 0.91$ and $p\big(\theta_{99}(x) \geq \theta_{99}(y)\big) = 0.42$. Though both x and y are non α-dominated (in the sense described in Sections 2.2 and 2.3), portfolio x is arguably better than portfolio y. The final selection procedure followed here is based on the previous argument: let A be the set of solutions' performances in the approximation to the PF, similarly to the α-dominance concept described in Sections 2.2 and 2.3, we say that x is non-dominated in A with degree β if and only if[4] $\min\limits_{y \in A-\{x\}} \left\{ \max\limits_{1 \leq j \leq k} \left\{ p\left(\theta_{\gamma_j}(x) \geq \theta_{\gamma_j}(y) \right) \right\} \right\} = \beta$. The proposed

[3] Portfolios x and y are two actual portfolios obtained in the experiments below.

[4] Here, we consider the minimum to represent conjunction, and the maximum to represent disjunction. In this sense, $\max\limits_{1 \leq j \leq k} \left\{ p\left(\theta_{\gamma_j}(x) \geq \theta_{\gamma_j}(y) \right) \right\}$ is interpreted as the credibility of x being no worse than y. Then, $\min\limits_{y \in A/x} \left\{ \max\limits_{1 \leq j \leq k} \left\{ p\left(\theta_{\gamma_j}(x) \geq \theta_{\gamma_j}(y) \right) \right\} \right\}$ is interpreted as the credibility of x being non-dominated by any solution in the PF.

approach takes the portfolio that maximizes β as the best portfolio among the portfolios in the approximation to the PF.

4. System validation

The selection of financial portfolios refers to the analysis of financial objects (e.g., stocks, funds, bonds) to allocate resources that maximize the impact on the objectives of the decision maker. Let us show now an application of the proposed approach to the allocation of resources among a set of stocks with the objective of maximizing the return.

4.1 Selecting confidence intervals

Stock portfolio selection consists of two stages (see Zopounidis *et al.*, 2015; Kiris and Ustun, 2012; and Xidonas *et al.*, 2009): stock valuation and portfolio optimization. The first stage chooses "the best" subset of stocks, while the second stage assigns a proportion of money to each of the chosen stocks. Here, we focus on the second stage.

Let $p_0^i, p_1^i, p_2^i, \cdots, p_{T_0}^i$ be the historical prices of stock i in $T_0 + 1$ periods of time (we use monthly prices here), the T_0 historical rate of returns (returns in the consecutive) of the stock are given by $r_t^i = (p_t^i - p_{t-1}^i)/p_{t-1}^i$; $t = 1, 2, \cdots, T_0$. It is relatively easy to construct an approximation to the probability distribution of the stock's return through r_t^i. Several authors (e.g., Markowitz, 1952; Markowitz, 1968; Greco *et al.*, 2013) use this probability distribution to forecast the return of the portfolio in the period $T_0 + 1$ and a risk measure representing the uncertainty of not achieving the forecasted return. Although there are different ways of forecasting the return of the portfolio, the most common approach in the literature is the expected return, $E(R(x))$. While the most commonly used risk measures are based on the volatility around $E(R(x))$.

Following Section 3.1, once the approximation to the distribution has been constructed we can obtain as many confidence intervals around the expected return as needed. For this illustrative example, we simulate a highly risk-averse DM that requests information on two intervals, one having 70% and the other 99% probability around the expected return. For this case, the approach must solve the problem given by

$$\underset{x \in \Omega}{\text{maximize}}(\{\theta_{70}(x), \theta_{99}(x)\}) \tag{5}$$

subject to

$\sum x_j = 1 \longrightarrow$ Budget constraint.

$x_j \geq 0 \longrightarrow$ Non-negativity conditions

$x_j \leq 0.4 \longrightarrow$ Bounds on individual stocks.

$(j = 1, \cdots, n)$.

Where

$\theta_{70}(x) = \{[\alpha_{70}, \beta_{70}]: P(\alpha_{70} \leq R(x) \leq \beta_{70}) = 0.70\}$, and

$\theta_{99}(x) = \{[\alpha_{99}, \beta_{99}]: P(\alpha_{99} \leq R(x) \leq \beta_{99}) = 0.99\}$.

Later, we compare the solutions to Problem (5) with the solutions obtained by a less risk-averse DM. Thus, we now simulate a lowly risk-averse DM that requests information on intervals having 30% and 50% probability around the expected return. For this case, the approach must solve the problem given by

$$\underset{x \in \Omega}{\text{maximize}} \big(\theta_{30}(x), \theta_{50}(x)\big) \tag{6}$$

subject to

$\sum x_j = 1 \longrightarrow$ Budget constraint.

$x_j \geq 0 \longrightarrow$ Non-negativity conditions

$x_j \leq 0.4 \longrightarrow$ Bounds on individual stocks.

$(j = 1, \cdots, n)$.

Where

$\theta_{30}(x) = \{[\alpha_{30}, \beta_{30}]: P(\alpha_{30} \leq R(x) \leq \beta_{30}) = 0.30\}$, and

$\theta_{50}(x) = \{[\alpha_{50}, \beta_{50}]: P(\alpha_{50} \leq R(x) \leq \beta_{50}) = 0.50\}$.

4.2 Experimental design

A market index is a way of measuring the value of a section of the stock market. More specifically, it is an aggregated value that is produced by combining various stocks of the market section. Since the market indexes arise from a mathematical construction, it is not possible to invest directly in them. However, it is a tool used by investors to describe the market and compare the performance of the portfolios.

There are two streams of thought to create stocks portfolios (Soe and Poirier, 2016; Gorgulho *et al*., 2011; Maginn *et al*., 2007): passive management and active management. The first stream states that "the past movement or direction of return of a stock, or of the market in general, cannot be used to predict its future movement" (Malkiel, 1973). And that in trying, the DM spends resources that in the long run can be rather detrimental. As a result, "there has been an accelerating trend in recent decades to invest in passively managed investment funds based on market indexes, known as index funds" (Soe and Poirier, 2016). As index funds try to replicate index holdings, they eliminate the need -and therefore many costs- for the research involved in active management. This makes indexes one of the main benchmarks in the selection of stock portfolios (see e.g., Gorgulho *et al*., 2011; Xidonas *et al*., 2009; Lim *et al*., 2014).

On the other hand, active management depends on analytical research, estimations, and the judgment and experience of the decision maker to form portfolios. The objective of active management is to outperform a reference index (Gorgulho *et al*., 2011). It can be done through the incorporation of decision-maker preferences, estimation of portfolio return (e.g., expected return), measurement of risk of not obtaining the return estimated (e.g., standard deviation), and the purchase of undervalued stocks (e.g., through financial indicators).

4.2.1 Dataset

We describe in this section the dataset used in our experiments. In these experiments, the performance of the approach is compared with that of a highly important market index, namely, the Dow Jones Industrial Average, DJIA. The DJIA index contains the stocks of 30 of the largest companies in the United States.

Following (Soe and Poirier, 2016), the main contraindication of using market indexes as benchmarks is that the profitability of portfolios is often compared to popular indexes such as DJIA, regardless of portfolio size or classification of its stocks. Most investors expect to reach or exceed the yields of these indexes over time. The problem with this expectation is that they are at a disadvantage because they are not "comparing apples to apples". That is, there is no guarantee that the characteristics of the stocks in the portfolio coincide with the characteristics of the stocks contained in the index. We avoid this trap by incorporating into the portfolio only the stocks of the index being considered as benchmark.

We use the historical monthly returns of the stocks in the DJIA index for the period April 1998-March 2016 (see e.g., Xidonas *et al*., 2009) to perform a back-testing strategy (cf. Ni, and Zhang, 2005). Each investment horizon goes from April of the current year to March of the following year because the yearly financial information is publicly available for the stock market in March (Lim, 2014). The reason for using this particular period is because the index shows upward and downward trends, so there are multiple scenarios to validate the approach. While the duration of the period is an approximate average of the horizons used in several articles of the literature revised. Finally, similarly to (Lim *et al*., 2014) and (Gorgulho *et al*., 2011), we use a sliding time window of 60 months/1 month. That is, we use five years for model training (e.g., we obtain metrics of the data set from April 1998 to March 2003) and one month for validation (e.g., we use the metrics obtained to create a portfolio and estimate its monthly performance in April 2003). The process is then repeated for each period of one month (in a sliding window manner) until the end of the evaluation period (see e.g., Gorgulho *et al*., 2011). In other words, we consider a buy and hold strategy (B & H), where we select the best stock portfolio of the current month by solving Problem (5) or Problem (6) and using the historical metrics of the previous five years. This portfolio is maintained over a one-month investment horizon. Each time we start a new investment horizon, we review the stock portfolio (i.e., select a new distribution of resources among the stocks) according to the corresponding horizon's valuation.

As done in other works (see e.g., Lwin *et al*., 2017; Almahdi *et al*., 2017; Cesarone *et al*., 2013; Gorgulho *et al*., 2011; Zhu *et al*., 2011), the historical monthly prices of the stocks and index were downloaded from the Yahoo! Finance database (Yahoo, 2017). DJIA index updated its listed stocks several times during the period considered. Thus, the data retrieving process starts by finding out the corresponding stocks to a specific year. The configuration of the historical data downloaded from the database is *Date, Open, High, Low, Close, Volume, and Adj. Close.* We use the *Close* parameter to calculate the returns. All data used in this work is available for consultation upon request.

4.3 Results

In this section, the results of the proposed approach are shown. First, we provide the results obtained when solving Problem (5); that is, when the DM is highly risk averse. Later, we show the results obtained when solving Problem (6); that is, when the DM is lowly risk averse. For both situations, we compare the solutions of the proposed approach using three benchmarks: the DJIA index, the mean-variance model (see Subsection 2.1), and the results in (Gorgulho *et al*., 2011). The latter comparison is valid given that the dataset used in that work is a subset of the one used in this paper.

4.3.1 Selecting portfolios with high risk aversion

4.3.1.1 Comparing with Dow Jones Industrial Average index

Tables 1 and 2 show the portfolios obtained by the approach that produce the most extreme returns when solving Problem (5); the worst return, obtained in February 2009, and the best return, obtained in July 2009. Both tables show the stocks in the portfolio, the actual return of these stocks and the proportion of resources assigned by the approach to each stock. Finally, both tables show also the return of the portfolio and the corresponding confidence intervals obtained in the simulation. The portfolio shown in Table 1 produces an actual return of $R(x) = -0.1243$, while its 70% confidence interval is [0.0004, 0.0141] and its 99% confidence interval is [-0.0389, 0.0188]. Note how the actual return of the portfolio is far from being within the confidence intervals. This is due to the high volatility produced by the crisis. Interestingly, the worst return obtained by the approach was not during the market crisis of October 2008. The return of DJIA index in this month was -0.1406 (the lowest in the whole period considered) while the return obtained by the approach was -0.0797. We believe this situation is due to a consistency in the losses of the stocks before October 2008. That is, the stocks of the DJIA index with the greatest losses (AA, AIG, CAT, …) had presented highly negative returns before October 2008, making the confidence intervals of the portfolios containing those stocks to have low values and the approach to neglect most of them. It was not until February 2009 that the volatility and the lack of consistency in the historical returns of the stocks had repercussions on the performance of the approach. But even when this was the worst performance of the approach in the whole period, it was actually not too far from the return of the DJIA index in the corresponding month, -0.1172. The aggressive recovery of the stocks in the following months allowed the approach to find its best performance of the entire period, 0.0986, in April 2009. This return was greater than each return produced by the index in the thirteen years.

Table 1. Portfolio created by the proposed approach in February 2009 when solving Problem (5).

$R(x) = -0.1243$ $x_{70}(x) = [0.0004, 0.0141]$ $X_{99}(x) = [-0.0389, 0.0188]$		
Stock	**Return**	x_i
Alcoa Corp (AA)	-0.2003	0
American International Group Inc (AIG)	-0.6719	0
American Express Company (AXP)	-0.2791	0
Boeing Co. (BA)	-0.2569	0
Bank of America Corporation (BAC)	-0.3997	0
Citigroup, Inc (C)	-0.5775	0
Caterpillar Inc. (CAT)	-0.2023	0
Chevron Corporation (CVX)	-0.1391	0.13
EI du Pont de Nemours & Co (DD)	-0.1829	0
Walt Disney Company (DIS)	-0.1891	0
General Electric Company (GE)	-0.2984	0
Home Depot, Inc. (HD)	-0.0297	0
HP Inc. (HPQ)	-0.1646	0.198
International Business Machines Corporation (IBM)	0.0041	0
Intel Corporation (INTC)	-0.0124	0
Johnson & Johnson (JNJ)	-0.1333	0
JPMorgan Chase & Co. (JPM)	-0.1043	0
Coca-Cola Company (KO)	-0.0438	0
McDonald's Corporation (MCD)	-0.0994	0.316
3M Co. (MMM)	-0.1549	0
Merck & Co., Inc. (MRK)	-0.1524	0.056
Microsoft Corporation (MSFT)	-0.0556	0
Pfizer Inc. (PFE)	-0.1557	0
Procter & Gamble Co. (PG)	-0.1161	0
AT&T Inc. (T)	-0.0345	0
United Technologies Corporation (UTX)	-0.1492	0
Verizon Communications Inc. (VZ)	-0.0449	0
Wal-Mart Stores Inc. (WMT)	0.0450	0
Exxon Mobil Corporation (XOM)	-0.1122	0.3

Note: The actual return of the portfolio in the above Table, $R(x)$, corresponds to the lowest return obtained in the whole period 2003-2016.

Table 2. Portfolio created by the proposed approach in July 2009 when solving Problem (5).

$R(x) = 0.0986$ $x_{70}(x) = [-0.0214, 0.0488]$ $X_{99}(x) = [-0.1024, 0.1522]$		
Stock	**Return**	x_i
Apple Inc. (AAPL)	0.1384	0
American Express Company (AXP)	0.2190	0
Boeing Co. (BA)	0.0096	0
Caterpillar Inc. (CAT)	0.1205	0
Cisco Systems, Inc. (CSCO)	0.3335	0.192
Chevron Corporation (CVX)	0.1802	0
EI du Pont de Nemours & Co (DD)	0.0486	0
Walt Disney Company (DIS)	0.2073	0
General Electric Company (GE)	0.0767	0
Goldman Sachs Group Inc. (GS)	0.1433	0
Home Depot, Inc. (HD)	0.0978	0
International Business Machines Corporation (IBM)	0.1203	0.395
Intel Corporation (INTC)	0.1294	0
Johnson & Johnson (JNJ)	0.1631	0
JPMorgan Chase & Co. (JPM)	0.0720	0.039
Coca-Cola Company (KO)	0.1331	0
McDonald's Corporation (MCD)	0.0385	0
3M Co. (MMM)	-0.0423	0.374
Merck & Co., Inc. (MRK)	0.1734	0
Microsoft Corporation (MSFT)	0.0733	0
Nike Inc. (NKE)	-0.0105	0
Pfizer Inc. (PFE)	0.0620	0
Procter & Gamble Co. (PG)	0.0863	0
Travelers Companies Inc. (TRV)	0.0560	0
UnitedHealth Group Inc. (UNH)	0.0495	0
United Technologies Corporation (UTX)	0.0483	0
Visa Inc. (V)	0.0436	0
Verizon Communications Inc. (VZ)	0.0297	0
Wal-Mart Stores Inc. (WMT)	0.0069	0
Exxon Mobil Corporation (XOM)	0.1384	0

Note: The actual return of the portfolio in the above Table, $R(x)$, corresponds to the highest return obtained in the whole 2003-2016.

Figure 3 shows the returns obtained by the approach and by the Dow Jones Industrial Average index in the period 2003-2016. The difference between these results is shown in Figure 4. Although this figure shows that there are several occasions where the difference is against the proposal (bars below zero), the number of times and magnitude of difference when the new approach outperforms the index is greater. Figure 5 confirms this through the accumulative return. Recall that the allocation of resources is performed on a monthly basis (the investment in maintained during one month and the returns are obtained and compared at the end of the month; later, the portfolio is reconfigured and a new allocation is performed), which implies that each of these figures actually provides 156 comparisons between the proposal and the reference index.

Figure 3. Returns produced in the period 2003-2016 by the DJIA index and the proposed approach when solving Problem (5).

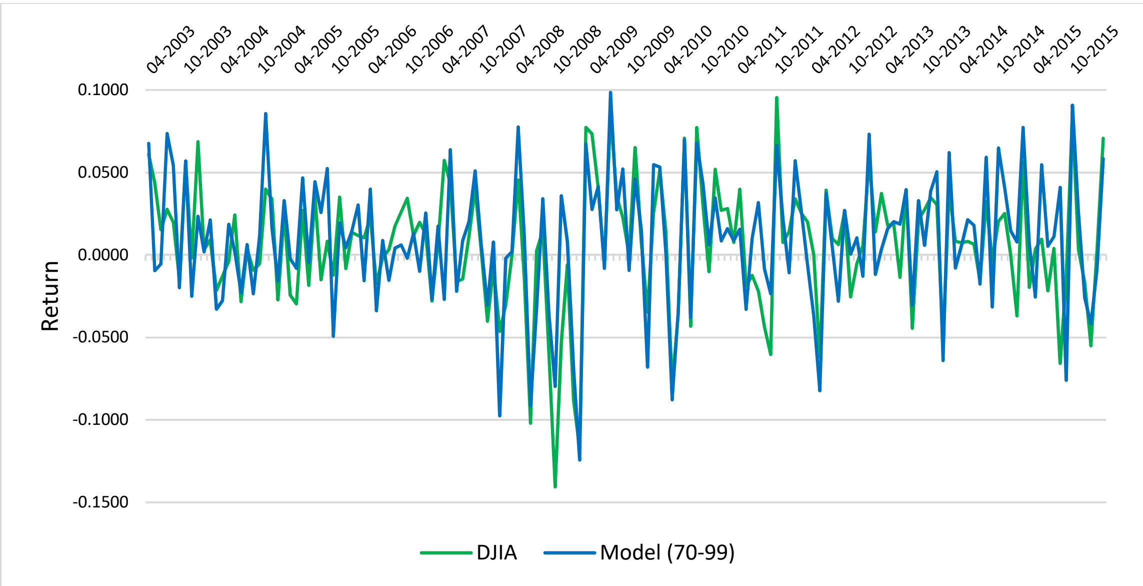


Figure 4. Difference of the returns obtained by the proposed approach when solving Problem (5) and the DJIA index in the period 2003-2016.

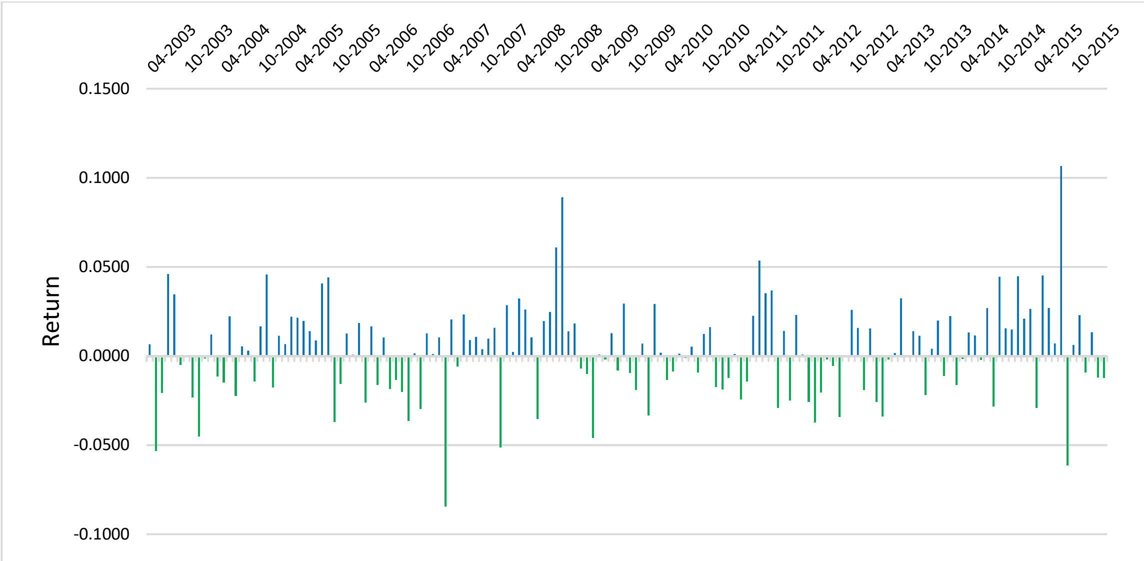
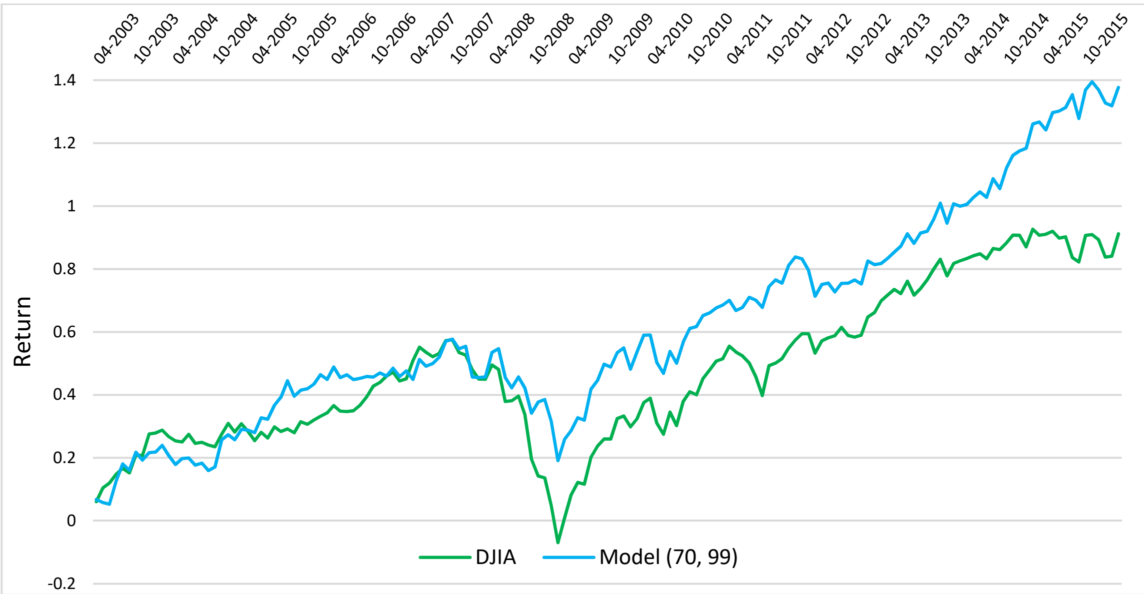


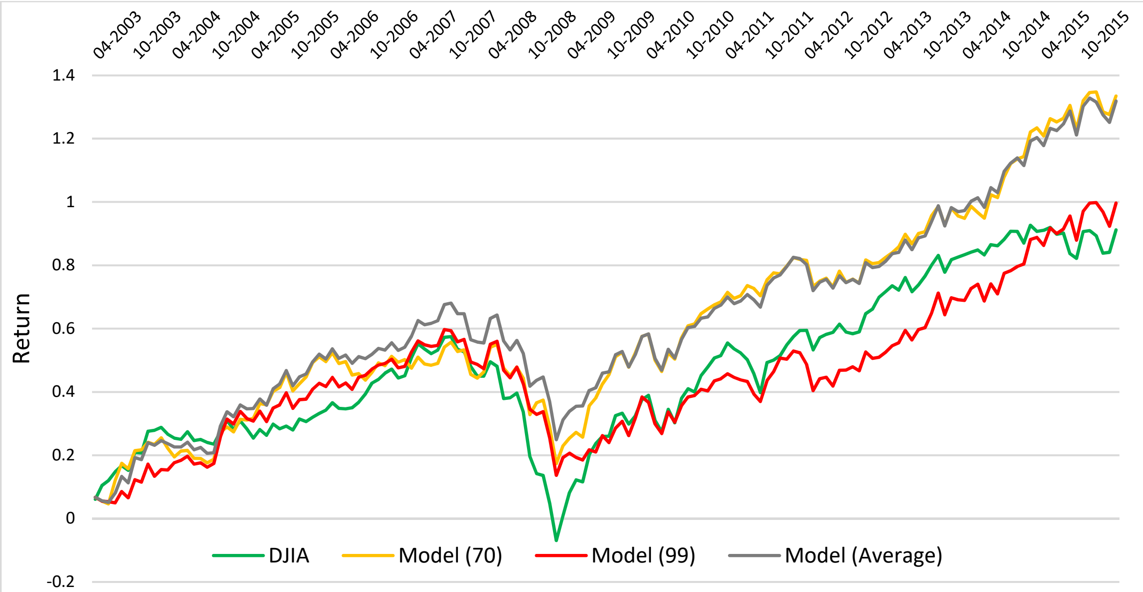
Figure 5. Accumulative return produced in the period 2003-2016 by the DJIA index and the proposed approach when solving Problem (5).



It is interesting to highlight that the decrease of the approach's accumulative return in the period Mar/2008-Feb/2009 is 0.27, while the decrease of the DJIA index in the same period is 0.52. This implies that the approach was more robust than the DJIA index in the worst crisis of the last years.

With the aim of analyzing the quality of the PF, we show the performance of its extremes (i.e., the portfolio that maximizes criterion θ_{99} and the portfolio that maximizes criterion θ_{70}) and the average of its solutions' performances. Figure 6 shows i) the accumulative return of the DJIA index (*DJIA*); ii) the accumulative return of the portfolio maximizing criterion θ_{70} (*Model (70)*); iii) the accumulative return of the portfolio maximizing criterion θ_{99} (*Model (99)*); and iv) the average accumulative return of the portfolios in the PF (*Model (Average)*).

Figure 6. Description of the PF obtained by the proposed approach when solving Problem (5).



4.3.1.2 Comparing with the mean-variance model

Now, we compare the results of the proposal with those of the mean-variance (MV) model (see Markowitz, 1952). Figure 7 shows the comparison. In this Figure, the approximation to the PF made by the MV *classical formulation* in a given month is described using the average of all the returns within the whole approximation (*MV (average)*). We also used the *risk aversion formulation* and, similarly to (Das *et al*., 2010), defined the high-risk aversion parameter as $\gamma = 4$ (see Subsection 2.1 for the definition of both formulations).

Figure 7. Accumulative return produced in the period 2003-2016 by the Mean-Variance model and the new approach when solving Problem (5).

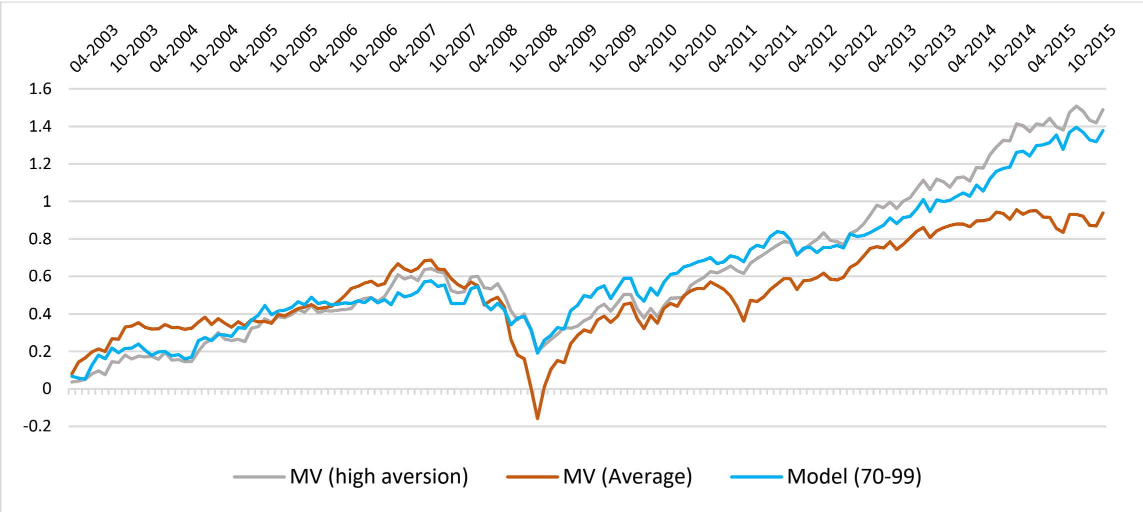
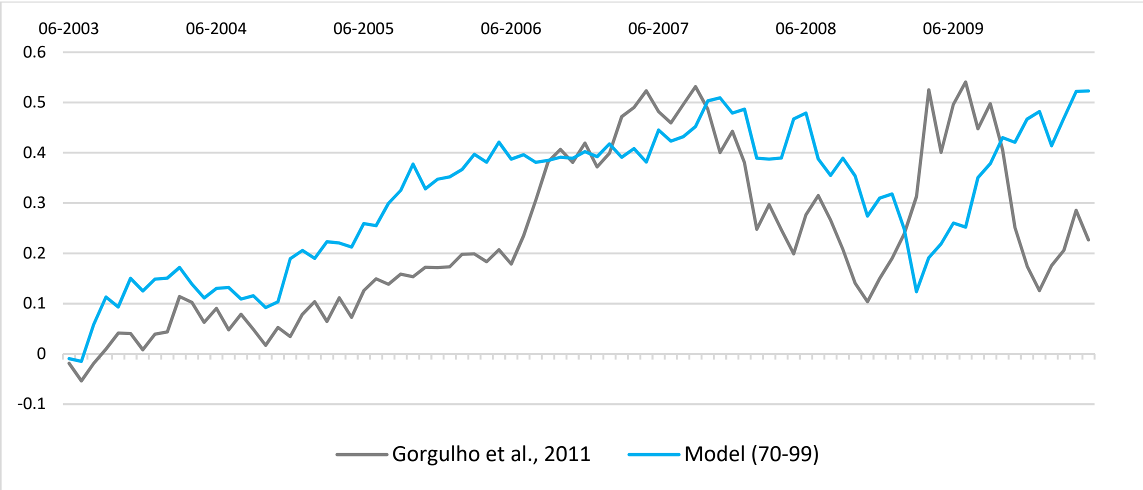


Figure 7 shows superiority of the proposed approach over the mean-variance when using the classical formulation. Nevertheless, it ends up being outperformed by the mean-variance when using the risk aversion formulation. An interesting result shown in this figure is the fall suffered by the mean-variance model during the 2008 crisis. In this period, its fall is appreciably steeper than that of our proposal. This, together with the also steeper rise of the mean-variance model, might indicate lack of representativeness of the DM's risk behavior.

4.3.1.3 Comparing with Gorgulho *et al*., 2011

Finally, we compare the performance of our proposal with that of a recently published work; namely, (Gorgulho *et al*., 2011) whose dataset is a subset of the one used here. Particularly, they used the returns of stocks within the DJIA index in the period 2003-2009. Hence, the comparison is in that specific period. The comparison of the results is shown in Figure 8.

Figure 8. Accumulative return produced in the period 2003-2009 by (Gorgulho *et al*., 2011) and the proposed approach when solving Problem (5).



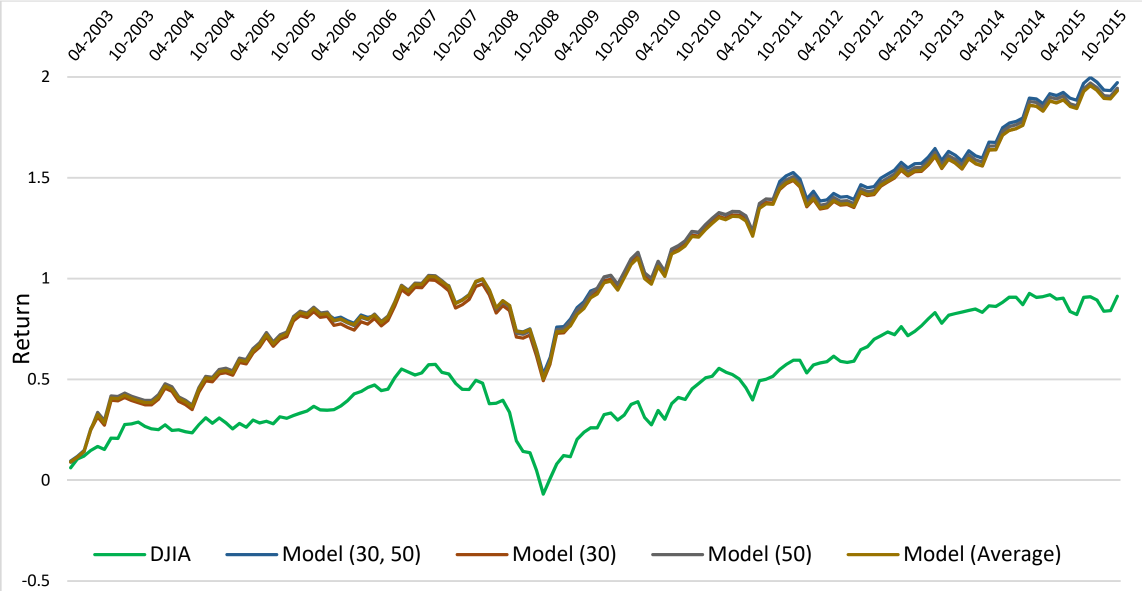
4.3.2 Selecting portfolios with low risk aversion

4.3.2.1 Comparing with Dow Jones Industrial Average index

We now present the results of the approach when solving Problem (6).

Figure 9 shows i) the accumulative return of the DJIA index (*DJIA*); ii) the accumulative return of the portfolio with the highest non-dominance degree from the PF, (*Model (30, 50)*); iii) the accumulative return of the portfolio maximizing criterion θ_{30} (*Model (30)*); iv) the accumulative return of the portfolio maximizing criterion θ_{50}, (*Model (50)*); and v) the average accumulative return of the portfolios in the PF, (*Model (Average)*);

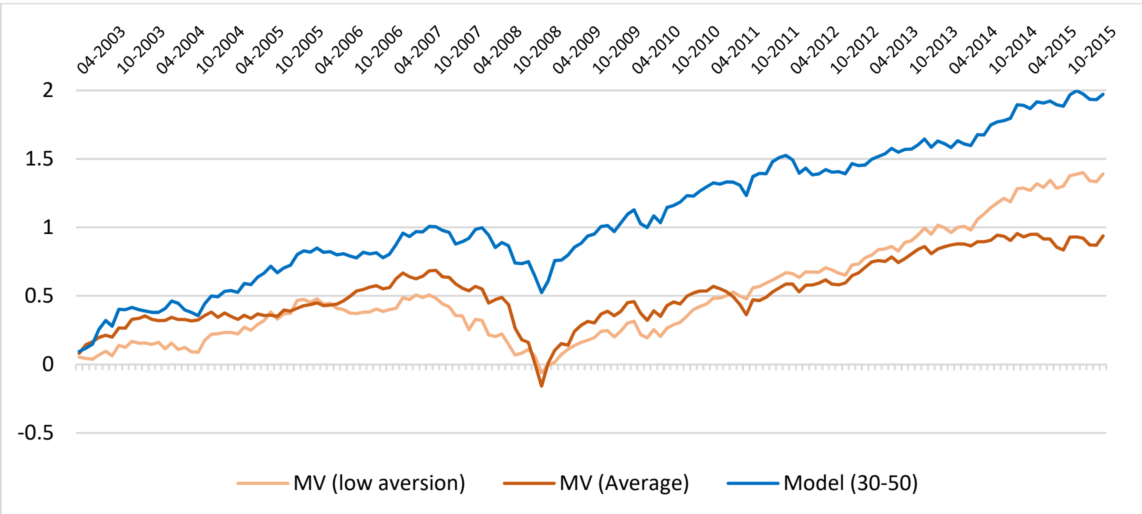
Figure 9. Performance of some solutions in the PF obtained by the proposed approach when solving Problem (6).



4.3.2.2 Comparing with mean-variance model

Figure 10 presents a comparison between the performance of the solutions found by the proposal when solving Problem (6) and the mean-variance model in its classic and risk aversion formulations. Here, similarly to (Das *et al.*, 2010), we defined the low risk aversion as $\gamma = 3$.

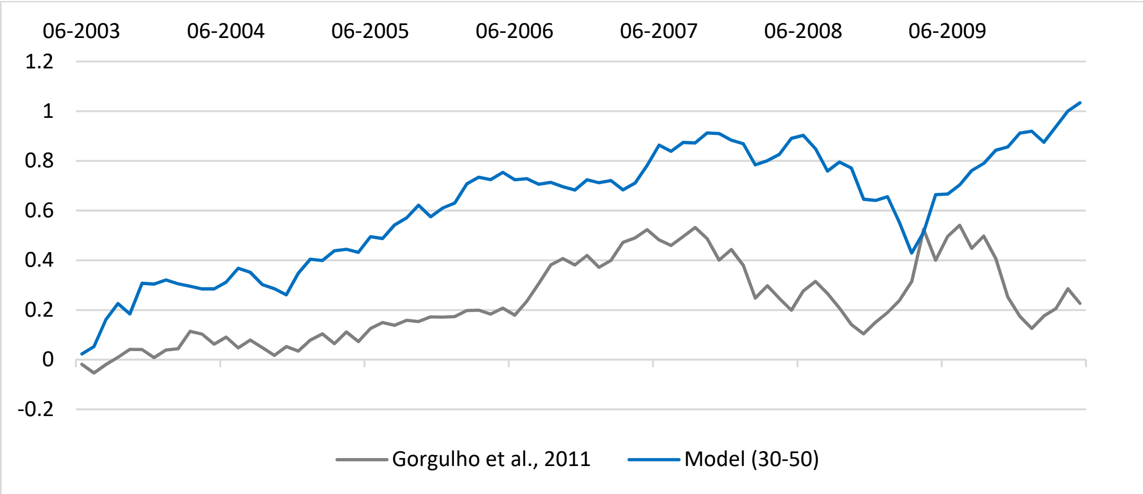
Figure 10. Accumulative return produced in the period 2003-2016 by the Mean-Variance model and the proposed approach when solving Problem (6).



4.3.2.3 Comparing with Gorgulho *et al.*, 2011

Figure 11 shows the comparison between the solutions of the proposal and the results provided by Gorgulho *et al.* (2011).

Figure 11. Accumulative return produced in the period 2003-2009 by (Gorgulho *et al.*, 2011) and the proposed approach when solving Problem (6).



4.3.3 Discussion

It is evidently that the performance of the approach when solving Problem (6) is better than the performance of the approach when solving Problem (5). This situation is due to the general uptrend of the market and the conservativism of the approach when solving Problem (5) that prevents it from taking advantage of the trend.

In order to support this claim, we develop an analysis of the last year of the crisis (Mar/2008-Feb/2009) and the subsequent year to the crisis (Feb/2009-Jan/2010) that allows us to see the effects of risk aversion embodied by both problems. We selected these years because they present the steepest fall and rise of the whole period.

From March 2008 to February 2009,

- the accumulative return of the portfolio from the PF that maximizes the 99% confidence interval fell from 0.47 to 0.14 (a difference of 0.33);
- the accumulative return of the portfolio from the PF that maximizes the 70% confidence interval fell from 0.46 to 0.17 (a difference of 0.29);

- the accumulative return of the portfolio from the PF that maximizes the 50% confidence interval fell from 0.91 to 0.51 (a difference of 0.40);
- the accumulative return of the portfolio from the PF that maximizes the 30% confidence interval fell from 0.90 to 0.49 (a difference of 0.41).

From February 2009 to January 2010,

- the accumulative return of the portfolio from the PF that maximizes the 99% confidence interval raised from 0.14 to 0.38 (a difference of 0.24);
- the accumulative return of the portfolio from the PF that maximizes the 70% confidence interval raised from 0.17 to 0.58 (a difference of 0.41);
- the accumulative return of the portfolio from the PF that maximizes the 50% confidence interval raised from 0.51 to 1.10 (a difference of 0.59);
- the accumulative return of the portfolio from the PF that maximizes the 30% confidence interval raised from 0.49 to 1.08 (a difference of 0.59).

(As a reference, the DJIA index fell 0.52 and raised 0.37.)

Hence, in these periods there was, although not clearly, a tendency to decrease losses in the downtrend as the probability of the intervals increases, and to increase profits in the uptrend as the probability of the intervals decreases. This indicates a correct modeling of the DM's conservativism.

We also see that, in general, the solutions with the best performance are those with the highest non-dominance degree.

Finally, we can see that the performance of the portfolios generated by the approach, and particularly those generated by solving Problem (6), clearly outperforms not just the Dow Jones Industrial Average index but also the performance of some portfolios built by other researchers in the literature (e.g., Markowitz, 1952; Gorgulho *et al*., 2011; Keçeci *et al*., 2016; Hochreiter, 2015).

5. Conclusions

We presented here an approach where confidence intervals around the expected returns are used as criteria to select portfolios. The optimization procedure is performed on the basis of the so-called interval analysis theory. Accordingly, we enhance a widely accepted multi-objective evolutionary algorithm based on decomposition, MOEA/D, to deal with parameters defined as interval numbers. Furthermore, we implement some improvements to increase the diversity of the evolutionary algorithm. The results show that these enhancements allow the evolutionary algorithm to satisfactorily deal with parameters described as intervals.

An extensive validation of the system was performed, where out-of-sample historical data from the stocks in the Dow Jones Industrial Average index was used to perform a back-testing strategy. The system was compared in 156 scenarios against the index, the classical and risk aversion formulations of the mean-variance optimization, and a recently published work (Gorgulho *et al*., 2011). We used two confidence intervals in the optimization process as criteria to represent the investor's behavior facing risk. And two different behaviors were simulated. First, a highly risk-averse investor, and later a lowly risk-averse investor.

The results shown in Figures 4-11 allow to conclude that the approach is effective in the construction of portfolios when the objective is maximization of return. The approach outperformed all the benchmarks in most of the 156 scenarios, giving a considerably better accumulated return after an optimization with 13 years of historical data. These results are strong out-of-sample evidence that confidence intervals provide useful characterizations of the portfolios' returns and their volatility. Furthermore, we confirm with these results that an active management can in fact be achieved and that greater advances should be sought in this stream of thought.

Comparable with the importance of the previous results, the analysis performed in Subsection 4.3.3 allows us to see that the approach presents robustness in the more critical period of the last years, specifically March 2008-February 2009. This is a crucial result in the experiments because the behaviors of the investors modeled are risk-averse. Our results confirm that i) the approach is adept to find portfolios by explicitly considering the DM's attitude facing risk, being conservative when the behavior is highly risk averse and taking good advantage of the uptrends when the behavior is lowly risk averse; and ii) confidence intervals were a useful risk measure in the 2008 crisis, since they helped to reduce losses in the period.

However, when considering the results of the whole period (156 months), we see that more actual returns of the portfolios than expected fell outside their respective confidence intervals, thus reducing the performance of the approach. This effect might be mitigated when confidence intervals are around alternative estimators other than the expected return, or when a more precise analysis of the probability within the intervals is made. A validation of the approach using different types of estimators of return, their combination and a method to scrutinize the probability within the intervals is deferred as future work.

Finally, even when the portfolios constructed by the approach offer acceptable confidence intervals, some of these portfolios are more preferred by the investor. The approach proposed in this work is not capable of identifying such portfolios, because the approximation to the Pareto Front gives us non-dominated solutions and we have not incorporated the investor's preferences on the objectives of Problem (5) and Problem (6). Thus, a way to incorporate the investor's preferences is also needed. This will be addressed in a new paper which is currently in preparation.

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