A Review of Features and Limitations of Existing Scalable Multi-Objective Test Suites

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***Abstract*—In multi-objective optimization, a scalable test problem is one that can be formulated for an arbitrary number of objectives. Scalable test problems evaluate the conceptual foundations of the so-called many-objective evolutionary algorithms. As an important class of problems, scalable test problems should contemplate a wide variety of features allowing us to evaluate and judge specific components of many-objective evolutionary algorithms. This, in fact, should promote the development of new strategies and/or methods in the design of many-objective optimization approaches. For this reason, the study of features and difficulties of this class of problems, plays a salient role in the development of many-objective approaches. As a result, a number of multi-objective scalable test problems have been proposed in recent years. In this paper, we present a review of features and limitations of existing multi-objective test problems formulated in continuous and unconstrained search spaces. We examine some features observed in some test problems which have not been properly discussed before. Additionally, we summarize a list of features and recommendations that should be considered in the design of scalable multi-objective test instances. Then, we preset a review of the state-of-the-art scalable test suites, including their features and limitations according to the recommended guidelines discussed herein. Finally, some possible paths for future research in this area are briefly discussed.**

I. Introduction

MULTI-OBJECTIVE evolutionary algorithms (MOEAs) have become an efficient and flexible tool to solve the so-called multi-objective optimization problems (MOPs). Notwithstanding the wide variety of evolutionary approaches that have shown success in solving problems with two and three objectives, several studies have revealed that most of the existing evolutionary approaches become inefficient and even impractical as the number of objectives increases [37], [28]. Optimization problems having more than three objectives are commonly referred to as many-objective optimization problems, and their solution with MOEAs has motivated a significant amount of research, as they appear in a wide number of real-world applications.[1] This has motivated the design of new strategies capable of solving in an efficient manner many-objective optimization problems, see for example [69], [15], [72]. In this regard, many-objective optimization is considered today as a very active research area.

[1]As an example of this class of problems, interested readers are referred to [51], [63], [52].

Researchers working on continuous many-objective optimization have adopted scalable multi-objective test problems—i.e., problems that can be formulated for an arbitrary number of objectives—to analyze the working principles of many-objective approaches. Different characteristics of scalable test problems have facilitated the study and evaluation of different mechanism into many-objective approaches. This, in fact, has motivated the design of scalable test problems with a large variety of features. As a consequence, a number of scalable test problems were proposed since the early days of many-objective optimization, see for example [18], [27]. Many are the criticisms towards the use of artificial test problems, as some researchers claim that such problems do not reflect the actual features of real-world problems. However, the properties of artificial test problems vary significantly between each other and they can be complemented among different problems. In this way, many-objective approaches can be tested on different problems with different features and the well-performing approaches become good candidates to solve a real-world problem. For this reason, a multi-objective test problem should include a variety of characteristics that simulate the properties observed in real-world problems.

In this paper, we review the recommendations and characteristics for the construction of continuous and unconstrained test problems that are available in the specialized literature. Particularly, we focus on [27], where a vast list of recommendations and features is well analyzed and discussed. However, due to the inherent advances in evolutionary multi-objective optimization, and as part of our contribution, we examine some features observed in some test problems which had not been explicitly discussed before. As a consequence, we summarize an extended list of characteristics and recommendations that should be considered when designing continuous and unconstrained scalable multi-objective test instances. In addition to this, we present a review of state-of-the-art scalable test problems including their features and limitations according to the features discussed in this paper. With this, we attempt to reflect some of the areas in which more research is needed and we establish such areas as possible paths for future research.

The rest of the paper is organized as follows. Section II describes the design approaches that have been used for multi/many-objective test problems, and introduces the recommendations and features suggested for the construction of such test instances. In Section III, a comprehensive review of scalable test suites is presented. Section IV exposes the features and limitations of the existing test suites. Finally, in Section V, we provide our conclusions and some possible paths for future research.

II. Multi-Objective Test Problems: Design Approaches, Recommendations and Features

A. Design Approaches

In this section, we describe the three different techniques which have been adopted in the construction of multi-objective test problems [18].

1) Multiple single-objective approach: This is an intuitive method that combines a number of single-objective optimization problems to formulate a multi-objective model. This strategy was extensively adopted in the early days of evolutionary multi-objective optimization research, see for example [59], [67], [65]. The main disadvantage of this approach is that the *Pareto set* (PS) and the *Pareto front* (PF) are unknown, and depending of the single-objective functions, they can be very difficult to state. This, in fact, complicates the analysis of results and the comparison of MOEAs may become unfair. Nonetheless, this methodology has been recently adopted to formulate new multi-objective test problems [4].

2) Bottom-Up approach: The bottom-up approach [18] is a flexible method that has facilitated the design of multi-objective test problems. In this approach, the *Pareto optimal front*, the *objective space* and the *decision space* are separately constructed. Concretely, the decision variables are splitted into two groups: "*position*" and "*distance*" parameters. The Pareto optimal surface is constructed by parametric functions (*position functions*) whose inputs are the position parameters. The objective space is stated by constructing an extreme boundary surface parallel to the Pareto optimal surface, so that the hyper-volume bounded by these two surfaces constitutes the attainable objective space. Finally, each decision variable vector is mapped into objective space. This task is carried out by defining linear/nonlinear functions where the inputs are the distance parameters. Such functions (known as *distance functions*) establish the distance of the objective vectors to the PF. Therefore, the difficulty to approximate solutions to the PF depends directly on the difficulty of solving such distance functions. Because of its flexibility, the bottom-up approach has been successfully employed in the construction of multi-objective test problems, particularly in the design of scalable test problems. However, most of the test suites adopting the bottom-up approach assume that position and distance parameters are completely uncorrelated—i.e. they can be easily identified—which is something hardly seen in real-world problems.

3) Constraint surface approach: This method was introduced to construct constrained multi-objective test problems [18]. Unlike the bottom-up approach that starts from a pre-defined Pareto optimal surface, the constraint surface approach first states the overall search space. Second, a number of linear/non-linear constraints involving the objective function values is added, thus erasing part of the objective space (i.e., restricting the search space). Finally, by defining linear/non-linear objective functions, the decision variable space is mapped into the objective space.

B. Recommendations and Features

The construction of multi-objective test problems should satisfy some requirements and should include characteristics aimed to evaluate specific components of MOEAs. In particular, when a test instance possesses different characteristics, the test problem should evaluate the robustness of a MOEA, i.e., the capability of a MOEA to solve a test problem with a certain number of features. Several criteria for the construction of multi-objective test instances have been discussed by a number of researchers, particularly in the pioneering works of Deb *et al.* [18] and Huband *et al.* [27]. Huband *et al.* [27] analyzed and justified different requirements which should be considered in the design of multi-objective test problems. Table I presents the seven recommendations (R1–R7) and the five features (F1–F5) discussed by Huband *et al.* [27]. However, because of the inherent progress on evolutionary multi-objective optimization, other features (F6–F8) are added and described below.

1) Feature 6 (F6): Difficult Pareto Set Topology: Okabe *et al.* [53] noticed that the Pareto optimal set of most multi-objective test instances, is defined by a piecewise linear function. However, as pointed out in [40], this limitation of test problems does not reflect the actual characteristics commonly observed in real-world problems. Okabe *et al.* [53] introduced the notion of predefined PS as an alternative to construct multi-objective test instances with arbitrary PS topologies. This notion was generalized to an arbitrary number of decision variables in [76], [40], in a set of test instances with complicated Pareto sets.

To observe the difficulties that can bring a complicated PS topology to a multi-objective problem, consider the following "*kite*" test problem, where all the objectives are to be minimized:

$$\begin{aligned} &F_{j=1:M-1}(\mathbf{x}) = (1+g(\mathbf{x})) \times x_j \\ &\qquad F_M(\mathbf{x}) = (1+g(\mathbf{x})) \times \left(1 - \prod_{i=1}^{M-1} \tfrac{9x_i+1}{10}\right) / \left(1 - 0.1^{M-1}\right) \\ &\qquad\qquad \mathbf{x} \in \Omega = [0,1]^n \text{ and } M < n \end{aligned} \tag{1}$$

In the above problem, a piecewise linear PS topology can be defined with

$$g(\mathbf{x}) = \textstyle\sum_{j=M}^{n} (x_j - 0.5)^2 \tag{2}$$

while a more complicated PS shape is formulated with

$$g(\mathbf{x}) = \textstyle\sum_{j=M}^{n} (x_j - \gamma_j(\mathbf{x}))^2 \tag{3}$$

such that

$$\gamma_j(\mathbf{x}) = \tfrac{1}{2(M-1)} \textstyle\sum_{i=1}^{M-1} \sin(2\pi x_i - 1 + \theta_j)^3 + \tfrac{1}{2} \tag{4}$$

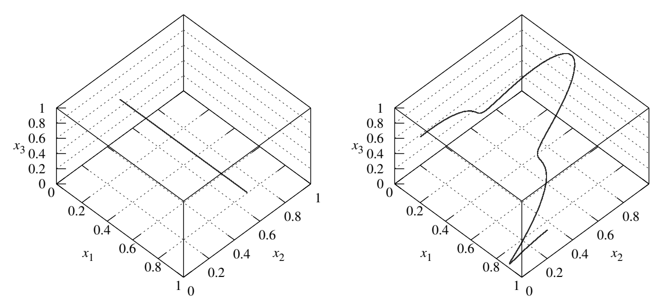
with $\theta_j = \frac{2\pi(j-M)}{n}$, for $j = M, \ldots, n$.

In the following experiment, let's consider the above test problem with $M = 2$ and $n = 20$. Thus, the PSs projections onto the space x_1, x_2 and x_3, for the two formulations of g can be appreciated in the plots of Fig. 1. In both formulations

TABLE I: Recommendations and features for multi-objective test problems

Recommendation (R) or Feature (F)	Comment
R1: No Extremal Parameters	Prevents exploitation by truncation based correction operators
R2: No Medial Parameters	Prevents exploitation by intermediate recombination
R3: Scalable Number of Parameters	Increases flexibility, demands scalability
R4: Scalable Number of Objectives	Increases flexibility, demands scalability
R5: Dissimilar Parameter Domains	Encourages EAs to scale mutation strengths appropriately
R6: Dissimilar Trade-off Ranges	Encourages normalization of objective values
R7: Pareto Optima Known	Facilitates the use of measures, analysis of results, in addition to other benefits
F1: Pareto Optimal Geometry	Convex, linear, concave, mixed, degenerate, disconnected, or some combination
F2: Parameter dependencies	Objectives can be separable or non-separable
F3: Bias	Substantially more solutions exist in some regions of fitness space than they do in others
F4: Many-to-one mappings	Pareto one-to-one/many-to-one, flat regions, isolated optima
F5: Modality	Uni-modal, or multi-modal (possibly deceptive multi-modality)
F6: Difficult Pareto Set Topology	Pareto set difficult to characterize
F7: Difficult Pareto Front Shape	Pareto optimal front difficult to estimate
F8: Correlation of Position and Distance Functions	Dependencies between position and distance functions
F9: Single Optimal Solution for a High Number of Objectives	Single objective solution for multiple objective functions
F10: Easy Configuration of Features in Scalable Test Problems	Easy way to configure the features of a scalable test problem

of the problem, the PF (for $M = 2$) is exactly the same and corresponds to a single linear piece contained into the square $[0, 1]^2$. The difficulties of solving the kite test problem with different PS topologies, can be observed by executing a multi-objective algorithm with the same parameter settings over the two different formulations of g. In this study, we consider a hypervolume-based MOEA (namely the "S Metric Selection Evolutionary Multi-Objective Algorithm", SMS-EMOA [2]) as the solver of this problem for the two formulations of g. SMS-EMOA is a steady state algorithm that generates a trial solution at a time, aiming to maximize the hypervolume of the non-dominated solutions found during the search. Since the PF of the kite test problem in both configurations of g is linear, the hypervolume indicator does not favor any convex or concave part of the PF, being a good candidate to carry out this experiment. In both scenarios, SMS-EMOA performed the same number of fitness function evaluations and used the same parameters settings. The performance evaluation was carried out by using the empirical attainment functions (EAFs) [48], and comparing the final set of solutions obtained by SMS-EMOA in the problem with the two formulations of g. The EAF provides the probability, estimated from several runs, that an arbitrary objective vector is dominated by, or equivalent to, a solution obtained by a single run of the algorithm. The difference between the EAFs for an algorithm solving two different problems with the same PF, identifies the regions of the objective space where an algorithm is able to perform better in a problem than in another one. In this particular case, the magnitude of the difference in performance of SMS-EMOA in a determined problem is plotted with a gray-colored scale as illustrated in Fig. 2. We can clearly see that for the complicated PS topology formulation, SMS-EMOA produces solutions that are very likely dominated almost everywhere by the solutions obtained in the problem with the simple PS topology (i.e., the formulation with a piecewise linear PS). This means that the kite test problem with the complicated PS topology is more difficult to solve than the one with the simple PS shape. As it was noticed by other authors, complicated PS topologies are meant to test specific mechanisms of MOEAs, particularly, parent selection and recombination [40]. For this reason, this feature should be thoughtfully considered in the design of multi-objective test problems.



(a) $g(\mathbf{x}) = \sum_{j=M}^{n}(x_j - 0.5)^2$ (b) $g(\mathbf{x}) = \sum_{j=M}^{n}(x_j - \gamma_j(\mathbf{x}))^2$, where $\gamma_j(\mathbf{x})$ is defined as in Eq. (4)

Fig. 1: Pareto optimal set for the kite test problems using different $g(\mathbf{x})$ functions

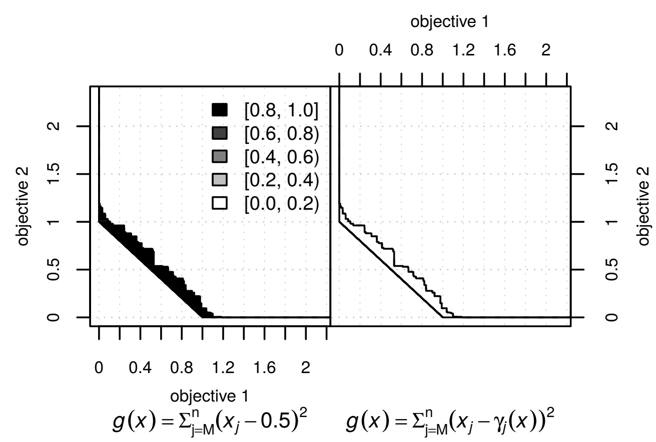
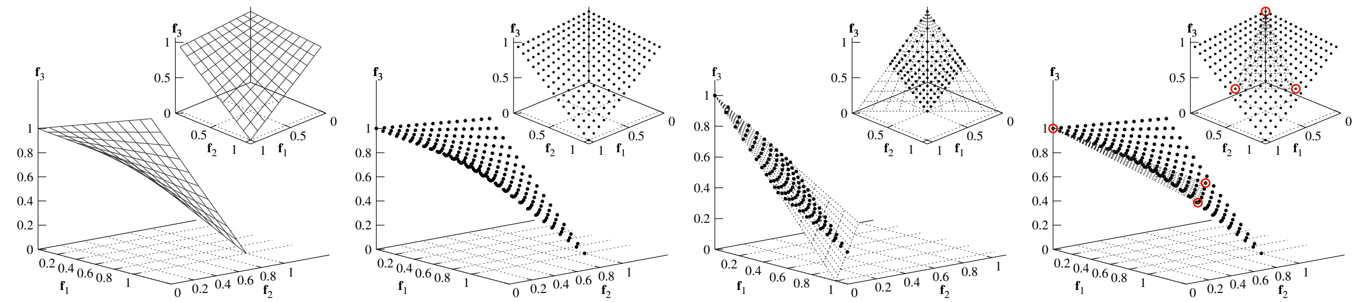


Fig. 2: EAF differences for SMS-EMOA solving the kite test problem (for $M = 2$) with the piecewise PS (lefthand plot) compared to SMS-EMOA solving the kite test problem with a complicated PS (righthand plot).

2) Feature 7 (F7): Difficult Pareto Front Shape: Pareto front geometries test the abilities of MOEAs to maintain spread solutions along the Pareto optimal surface.



(a) Pareto optimal front for the kite test problem
(b) A set of Pareto optimal solutions ($P^\star$) for the kite test problem
(c) Weights for the kite test problem which yield solutions in $P^\star$, under the NBI approach
(d) Hyperplane obtained in $P^\star$ using extreme points obtained by the ASF

Fig. 3: (a) Pareto optimal front, (b) optimal solutions ($P^\star$), (c) weights yielding $P^\star$, and (d) hyperplane found in $P^\star$ using the ASF for the "*kite*" test problem

Huband *et al.* [27] summarized several geometries involving curvatures (concave/linear/convex/mixed) and special cases (degenerate/disconnected) of Pareto optimal fronts (see *Feature 1* in [27]). Recently, studies have shown and treated the difficulties to solve multi-objective test problems with disconnected or degenerate PFs, see for example [55], [41], [6], [35].

In the case of test problems with connected and non-degenerate PFs, their difficulties have not been clearly established. Nonetheless, the PF of several state-of-the-art test problems with this type of geometries have similar shapes. Concretely, their Pareto optimal fronts can be characterized by an $(M-1)$-dimensional simplex (see Section IV-A).

To settle down difficulties for test problems with connected and non-degenerate PFs, consider the kite test problem formulated in Equation (1). This test problem has its ideal point in $\mathbf{z} = (0, \ldots, 0)$ and its nadir point in $\mathbf{n} = (1, \ldots, 1)$. Its Pareto optimal front is a connected and non-degenerate surface which is illustrated, for $M = 3$, in Fig. 3a.

Several are the complications of solving multi-objective optimization problems. Particularly, to find a proper representation of the real PF is a hard task which has been studied throughout the development of evolutionary approaches. Some techniques adopted by evolutionary approaches to reach a relative good representation of the PF are described below.

Diversity assessment: Diversity assessment of a set of non-dominated solutions is one of the most popular strategies that for several years, has been adopted by MOEAs to achieve a proper representation of the PF of a MOP. Usually, diversity assessment is employed together with a convergence criterion (normally, Pareto dominance) as part of the survival mechanism of a MOEA. Some methods that have been proposed throughout the years include: *fitness sharing and niching* [14], *clustering* [80], and *crowding distance* [16], among many others. Although these methods were very popular in the first decade of the 2000s, their use has decreased because of the difficulty of measuring diversity in a set of non-dominated solutions, as pointed out by several researchers [22], [25], [23], particularly in high-dimensional objective spaces [45], [68].

Performance Indicators: Other strategies employed by MOEAs to achieve an adequate representation of the PF are related to performance indicators. With its emergence, the Indicator-Based Evolutionary Algorithm (IBEA) [78] posed the possibility to optimize a performance indicator in the evolutionary process of MOEAs. As it is well-known, there exists a large number of indicators to assess the performance of MOEAs, see for example [82], [54], [34]. Such indicators are able to assess, in different ways, convergence and diversity, or both of them at the same time. In particular, a good representation of the PF can be achieved by using performance indicators such as the hypervolume [81], IGD [65], and Δ_p [61], among others. However, these indicators become impractical when the number of objectives increases and most of them require a reference set of the PF (as the case of IGD, Δ_p, and their variants) which is difficult to obtain *a priori*.

Decomposition: In the last decade, scalarization functions have been employed by several evolutionary approaches giving rise to the well-known MOEAs based on decomposition. Decomposition approaches rely on solving a number of scalarization functions which are formulated by an even number of weight vectors. Such scalarization functions are solved through the search by approximating solutions towards the real PF. Decomposition-based MOEAs have been found to be very efficient in solving complicated test problems, see for example [73], [40], [74]. In addition, having a well-distributed set of weight vectors, a proper representation of the complete PF can be reached in some multi-objective problems.

Convex hull of individual minima: A strategy recently adopted by MOEAs is related to the convex hull of individual minima (CHIM) [11], [12]. The idea behind these approaches consists in finding M (number of objectives) extremes of the PF and tracing a hyperplane (commonly referred to as CHIM) defined by such extremes. The constructed hyperplane corresponds to an $(M-1)$-simplex which is used in different ways to approximate solutions towards the Pareto optimal front. A discretization of the CHIM is reached by defining a set of weight vectors, which serve as direction guides in the search. Therefore, a proper representation of the PF can be achieved with a well-distributed set of weight vectors. Approaches adopting the idea of CHIM are, for example, the Convex Hull Multi-Objective Evolutionary Algorithm (CH-MOEA) [71], the Nondominated Sorting Genetic Algorithm III (NSGA-

III) [15], and the Reference Indicator-Based Evolutionary Multi-Objective Algorithm (RIB-EMOA) [72], among others.

Approaches based on decomposition and the CHIM, share one property: they depend on the definition of a set of weight vectors. Particularly, when the Pareto optimal front of a MOP can be characterized by a well-distributed set of weight vectors, **decomposition-** and **CHIM-**based MOEAs become highly effective since a well-distributed set of weight vectors can be defined *a priori*.[2]

In the case of the three-objective kite test problem, a well-distributed set of Pareto optimal solutions ($P^\star$) is shown in Fig. 3b. Such solutions can, in fact, be reached by using a number of scalarization functions defined by an even number of weight vectors. Fig. 3c illustrates the weight vectors that yield, under the NBI approach [12], the well-distributed optimal solutions in $P^\star$. Such weight vectors are, as can be seen, not well distributed and do not cover the complete space of weight vectors. In Fig. 3d, the hyperplane formed by the approximated extremes (solutions in red) found into the PF representation $P^\star$ is illustrated.[3] As can be seen, the points ξ^i do not correspond to the real extremes of the PF and the hyperplane does not cover the entire PF. Note however, that finding the extremes of the PF is by itself a hard problem for difficult Pareto shapes. Thus, approaches based on decomposition or the idea of CHIM are not appropriate to deal with this type of problems. In general, difficult PFs are those having geometrical shapes which can not be estimated *a priori*.

3) Feature 8 (F8): Correlation of Position and Distance Functions: In the bottom-up approach, position and distance functions can be distinguished. Position functions define the PF of a test problem, while the PS is defined by the distance functions. Together, position and distance functions, determine the final coordinates of an objective vector in feasible objective space. In the case of real-world problems and test problems based on multiple single-objective functions, parameters related to the position and distance functions can be difficult to identify.

To better understand the correlation between position and distance functions, let's consider the *kite* test problem defined in equation (1).

Following the bottom-up approach, the decision variable $\mathbf{x} = (x_1, \ldots, x_n)$ is split into position $\mathbf{x}_I = (x_1, \ldots, x_{M-1})$ and distance $\mathbf{x}_{II} = (x_M, \ldots, x_n)$ parameters. The functions that define the Pareto optimal front (*position functions*) are:

$$\begin{aligned} h_{j=1:M-1}(\mathbf{x}_I) &= x_j \\ h_M(\mathbf{x}_I) &= \left(1 - \prod_{i=1}^{M-1} \frac{9x_i + 1}{10}\right) / \left(1 - 0.1^{M-1}\right) \end{aligned} \quad (5)$$

[2]A set of weight vectors properly distributed can be reached by simplex-lattice design [60], uniform design [21] or low-discrepancy sequences [70], among other strategies.

[3]Such extremes ξ^is ($i = 1, \ldots, M$) are approximated by finding the solutions in $P^\star$ that minimize the following achievement scalarization function (ASF): $\xi^i = \arg\min_{\mathbf{x}\in P^\star} \max_{j=1}^{M} \left((f_j(\mathbf{x}) - z_j)/e^i_j\right)$, where $\mathbf{e}^i = (e^i_1, \ldots, e^i_k)^T$ is the i^{th} canonical basis in $\mathbb{R}^M$. When $e^i_j = 0$, it is set as: $e^i_j = 1 \times 10^{-6}$.

Considering the simple PS topology as described by equation (2), the *distance function* is given by $g(\mathbf{x}_{II}) = \sum_{i=M}^{n} (x_i - 0.5)^2$. Objective values are obtained by a composition of position and distance functions. These two functions (position and distance) give the absolute position of a solution into the objective space. A composition of position and distance functions can be done in many ways. Next, we mention three approaches which have been commonly adopted in the design of multi-objective test problems:

1) *Multiplicative approach [18]* [4]*:*

$$f_{i=1:M}(\mathbf{x}) = h_i(\mathbf{x}_I) \times (1 + g(\mathbf{x}_{II}))$$

2) *Additive approach [27]:*

$$f_{i=1:M}(\mathbf{x}) = h_i(\mathbf{x}_I) + g(\mathbf{x}_{II})$$

3) *Modular approach [58]:*

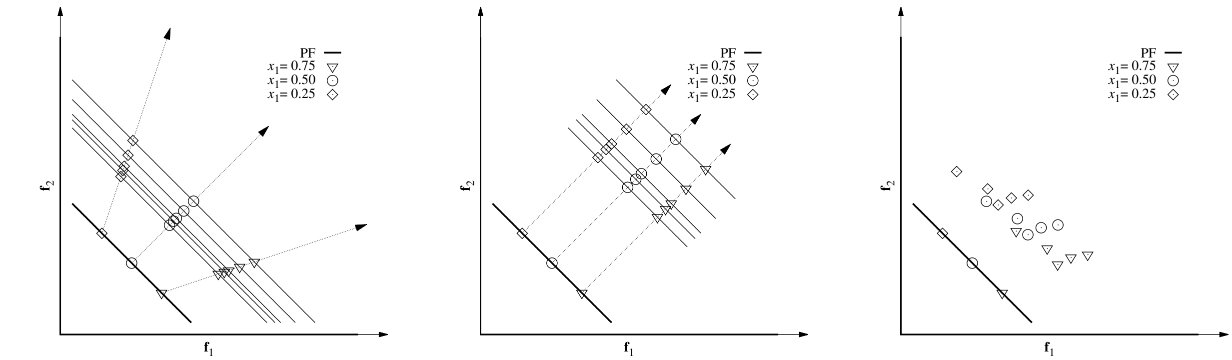
$$f_{i=1:M}(\mathbf{x}) = h_i(\mathbf{x}_I) + g_i(\mathbf{x}_{II}|J_i)$$

where $g_i(\mathbf{x}_{II}|J_i) = \sum_{j\in J_i} (x_j - 0.5)^2$, such that $J_i = \{j| \mod (j - M + 1 - i, M) = 0, j = M, \ldots, n\}$.

To observe the effect of the above approaches, consider the case with $M = 2$ and $n = 10$ which implies that $\mathbf{x}_I = x_1$ and $\mathbf{x}_{II} = (x_2, \ldots, x_{10})$. The Pareto optimal set (in all the approaches) is $x_1 \in [0, 1]$ and $x_{j=2:10} = 0.5$. Let $x_1 \in \{0.25, 0.5, 0.75\}$ and $\mathbf{x}^{(1)}_{II}, \ldots, \mathbf{x}^{(5)}_{II}$ be three position values and five random distance vectors, respectively. The objective values of all pairs of position and distance vectors ($\mathbf{x}_I$'s and $\mathbf{x}_{II}$'s) for each approach are shown in Fig. 4. The objective vectors yielded by the multiplicative and additive approaches are shown in Figs. 4a and 4b, respectively. As can be seen, the five objective vectors can be sorted at different levels of Pareto dominance. In both approaches, the distance functions modify only the distances of solutions to the PF without affecting their relative positions with respect to the PF. In this way, a domination order can be observed for different distance vectors as shown in Figs. 4a and 4b. This means that the relative position to the PF does not depend on the distance functions.

On the other hand, in the modular approach (see Fig. 4c), objective vectors do not necessarily follow a dominance order. For different distance parameters (i.e., different values of the distance functions), the relative position of solutions with respect to the PF is modified. Analogously, changing the position parameters (i.e., different values for position functions) implies a change in the distance. This means that the position and distance functions have a direct dependency. Other strategies showing greater or lower correlation between position and distance functions are, for example, those introduced in [9], [49]. This property makes, in fact, more realistic the design of multi-objective test problems.

[4]The approach employed in the kite test problem.



(a) Multiplicative approach: $f_{i=1:2}(\mathbf{x}) = h_i(\mathbf{x}_I) \times (1 + g(\mathbf{x}_{II}))$
(b) Additive approach: $f_{i=1:2}(\mathbf{x}) = h_i(\mathbf{x}_I) + g(\mathbf{x}_{II})$
(c) Modular approach: $f_{i=1:2}(\mathbf{x}) = h_i(\mathbf{x}_I) + g_i(\mathbf{x}_{II}|J_i)$

Fig. 4: Correlation of position and distance functions in different approaches

4) Feature 9 (F9): Single Optimal Solution for a High Number of Objectives: Correlation among objectives is a feature inherent to real-world multi-objective problems. In the specialized literature we can find problems for which the objectives are either negatively or positively correlated (according to Spearman's coefficient), see for example [33], [7], [24]. A positive correlation between a pair of objectives refers to the fact that the optimization of an objective implies the optimization of the other one. Analogously, a negative correlation means that the optimization of an objective implies the deterioration of another one, i.e., there exists a clear trade-off (conflict) between the objectives.

In the case of some artificial test problems (such as DTLZ and WFG), pairs of objectives are positively correlated. Furthermore, in some test problems, the optimal solution is the same for the objectives positively correlated or it corresponds to the same objective value. From the correlation standpoint, DTLZ and WFG can be seen as a particular case of this class of problems. In general, it is desirable to have test problems with various degrees of correlation, where the optimal solution for each objective does not need to be the same even for a strong positive correlation. It should be noted that different correlation (positive or negative) among objectives makes possible different PF's shapes, see the works reported in [66], [41]. On the other hand, the correlation among objectives can affect the size of the Pareto optimal set. Actually, in the discrete case, the size of the Pareto optimal set increases with negative correlation and decreases with positive correlation, see the study reported in [66]. In the continuous case (under certain numerical precision), a similar situation is expected to occur, i.e., the size of the Pareto optimal set increases with negative correlation and decreases otherwise.

This property of artificial test problems, i.e., the presence of positive correlation among different objectives, allows to evaluate certain types of approaches. It is known that the presence of positive correlation places obstacles for the good performance of approaches based on the reduction of objectives [5], [3]. Particularly, approaches based on reduction of objectives can fail when dealing with scalable test problems where the optimal solution is the same for some objectives positively correlated. Such is the case of DTLZ2 where removing an objective function results in collapsing the remaining objectives to the same single optimal solution. For this reason, in [5], a variant of DTLZ2 (namely DTLZ2$_{BZ}$) was proposed where the positive correlation is maintained but the single optimal solution for the objectives positively correlated is avoided. Thus, we shall say that a scalable test problem adheres to the property that objectives positively correlated collapses into a single optimal solution, if the objectives collapse to a single optimal solution when an objective is removed from the original formulation of the problem.

5) Feature 10 (F10): Easy Configuration of Features in Scalable Test Problems: As pointed out early, scalable test problems have helped us to understand different components of MOEAs. Although they have been criticized as they are artificial and they rarely reflect the true conditions of real-world problems, their properties vary significantly with respect to each other, and they can be complemented among themselves. Therefore, the well-performing evolutionary approaches should be good candidates to solve real-world problems. For this reason a scalable test suite should contemplate a vast number of characteristics simulating some properties that have been observed in real-world applications. Assumed in such a way, since the early developments in this area, researchers have considered distinct features in the design of scalable test problems, such as those discussed in [18], [27]. Currently, most of the existing test suites incorporate problems with different features which cannot be modified by a standard user. Thus, if a test problem is formulated with a given PF geometry being separable and unimodal, a conventional user can hardly reformulate such problem keeping the same PF geometry but making it non-separable and multi-modal. Therefore, a flexible configuration of scalable test problems gives to the users the possibility of configuring a scalable test problem with its desirable properties going from easy to difficult. Thus, the easy configuration of a test problem with its desirable characteristics is an issue considered in the list of features of scalable test problems.

III. Scalable Test Suites

A scalable multi-objective test problem refers to a problem which can be formulated with an arbitrary number of objectives. Generalized in [39], the multi-objective sphere model [59] (SPH-m in [79]) is, perhaps, the first scalable test problem reported in the specialized literature. Throughout the years, researchers have introduced test instances to evaluate the working principles of many-objective evolutionary algorithms—see for example [38], [76], [69], [43]. However, the design of scalable problems as a set of test problems or a "*test suite*", providing different difficulties to promote the design of robust evolutionary approaches (in the context of many-objective optimization), has been much less investigated. Next, a review of scalable test suites reported in the specialized literature is presented.

1) Deb et al. *test suite:* The **Deb-Thiele-Laummans-Zitzler** (DTLZ) suite [18] constitutes the first set of test problems that is scalable in both the number of decision variables and the number of objective functions. This set of problems, provides seven unconstrained and two constrained scalable test problems with different characteristics. Focusing on unconstrained test problems—the scope of this paper—this testbed introduces characteristics such as uni-modal (DTLZ2 and DTLZ4), multi-modal (DTLZ1, DTLZ3 and DTLZ7) and biased (DTLZ4 and DTLZ6) test instances which aim to test the abilities of MOEAs to approximate solutions towards the real PF. Four continuous and non-degenerate PFs (DTLZ1–DTLZ4), two degenerate PFs (DTLZ5–DTLZ6) and a disconnected PF (DTLZ7) summarize the Pareto optimal surfaces in this test suite. In spite of its popularity, the DTLZ test suite has several weaknesses which have been exposed by a number of authors [27], [40]. Remarkable shortcomings are, for instance, the absence of variable linkages or parameter dependencies, the simple topologies of their PSs, and the cushy identification of distance and position parameters.

2) Huband et al. *test suite:* To overcome some of the disadvantages of the DTLZ test suite, Huband *et al.* [27] proposed a set of nine scalable test problems called the **Walking-Fish-Group** (WFG) test problems. These problems incorporate important properties simulating several features commonly observed in real-world problems (including non-separable, multi-modal, deceptive, and biased test problems). The WFG test problems embody seven connected and non-degenerate PFs (WFG1 and WFG4–WFG9), a problem with a disconnected PF (WFG2) and a degenerate test instance (WFG3). An important advantage of this toolkit, is the possibility to construct new test problems from a combination of shape functions (which defines the PF) and a set of transformations (which determine the search space). The WFG test problems established a well-defined test suite that, in the last decade, has become (together with the DTLZ test suite) widely adopted in the analysis and study of many-objective evolutionary approaches. Similar to DTLZ, in the WFG test problems, distance and position parameters can be easily identified.

3) Emmerich and Deutz test suite: In order to design multi-objective test problems with different Pareto optimal fronts, Emmerich and Deutz [20] introduced a scalable test suite based on the **Lamé superspheres** (LSS). Although this methodology is limited to design Pareto optimal geometries with spherical shapes, it can be considered as the first study focused on the Pareto shape of scalable test problems. Something remarkable about this proposal are the mirror test problems which adopt an inverted sphere as the Pareto optimal front of the proposed multi-objective test problems. Even though the use of mirror spheres had already been adopted as a Pareto optimal surface in [27], the parameter γ of the Lamé spheres is able to modify the convexity/concavity degree in these test problems. In addition to new Pareto optimal shapes, the Lamé superspheres test suite incorporates features such as multi-modality and many-to-one mapping which introduce additional difficulties for solving these problems using a MOEA. Since the LSS test suite adopts the DTLZ framework, distance and position parameters can also be easily identified.

4) Saxena et al. *test suite:* Saxena *et al.* [58] extended the principle of complicated PSs (initially introduced for two- and three-objective problems [76], [40]) to scalable multi-objective test problems. The **Saxena-Zhang-Duro-Tieari** (SZDT) test suite introduces seven unconstrained test problems and the possibility of designing new test problems by choosing a combination between Pareto optimal shapes and complicated PS topologies. The Pareto optimal fronts for all these test problems are defined in one and two dimensions, i.e., they become degenerate for more than two and three objectives, respectively. Regarding the Pareto optimal fronts, four continuous and connected surfaces including convexity and concavity generalize the Pareto shapes in this test suite. The convergence difficulties in this benchmark are specifically stated by the topology of the PSs. The absence of multi-modality and non-separability, are the shortcomings in this test suite. However, the use of the modular approach [40] in this testbed, makes difficult to determine the position and distance parameters, which becomes an advantage over the previous test suites.

5) Cheng et al. *test suite:* As pointed out in [27], [17], variable linkages should be considered in the construction of multi-objective test problems. This feature is particularly important in test instances because, as remarked in [57], it makes it more difficult for a MOEA to properly exploit optimal solutions. Cheng *et al.* [9] introduced a set of nine test problems specially designed to test MOEAs for large scale optimization (i.e., for multi-objective problems with a large number of decision variables). In the **Large Scale Multi-Objective Problems** (LSMOPs), variable dependencies are stated by two variable linkage functions (linear and nonlinear). In addition to the dependencies among variables, this test suite introduces correlation between decision variables and objectives by means of a correlation matrix. Although the test problems are scalable to an arbitrary number of objectives, this test suite is limited to three Pareto optimal shapes, concretely, the PFs from DTLZ1 (normalized in objective function space), DTLZ2, and DTLZ7.

6) Masuda et al. *test suite:* Masuda *et al.* [49] proposed a toolkit to generate scalable test problems. This test suite is mainly focused on the design of different Pareto optimal shapes. The methodology introduced in this approach allows the design of Pareto optimal surfaces by using a finite number of vertices. Such vertices state the Pareto optimal front whose shape can be defined as linear, concave, or convex. Although only two test problems were instantiated, the toolkit provides a methodology for designing scalable test problems with Pareto optimal surfaces having an arbitrary number of vertices. A remarkable aspect of the **Masuda-Nojima-Ishibuchi** (MNI) test suite, is that at different distances from the PF, different PF shapes can be produced.

7) Scalable Multi-Objective Test Suites for Visual Examination of Multi-objective Search: As a generalized version of the single polygon problems of Köppen and Yoshida [38] and the multi-line (or multi-curve) problems of Rudolph *et al.* [56], Ishibuchi *et al.* [29] proposed a set of many-objective test problems in two- and three-dimensional decision spaces. These test problems allow the visual examination (in the decision space) of multi-objective algorithms when approximating solutions towards the Pareto optimal set. Although the test problems presented in this study were initially formulated in low-dimensional decision spaces, the authors drew the possibility to construct test problems in high-dimensional spaces by specifying multiple points with the required dimensionality. Such idea was implemented in the many-objective test suite formulated in a high-dimensional decision space [32]. Inspired by the above test suites, Li *et al.* [46] proposed the construction of a test problem whose Pareto optimal solutions lie in a rectangle (in the two-variable decision space) and, more importantly, are similar (in the sense of Euclidean geometry) to their images in the four-dimensional objective space. As a generalization of Li *et al.* 's test problem, in [44], a class of multi-objective test problems scalable in the number of objectives (called multi-line distance minimization problems (ML-DMP)) was introduced. Two are the main characteristics in this test suite: 1) the Pareto optimal solutions lie in a regular polygon in a two-dimensional decision space, and 2) these solutions are similar (in the sense of Euclidean geometry) to their images in high-dimensional spaces.

It could be argued whether the properties included in the above test suites are observed in real-world multiobjective problems. However, these test suites allow understanding how a multi-objective algorithm can achieve a good representation of the true PF by visualizing the front in a degenerate surface.

8) Other test Suites Scalable in the Number of Objectives: In the following, we briefly describe other scalable test suites available in the specialized literature.

Ishibuchi et al. *test suite:* Ishibuchi *et al.* [31] proposed minus versions of the DTLZ and WFG test problems (namely minus-DTLZ (DTLZ^{-1}) and minus-WFG (WFG^{-1}), respectively) as scalable test problems with clear differences from their original versions. These test problems stand out mainly because the Pareto optimal fronts of the original DTLZ and WFG test problems are inverted to obtain a similar effect as in the mirror LSS test problems [20]. However, in this test suite, different geometries (the geometries used in the DTLZ and WFG test problems) are employed instead of being limited to the superspheres as in the case of the mirror LSS test problems. Some important key points to consider are the following: 1) all the test problems maintain the same properties respect to the difficulties of the distance functions in DTLZ and WFG test problems, respectively; and 2) different test problems promote the design of diversity mechanisms to achieve a proper representation of the inverted DTLZ and WFG Pareto optimal fronts.

Cheng et al. *test suite:* In [10], a compilation of 15 test problems was presented as a scalable test suite, called MaF. In this test suite, the authors' intention is to compile a set of test problems with different features in order to evaluate many-objective evolutionary approaches. Most of the test problems included in this test suite were taken from already formulated test problems such as WFG, DTLZ, and ML-DMP, among other works. Thus, a wide variety of features can be found in this test suite which, indeed, shall be able to assess the robustness of many-objective evolutionary approaches.

9) On the Scalability of Decision Variables in Multi-Objective Test Suites: Through the development of multi-objective test suites, the scalability in the number of decision variables has been one of the main features contemplated in early investigations in this area. Thus, pioneering design approaches such as Deb's toolkit [13], and Ziztler et al.'s (ZDT) test suite [77], marked the beginning in the design of multi-objective test suites scalable in the number of parameters (i.e., decision variables). In particular, with the advent of the bottom-up approach [18] (adopted in the previous test suites), the design of multi-objective test suites has become much more accessible facilitating the construction of the PF and PS (they are separately designed). Currently, most of the multi-objective test suites adopting the bottom-up approach are scalable in the number of decision variables. Particularly, multi-objective test suites presented in Table II are scalable in both the number of objectives and the number of decision variables. Other multi-objective test suites scalable in the number of decision variables (but not in the number of objectives) are the test suite with complicated PSs [40], [76], the multi-objective test suite for robust optimization [50], and the test problems with degenerate PFs [41][5]. Although these test problems are not scalable in the number of objectives, their use can be seen in some studies concerning to the so-called large-scale multi-objective optimization, such as the work reported in [1], [64].

IV. Features and Limitations of Existing Scalable Test Problems

A. Features of Existing Scalable Test Problems

As indicated before, the features of a test problem are meant to evaluate the working principles of MOEAs. The features of some scalable test problems presented before, have been examined in the past by some researchers [27], [49]. Roughly, the features of a multi-objective test problem can be classified in two groups: (i) those related to the search space and (ii) those related to the Pareto optimal surface.

Features of the search space have been discussed by some authors [27], [40]. Such features are relatively important because they place obstacles that complicate the exploration of Pareto optimal solutions. In this regard, state-of-the-art test suites have adopted several features, such as multi-modal, non-separable, deceptive, and biased search spaces. Regarding the Pareto optimal front, Huband *et al.* [27] suggested the use of different geometries including curves (linear/convex/concave/mixed), disconnected and degenerate

[5] Furthermore multi-objective test suites scalable in the number of objectives can be found in [27], [75], [47], [42].

surfaces. An overview of Pareto optimal fronts (for three objectives, i.e., $M = 3$) of all non-degenerate test problems included in the above test suites is presented in Fig. 5.[6]

To summarize the properties of the above test problems, Table II exposes their attributes considering the suggested recommendations and features included in Table II. Note that the test problems DTLZ4, DTLZ5 and WFG3 were excluded due to the inconsistencies observed by some authors [27], [30]. In Table II, symbols "✓" and "✗" indicate whether a recommendation is adhered to, while "+" and "−" indicate the presence or absence of a given feature. In the case of features, the following abbreviations are included: "S" for separable, "NS" for non-separable, "U" for unimodal, "M" for multi-modal, and "D" for deceptive. The recommendations and features apply to the whole multi-objective test problem based on the hardest feature found in any of its objective functions. For example, if an objective function is non-separable/multi-modal/deceptive/biased and any other objective is separable/unimodal/non-deceptive/unbiased, the problem is said to be non-separable/multi-modal/deceptive/biased.

B. Limitations of Existing Scalable Test Problems

In Section II, we presented different design approaches as well as some recommendations and features for the construction of multi-objective test problems. A first issue to observe, is that all the test problems reviewed in the previous section, follow the bottom-up approach. This form to formulate multi-objective test problems splits the construction of the PF and the design of the search space which, in fact, facilitates the construction of multi-objective problems specially in high-dimensional objective spaces. The literature review presented in Section III, exhibits the test problems that follow the recommendations and features established in Section II-B, see the summary of such properties in Table II. As discovered in Section IV-A, recommendations (R1–R7) are partially covered by most of the test problems, being the WFG test suite, the only set of problems that satisfies entirely such requirements. In the case of features related to the search space (F2–F5), most of the test problems do not adhere or cannot fit in a specific or desirable combination of features, as can be seen in Table II. While these features can be studied separately, there is no reason to assume that a real-world problem does not adhere simultaneously to several of these features at the same time.

Although one might doubt the existence of multi-objective problems having a combination of characteristics different from the ones formulated in the existing scalable test suites, according to the *No-Free Lunch* theorem, this overestimation does not hold. In other words, there is an immense number of formulated and unformulated real-world problems and it is reasonable to think that any of them may have a wide variety of features not contemplated in any already formulated artificial test problem. Thus, the inflexibility of configuring (in an easy way) scalable test problems with a desirable combination of features, becomes also a limitation of the existing scalable test suites.

On the other hand, difficult PS topologies (F6) are not considered by most of the test suites, which, as pointed out by some authors [40], [58], becomes a limitation because these test problems do not reflect the features observed in real-world problems, see the work reported in [36], [26].

An important issue to consider in scalable test problems refers to the shape of the Pareto optimal front. In this regard, the Pareto optimal fronts of the existing scalable test problems combine a variety of different geometries including convexity, concavity and/or linearity.

In Fig. 5, it can be seen that the Pareto optimal fronts from Figs. 5a, 5b, 5d, 5f, and 5g, can be characterized by an $(M-1)$-simplex. As discussed in Section II-B2, test problems having this type of shapes are easy to solve for some evolutionary approaches. However, there is no reason to assume that real-world problems have this type of shapes. In the specialized literature, we can find several MOPs in which their PF approximations draw strange geometries that do not follow exactly the shape of an $(M-1)$-simplex, see for example the problems presented in [8], [19], [62]. Most of these PFs do not follow the property of difficult PF shape (F7) that has been suggested to evaluate diversity mechanisms in MOEAs. This, in fact, becomes a limitation of the constructed test problems and motivates to design new geometries different from those included in the state-of-the-art test suites.

Another important property that should be considered in the construction of scalable test problems is regarding the correlation between position and distance functions (F8). From Table II, it can be seen that most of the test problems do not follow this property which complicates the identification of position and distance parameters. Although there exist approaches employed to correlate position and distance functions (e.g. the modular approach), the investigation and development of a more flexible design approach for constructing scalable test problems—where position and distance variables are indistinguishable and the true PS and PF can be analytically known—is in fact a good path for future investigations.

From Table II, it can be seen that most of the test problems adhere to the property that a single solution is optimal for multiple objectives positively correlated. As discussed before, the presence of positive correlation among different objectives, allows to evaluate certain types of approaches. Particularly, the presence of positive correlation places obstacles for the good performance of approaches based on the reduction of objectives [5], [3]. In general, it is desirable to have test problems with different degrees of correlation among the objectives.

As we can see from the review of the test problems analyzed in this paper, the properties of scalable test problems vary significantly between each other and they can be complemented among different problems. We consider that many-objective approaches can be tested on different scalable test problems and the well-performing approaches should be good candidates to solve a real-world problem. However, as we pointed out before, this does not necessarily hold in real-world problems. There are different reasons for this observable behavior. One of

[6]Note that to provide a better view, the Pareto optimal fronts are normalized in the cube $[0,1]^M$

TABLE II: Properties of DTLZ, WFG, LSS, SZDT, LSMOP, and MNI Test Problems

MOP	R1: No Extremal	R2: No Medial	R3: # Parameters	R4: # Objectives	R5: Diss. Domains	R6: Diss. Tradeoffs	R7: Optima Known	F1: PF Geometry	F2: Separability	F3: Bias	F4: Many-to-one	F5: Modality	F6: Difficult PS	F7: Difficult PF	F8: Corr. Pos.–Dist. Func.	F9: Single Opt. Sol.	F10: Easy Conf. of Features
DTLZ1	✓	✗	✓	✓	✗	✗	✓	linear	S	−	+	M	−	−	−	+	−
DTLZ2	✓	✗	✓	✓	✗	✗	✓	concave	S	−	+	U	−	−	−	+	−
DTLZ3	✓	✗	✓	✓	✗	✗	✓	concave	S	−	+	M	−	−	−	+	−
DTLZ4	✓	✗	✓	✓	✗	✗	✓	concave	S	+	+	U	−	−	−	+	−
DTLZ7	✗	✓	✓	✓	✗	✗	✓	mixed, disconnected	S	−	−	M	−	+	−	−	−
WFG1	✓	✓	✓	✓	✓	✓	✓	mixed	S	+	$+^{\dagger}$	U	−	−	−	+	$+^{\S}$
WFG2	✓	✓	✓	✓	✓	✓	✓	convex, disconnected	NS	−	$+^{\dagger}$	M	−	+	−	+	$+^{\S}$
WFG4	✓	✓	✓	✓	✓	✓	✓	concave	S	−	$+^{\dagger}$	M	−	−	−	+	$+^{\S}$
WFG5	✓	✓	✓	✓	✓	✓	✓	concave	S	−	$+^{\dagger}$	D	−	−	−	+	$+^{\S}$
WFG6	✓	✓	✓	✓	✓	✓	✓	concave	NS	−	$+^{\dagger}$	U	−	−	−	+	$+^{\S}$
WFG7	✓	✓	✓	✓	✓	✓	✓	concave	S	$+^{\dagger}$	$+^{\dagger}$	U	−	−	−	+	$+^{\S}$
WFG8	✓	✓	✓	✓	✓	✓	✓	concave	NS	$+^{\dagger}$	$+^{\dagger}$	U	−	−	−	+	$+^{\S}$
WFG9	✓	✓	✓	✓	✓	✓	✓	concave	NS	$+^{\dagger}$	$+^{\dagger}$	D	−	−	−	+	$+^{\S}$
LSS	✓	✗	✓	✓	✗	✗	✓	linear/convex/concave†	NS	−	+	U/M†	−	−	−	+	−
mirror-LSS	✓	✗	✓	✓	✗	✗	✓	linear/convex/concave†	NS	−	+	U/M†	−	+	−	−	−
SZDT1	✓	✓	✓	✓	✗	✗	✓	degenerate (for $M > 2$)	S	+	−	U	+	+	+	−	$+^{\S}$
SZDT2–SZDT5	✓	✓	✓	✓	✗	✗	✓	degenerate (for $M > 2$)	S	−	−	U	+	+	+	−	$+^{\S}$
SZDT6	✓	✓	✓	✓	✗	✗	✓	degenerate (for $M > 3$)	S	−	+	U	+	+	+	+	$+^{\S}$
SZDT7	✓	✓	✓	✓	✗	✗	✓	degenerate (for $M > 3$)	S	−	+	U	+	+	+	−	$+^{\S}$
LSMOP1	✓	✓	✓	✓	✗	✗	✓	linear	S	−	+	U	−	−	+	+	$+^{\S}$
LSMOP2–LSMOP4	✓	✓	✓	✓	✗	✗	✓	linear	NS	−	+	M	−	−	+	+	$+^{\S}$
LSMOP5	✓	✓	✓	✓	✗	✗	✓	concave	S	−	+	U	−	−	+	+	$+^{\S}$
LSMOP6–LSMOP8	✓	✓	✓	✓	✗	✗	✓	concave	NS	−	+	M	−	−	+	+	$+^{\S}$
LSMOP9	✓	✓	✓	✓	✗	✗	✓	mixed, disconnected	S	−	−	M	−	+	+	−	$+^{\S}$
MNI1	✓	✓	✓	✓	✗	✗	✓	convex	NS	−	−	U	+	+	+	−	−
MNI2	✓	✓	✓	✓	✗	✗	✓	concave	NS	−	−	M	+	+	+	−	−

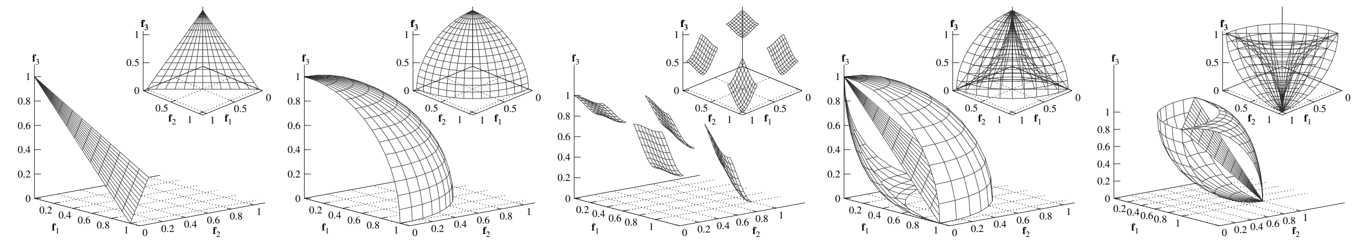
†Parameter dependent (see the specification of the concerned test problem)
§Possibility of reconfiguring the features by modifying the source code (its modification could be not straightforward)

the main reasons is that some real-world problems may present features that are not included in the currently available test suites. However, since the features of a real-world problem can not be known *a priori*, our recommendation is still to consider the use of MOEAs that offer the highest robustness (i.e., good performance in most test problems), as the first choice to deal with real-world problems.

V. Conclusions

Scalable test problems are regularly employed to evaluate and understand the working principles of many-objective evolutionary approaches. From a practical point of view, scalable test problems provide several advantages to assess the performance of MOEAs, including the fact that it is possible to know the Pareto optimal solutions of such problems. As discussed throughout the paper, properties of the existing test problems should simulate the characteristics observed in real-world problems. This paper has presented a review of the existing scalable test suites that have been developed in the headway of many-objective optimization. Additionally, a review of recommendations and features for the design of scalable test problems was presented. As an important part of our contribution, new features were added to the list of properties (see Table I) which were analyzed and discussed in Section II-B. We noticed that in spite of the wide variety of scalable test problems found in the specialized literature, most of them do not consider simultaneously a difficult PS and a difficult PF.

As discussed in Section II-B, this becomes a limitation for the development of new strategies able to solve problems with these features. On the other hand, the correlation of position and distance functions is also not considered by the majority of the currently available test problems, which becomes a limitation since distinguishing position and distance parameters in real-world problems turns out to be very difficult. Finally, flexibility should be considered, given that currently, only three existing scalable test suites consider this characteristic, limiting the configuration of new test problems based on the already formulated problems. These issues, in fact, mark some of the possible lines of research in the design of new multi/many-objective test problems.



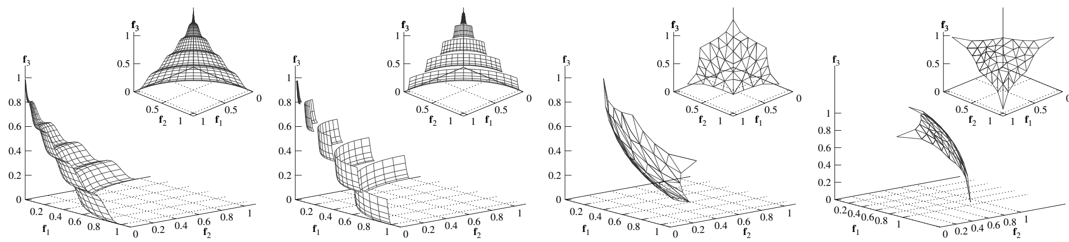
(a) PF for: DTLZ1 LSMOP1–LSMOP4

(b) PF for: DTLZ2–DTLZ4, WFG4–WFG9, LSMOP5–LSMOP8

(c) PF for: DTLZ7 and LSMOP9

(d) PF for: LSS with $\gamma = 1/2, 1, 2$, respectively

(e) PF for: Mirror LSS with $\gamma = 2, 1, 1/2$, respectively



(f) PF for WFG1

(g) PF for WFG2

(h) PF for MNI1

(i) PF for MNI2

Fig. 5: Summary of Pareto optimal fronts for the state-of-the-art scalable test suites

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