Supplementary Material of the Paper "Challenging Test Problems for Multi- and Many-Objective Optimization"

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Abstract

This supplementary material provides a complementary explanation of the ZCAT test suite. As described in the paper, the construction of the ZCAT test problems depends on the formulation of different functions which provide the difficulties and features of each problem. Throughout the appendices included in this supplementary material, we provide detailed information on the use of the ZCAT test suite and the construction of test instances. Additionally, we show more features of the proposed ZCAT suite not included in the paper. For a general overview of the ZCAT settings and the behavior of popular multi-objective evolutionary algorithms on different ZCAT test problems, the readers are referred to Section 5 of the paper.

Keywords: multi-objective optimization, scalable multi-objective test problems, evolutionary multi-objective algorithms.

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Appendix A. Three Examples of Test Problems Construction

The construction of the ZCAT test problems consists of a composition of different functions which establish the difficulties to each problem. In the paper, Section 3 introduces the complete set of functions employed by the proposed test suite. The construction of a ZCAT test problem consists of five steps that consider:

1. Problem configuration, i.e., state the `Level`, `Complicated_PS`, `Bias`, and `Imbalance` flags.
2. Definition of the search space.
3. Definition of the position functions α.
4. Definition of the PS topology (g function).
5. Definition of the distance functions β considering the `Level`, `Complicated_PS`, `Bias`, and `Imbalance` flags.

In the following, we detail the construction of three test problems to clarify the way in which the different components of the ZCAT test suite can be employed.

Appendix A.1. ZCAT3

1. Problem configuration:
   (a) `Level=1`
   (b) `Complicated_PS=True`
   (c) `Bias=True`
   (d) `Imbalance=False`
2. Definition of the search space:

$$\Omega = \prod_{i=1}^{n}[a_i, b_i] = \prod_{i=1}^{n}\left[-\frac{i}{2}, \frac{i}{2}\right]$$

   - Decision variable $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$;
   - Let $\mathbf{y} = \left(\frac{x_1 - a_1}{b_1 - a_1}, \ldots, \frac{x_n - a_n}{b_n - a_n}\right)$, then: $\mathbf{y}_I = (y_1, \ldots, y_m)$ and $\mathbf{y}_{II} = (y_{m+1}, \ldots, y_n)$;
   - $m = M - 1$;

3. Definition of the position functions:

   - α functions for ZCAT3:

$$
\begin{aligned}
\alpha_1(\mathbf{y}_I) &= 1^2 \times \frac{1}{M-1} \sum\nolimits_{j=1}^{M-1} y_j \\
\alpha_{i=2:M-1}(\mathbf{y}_I) &= i^2 \times \frac{1}{M-i+1} \left[\left(\sum\nolimits_{j=1}^{M-i} y_j \right) + \right. \\
&\quad \left. (1 - y_{M-i+1}) \right] \\
\alpha_M(\mathbf{y}_I) &= M^2 \times (1 - y_1)
\end{aligned}
$$

4. Definition of the PS topology:

   - g function for ZCAT3 (`Complicated_PS=True`):

$$
g(\mathbf{y}_I|m) = (g_{m+1}(\mathbf{y}_I|m), \ldots, g_n(\mathbf{y}_I|m))
$$

   where:

$$
g_j(\mathbf{y}_I|m) = \frac{1}{2m} \sum\nolimits_{i=1}^{m} y_i^2 \times \sin(4.5\pi y_i + \theta_j) + \frac{1}{2}
$$

   and $\theta_j = \frac{2\pi(j-(m+1))}{n}$, for $j = m+1, \ldots, n$.

5. Definition of the distance functions:

   - Bias problem (`Bias=True`):

     Let $z_{m+1:n} = (y_{m+1} - g_{m+1}, \ldots, y_n - g_n)$, then

$$
w_{m+1:n} = (|z_{m+1}|^{0.05}, \ldots, |z_n|^{0.05})
$$

   - β functions (`Level=1`):

$$
\begin{aligned}
\beta_i(z_{m+1:n}) &= i^2 \times Z_1(w_{m+1:n}|J_i) \\
&= i^2 \times \frac{10}{|J_i|} \sum_{j \in J_i} w_j^2
\end{aligned}
$$

   where $J_i = \{j| \mod(j-m-i, M) = 0\}$ for $i = 1, \ldots, M$ and $j =$

$m+1,\dots,n$.

With the above specifications, the M-objective test problem is formulated by

$$f_{i=1:M}(\mathbf{y}) = \alpha_i(\mathbf{y}_I) + \beta_i(\mathbf{y}_{II})$$

Therefore, the PF of ZCAT3 is

$$\left\{(\alpha_1(\mathbf{y}_I),\dots,\alpha_M(\mathbf{y}_I)) \in \mathbb{R}^M | \mathbf{y}_I \in [0,1]^m\right\}$$

and its PS in the normalized decision space is

$$\{\mathbf{y} \in [0,1]^n | \mathbf{y}_I \in [0,1]^m, \mathbf{y}_{II} = g(\mathbf{y}_I|m)\}$$

Appendix A.2. ZCAT7

1. Problem configuration:
   (a) `Level=1`
   (b) `Complicated_PS=False`
   (c) `Bias=False`
   (d) `Imbalance=True`
2. Definition of the search space:

$$\Omega = \prod_{i=1}^{n}[a_i,b_i] = \prod_{i=1}^{n}\left[-\frac{i}{2},\frac{i}{2}\right]$$

   - Decision variable $\mathbf{x} = (x_1,\dots,x_n) \in \Omega$;
   - Let $\mathbf{y} = \left(\frac{x_1-a_1}{b_1-a_1},\dots,\frac{x_n-a_n}{b_n-a_n}\right)$, then: $\mathbf{y}_I = (y_1,\dots,y_m)$ and $\mathbf{y}_{II} = (y_{m+1},\dots,y_n)$;
   - $m = M-1$
3. Definition of the position functions:

- α functions for ZCAT7:

$$\begin{aligned}\alpha_{i=1:M-1}(\mathbf{y}_I) &= i^2 \times y_i \\ \alpha_M(\mathbf{y}_I) &= M^2 \times \left[\frac{1}{2(M-1)\times(0.5)^5}\times \right. \\ &\left. \sum_{j=1}^{M-1}(0.5-y_j)^5 + \frac{1}{2}\right]\end{aligned}$$

4. Definition of the PS topology

- g function for ZCAT7 (`Complicated_PS=False`):

$$g(\mathbf{y}_I|m) = (g_{m+1}(\mathbf{y}_I|m), \ldots, g_n(\mathbf{y}_I|m))$$

where $g_j(\mathbf{y}_I|m) = 0.2210$, for $j = m+1, \ldots, n$.

5. Definition of the distance functions:

- Bias problem (`Bias=False`):

Let $z_{m+1:n} = (y_{m+1} - g_{m+1}, \ldots, y_n - g_n)$, then

$$w_{m+1:n} = z_{m+1:n}$$

- β functions (`Imbalance=True`):

$$\beta_i(z_{m+1:n}) = i^2 \times \begin{cases} Z_4(w_{m+1:n}|J_i), & \text{mod (i,2)} = 0 \\ Z_1(w_{m+1:n}|J_i), & \text{otherwise} \end{cases}$$

where $J_i = \{j| \mod(j-m-i, M) = 0\}$ for $i = 1, \ldots, M$ and $j = m+1, \ldots, n$.

$$Z_1(w_{m+1:n}|J_i) = \frac{10}{|J_i|}\sum_{j\in J_i} w_j^2$$

$$\begin{aligned}Z_4(w_{m+1:n}|J_i) = \frac{10}{2e-2}\Big[&e^{\max_{j\in J_i}\{|w_j|\}^{0.5}} \\ &-e^{\frac{1}{|J_i|}\sum_{j\in J_i}\frac{1}{2}[\cos((2K-1)\pi w_j)+1]} - 1 + e\Big]\end{aligned}$$

With the above specifications, the M-objective test problem is formulated by

$$f_{i=1:M}(\mathbf{y}) = \alpha_i(\mathbf{y}_I) + \beta_i(\mathbf{y}_{II})$$

Therefore, the PF of ZCAT7 is

$$\left\{(\alpha_1(\mathbf{y}_I), \ldots, \alpha_M(\mathbf{y}_I)) \in \mathbb{R}^M | \mathbf{y}_I \in [0,1]^m\right\}$$

and its PS in the normalized decision space is

$$\{\mathbf{y} \in [0,1]^n | \mathbf{y}_I \in [0,1]^m, \mathbf{y}_{II} = g(\mathbf{y}_I|m)\}$$

Appendix A.3. ZCAT20

1. Problem configuration:
   (a) `Level=2`
   (b) `Complicated_PS=False`
   (c) `Bias=False`
   (d) `Imbalance=False`
2. Definition of the search space:

$$\Omega = \prod_{i=1}^{n}[a_i, b_i] = \prod_{i=1}^{n}\left[-\frac{i}{2}, \frac{i}{2}\right]$$

   - Decision variable $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$;
   - Let $\mathbf{y} = \left(\frac{x_1 - a_1}{b_1 - a_1}, \ldots, \frac{x_n - a_n}{b_n - a_n}\right)$, then: $\mathbf{y}_I = (y_1, \ldots, y_m)$ and $\mathbf{y}_{II} = (y_{m+1}, \ldots, y_n)$;
   - $m = \begin{cases} 1, & y_1 \in [0.1, 0.4] \text{ or } y_1 \in [0.6, 0.9] \\ M-1, & \text{otherwise} \end{cases}$
3. Definition of the position functions:

- α functions for ZCAT20:

$$
\begin{aligned}
\alpha_{i=1:M-1}(\mathbf{y}_I) &= i^2 \times \begin{cases} y_1, & m = 1 \\ y_i, & \text{otherwise} \end{cases} \\
\alpha_M(\mathbf{y}_I) &= M^2 \times \begin{cases} \frac{(0.5-y_1)^5+0.5^5}{2\times(0.5)^5}, & m = 1 \\ \frac{\sum_{j=1}^{M-1}(0.5-y_j)^5}{2(M-1)\times(0.5)^5} + 0.5, & \text{otherwise} \end{cases}
\end{aligned}
$$

4. Definition of the PS topology:

   - g function for ZCAT20 (`Complicated_PS=False`):

$$g(\mathbf{y}_I|m) = (g_{m+1}(\mathbf{y}_I|m), \dots, g_n(\mathbf{y}_I|m))$$

   where $g_j(\mathbf{y}_I|m) = 0.2210$, for $j = m+1, \dots, n$.

5. Definition of the distance functions:

   - Bias problem (`Bias=False`):

   Let $z_{m+1:n} = (y_{m+1} - g_{m+1}, \dots, y_n - g_n)$, then

$$w_{m+1:n} = z_{m+1:n}$$

   - β functions (`Level=2`):

$$
\begin{aligned}
\beta_i(z_{m+1:n}) &= i^2 \times Z_2(w_{m+1:n}|J_i) \\
&= i^2 \times 10 \times \max_{j \in J_i}\{|x_j|\}
\end{aligned}
$$

   where $J_i = \{j| \mod (j-m-i, M) = 0\}$ for $i = 1, \dots, M$ and $j = m+1, \dots, n$

With the above specifications, the M-objective test problem is formulated by

$$f_{i=1:M}(\mathbf{y}) = \alpha_i(\mathbf{y}_I) + \beta_i(\mathbf{y}_{II})$$

Therefore, the PF of ZCAT20 is

$$\left\{(\alpha_1(\mathbf{y}_I),\ldots,\alpha_M(\mathbf{y}_I)) \in \mathbb{R}^M | y_1 \in \Omega_1 \cup \Omega_2\right\} \cup$$
$$\left\{(\alpha_1(\mathbf{y}_I),\ldots,\alpha_M(\mathbf{y}_I)) \in \mathbb{R}^M | \mathbf{y}_I \in \Omega_3 \cup \Omega_4 \cup \Omega_5\right\}$$

and its PS in the normalized decision space is

$$\{\mathbf{y} \in [0,1]^n | y_1 \in \Omega_1 \cup \Omega_2, \mathbf{y}_{II} = g(y_1|1)\} \cup$$
$$\{\mathbf{y} \in [0,1]^n | \mathbf{y}_I \in \Omega_3 \cup \Omega_4 \cup \Omega_5, \mathbf{y}_{II} = g(\mathbf{y}_I|M-1)\}$$

such that $\Omega_1 = [0.1, 0.4]$, $\Omega_2 = [0.6, 0.9]$, $\Omega_3 = [0, 0.1) \times [0,1]^{m-1}$, $\Omega_4 = (0.4, 0.6) \times [0,1]^{m-1}$, and $\Omega_5 = (0.9, 1] \times [0,1]^{m-1}$.

The source code of the proposed test suite can be obtained at `https://github.com/evo-mx/ZCAT`.

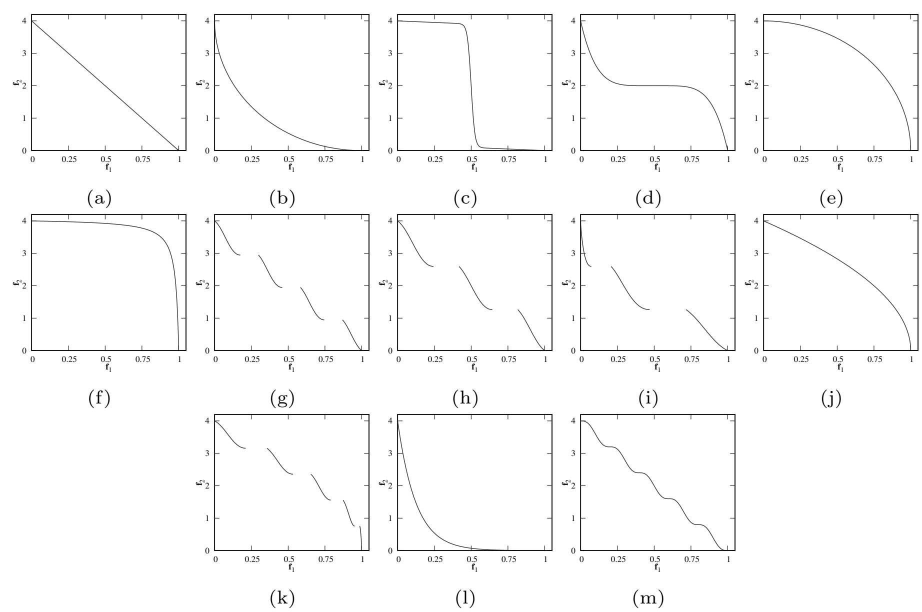
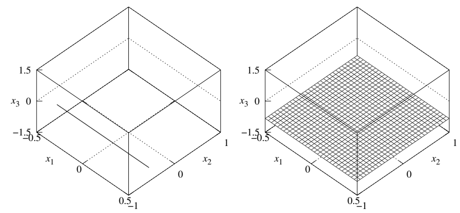


Figure A.1: Pareto fronts of ZCAT test problems using $M = 2$ for: (a) ZCAT1, ZCAT3, ZCAT4, and ZCAT5, (b) ZCAT2, (c) ZCAT6, (d) ZCAT7, ZCAT18, and ZCAT20, (e) ZCAT8 and ZCAT9, (f) ZCAT10, (g) ZCAT11, (h) ZCAT12 and ZCAT15, (i) ZCAT13, (j) ZCAT14, (k) ZCAT16, (l) ZCAT17, and (m) ZCAT19.



(a) (b)

Figure C.2: Pareto sets for the ZCAT test problems using `Complicated_PS=False`. (a) PS for the two-objective problems. (b) PS for the three-objetive problems.

Appendix B. Pareto Optimal Fronts for the two-objective ZCAT Formulation

The proposed test suite is characterized by the formulation of peculiar Pareto fronts. In Figure 2 from the paper, we show the Pareto fronts for the three-objective formulation of the proposed test suite. In this appendix, we include in Figure A.1, the corresponding Pareto optimal fronts for the two-objective formulation of the proposed problems. As can be seen, the ZCAT test problems provide a vast number of Pareto shapes even in their two-objective formulation.

Appendix C. Pareto Optimal Sets for the Two-objective ZCAT Test Problems

The ZCAT test problems provide different Pareto set topologies that can be switched *on* and *off* according to the user's preferences. Such behavior in a test problem can be controlled by means of the flag `Complicated_PS`. In the particular case of `Complicated_PS=False`, the PS topology of any test problem shall be piecewise linear. In Figure C.2, we can see the Pareto set for the ZCAT test problems using `Complicated_PS=False` for two and three objectives. As a complement to the Pareto sets projections illustrated for three objectives in Figure 3 from the paper, in this supplementary material, Figure C.3 shows the projections of the complicated Pareto sets for the two-objective ZCAT test problems.

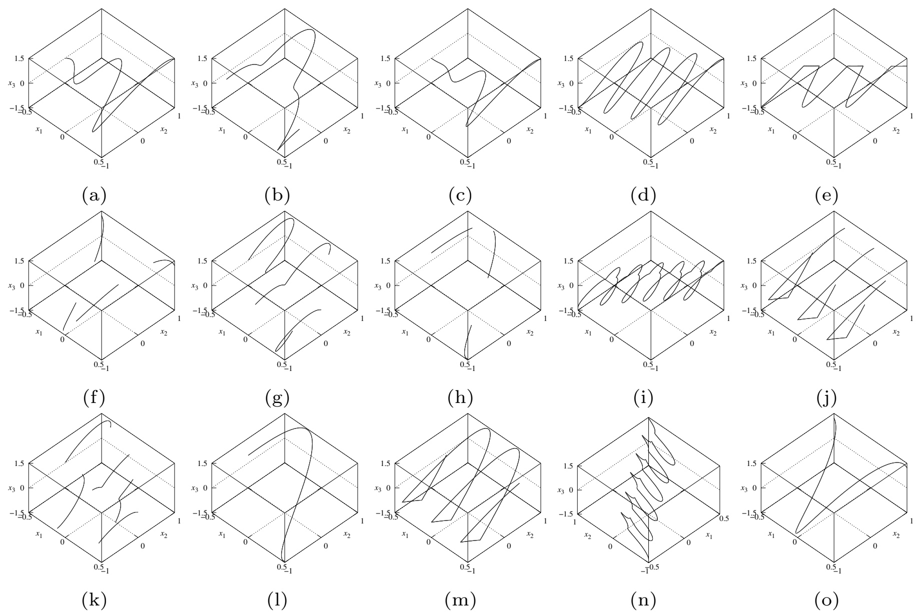


Figure C.3: Pareto sets of ZCAT test problems using $M = 2$ for: (a) ZCAT1 and ZCAT6, (b) ZCAT2 and ZCAT7, (c) ZCAT3 and ZCAT8, (d) ZCAT4 and ZCAT9, (e) ZCAT5 and ZCAT10, (f) ZCAT11, (g) ZCAT12, (h) ZCAT13, (i) ZCAT14, (j) ZCAT15, (k) ZCAT16, (l) ZCAT17, (m) ZCAT18, (n) ZCAT19, and (o) ZCAT20.

Appendix D. Difficulty Levels of the Test Problems: Z Functions

In the ZCAT test suite, the Z functions state the difficulty of approximating Pareto solutions for a multi-objective solver. The user can configure this function according to his/her preferences. Formulations of Z functions are presented in Table 4 from the paper. In the formulation of a ZCAT test problem, the parameter `Level` switches between different Z functions according to the description provided in Section 4.3.1 from the paper. To illustrate each Z function, Figure D.4 shows each unidimensional Z function and its corresponding contour lines for the two-dimensional Z function.

Appendix E. Features of the ZCAT Test Problems

In order to facilitate the reading of properties in the proposed test problems, Table E.1 summarizes the recommendations and features adhered to for each

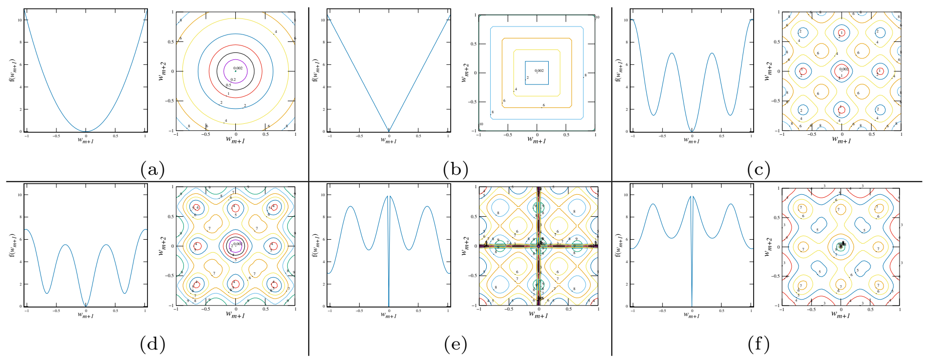


Figure D.4: Unidimensional Z function (*left*) and its corresponding contour lines for the two-dimensional Z function (*right*) of: (a) Z_1. (b) Z_2. (c) Z_3 using $K = 2$. (d) Z_4 using $K = 2$. (e) Z_5. (f) Z_6.

test instance. We refer to the set of properties discussed by Huband et al. [1] and Zapotecas et al. [2]. Additionally, we include in the list of features the imbalance of objectives discussed by Liu et al. [3]. In Table E.1, symbols "✓" and "✗" indicate whether a recommendation is adhered to, while "+" and "–" indicate the presence or absence of a given feature. In the case of features, the following abbreviations are included: "S" for separable, "NS" for non-separable, "U" for unimodal, "M" for multi-modal, and "D" for deceptive.

References

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[2] S. Zapotecas-Martínez, C. A. Coello Coello, H. E. Aguirre, K. Tanaka, A Review of Features and Limitations of Existing Scalable Multiobjective Test Suites, IEEE Transactions on Evolutionary Computation 23 (1) (2019) 130–142.

[3] H. Liu, L. Chen, K. Deb, E. D. Goodman, Investigating the Effect of Imbalance Between Convergence and Diversity in Evolutionary Multiobjective

Table E.1: Properties of the proposed ZCAT test suite

MOP	R1: No Extremal	R2: No Medial	R3: # Parameters	R4: # Objectives	R5: Diss. Domains	R6: Diss. Tradeoffs	R7: Optima Known	F1: PF Geometry	F2: Separability	F3: Bias	F4: Many-to-one	F5: Modality	F6: Difficult PS	F7: Difficult PF	F8: Corr. Pos.–Dist. Func.	F9: Single Opt. Sol.	F10: Easy Conf. of Features	F11: Imbalance
ZCAT1	✓	✓	✓	✓	✓	✓	✓	concave	S/NS†	+/−†	+	U/M/D†	+/−†	−	+	+	+	+/−†
ZCAT2	✓	✓	✓	✓	✓	✓	✓	convex	S/NS†	+/−†	+	U/M/D†	+/−†	−	+	+	+	+/−†
ZCAT3	✓	✓	✓	✓	✓	✓	✓	concave	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT4	✓	✓	✓	✓	✓	✓	✓	linear	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT5	✓	✓	✓	✓	✓	✓	✓	convex	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT6	✓	✓	✓	✓	✓	✓	✓	mixed	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT7	✓	✓	✓	✓	✓	✓	✓	mixed	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT8	✓	✓	✓	✓	✓	✓	✓	linear	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT9	✓	✓	✓	✓	✓	✓	✓	concave	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT10	✓	✓	✓	✓	✓	✓	✓	concave	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT11	✓	✓	✓	✓	✓	✓	✓	mixed, disconnected	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT12	✓	✓	✓	✓	✓	✓	✓	mixed, disconnected	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT13	✓	✓	✓	✓	✓	✓	✓	mixed, disconnected	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT14	✓	✓	✓	✓	✓	✓	✓	degenerate§	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT15	✓	✓	✓	✓	✓	✓	✓	disconnected, degenerate§	S/NS†	+/−†	−	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT16	✓	✓	✓	✓	✓	✓	✓	disconnected, degenerate§	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT17	✓	✓	✓	✓	✓	✓	✓	convex, degenerate§	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT18	✓	✓	✓	✓	✓	✓	✓	mixed, degenerate§	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT19	✓	✓	✓	✓	✓	✓	✓	mixed, degenerate§	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†
ZCAT20	✓	✓	✓	✓	✓	✓	✓	mixed, degenerate§	S/NS†	+/−†	+	U/M/D†	+/−†	+	+	−	+	+/−†

†Parameter dependent (see the specification of the concerned test problem)
§Degenerate for $M > 2$

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