# The Central Conic Sections Revisited

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If a central conic section is given by the general quadratic equation

$$f(x,y)=ax^2+2nxy+by^2+2hx+2ky+c=0, \tag{1}$$

how do we find its center, the equations of its principal axes and the radii along these axes? Usually by translating and rotating the coordinate axes. Here, however, we will use a combination of techniques that will shed more light on this topic.

First, we will get the center by using the fact that it is the midpoint of any diameter passing through it. Assume that the center of the conic is $(x_0, y_0)$. Then the equation of the diameter parallel to the $x$-axis is $y=y_0$, and if we substitute that in (1), we get $ax^2+(2ny_0+2h)x+by_0^2+2ky_0+c=0$. If the roots of this equation are $x_1$ and $x_2$, then the $x$ value of the center, $x_0$, is given by

$$x_0=\frac{1}{2}(x_1+x_2)=-\frac{1}{2}\left(\frac{\text{coefficient of } x}{\text{coefficient of } x^2}\right).$$

Thus,

$$x_0=-\frac{1}{2}\left(\frac{2ny_0+2h}{a}\right), \quad \text{that is,}$$

$$ax_0+ny_0+h=0. \tag{2}$$

Similarly if we use the diameter $x=x_0$, we get

$$nx_0+by_0+k=0. \tag{3}$$

Equations (2) and (3) will determine the center $(x_0, y_0)$ of the conic given by (1), and it is easy to see that these equations are equivalent to $\partial f/\partial x=0$ and $\partial f/\partial y=0$.

One may ask: What's so special about the diameters represented by (2) and (3)? You may notice that if $(x_0, y_0)$ was a midpoint of any chord parallel to the $x$-axis, it also would have satisfied the equation $ax+ny+h=0$. Hence this equation represents the diameter bisecting all chords parallel to the $x$-axis. The endpoints of the diameter, being the limiting cases of the chords, are the points where the tangents parallel to the $x$-axis touch the conic, and that explains why these points satisfy the condition $\partial f/\partial x=0$. By the same token $nx+by+k=0$ represents the diameter bisecting the chords parallel to the $y$-axis. FIGURE 1 depicts such two diameters in a case of an ellipse.

Solving equations (2) and (3) we get $x_0=H/C$ and $y_0=K/C$, where $H,K,C$ are the cofactors of $h,k,c$, respectively, in the determinant

$$\Delta=\begin{vmatrix} a & n & h \\ n & b & k \\ h & k & c \end{vmatrix}.$$

Of course, if $C=0$, this indicates that we are dealing with noncentral conic section, i.e. either a parabola or a pair of parallel lines.

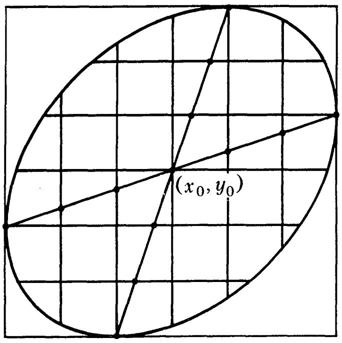



FIGURE 1

If we replace $x$ and $y$ of (1) by $x+x_0$ and $y+y_0$ to translate the origin to the center of the conic we get

$$ax^2+2nxy+by^2+x_0(ax_0+ny_0+h)+y_0(nx_0+by_0+k)+hx_0+ky_0+c=0.$$

Making use of (2), (3), $x_0=H/C$, $y_0=K/C$, and $hH+kK+cC=\Delta$ the equation of the conic, with its center now at the origin, becomes

$$ax^2+2nxy+by^2+\frac{\Delta}{C}=0$$

or

$$ax^2+2nxy+by^2+\frac{\Delta}{ab-n^2}=0.$$

If $\Delta=0$ then the equation represents a pair of straight lines. Let us assume that $\Delta\neq 0$ and proceed to find the extreme radii of the conic in the nontrivial case when $n\neq 0$. To that end we use polar coordinates and in this case the previous equation takes the form

$$r^2(a\cos^2\theta+2n\sin\theta\cos\theta+b\sin^2\theta)+\frac{\Delta}{ab-n^2}=0.$$

Since $\Delta/(ab-n^2)$ is constant then $r$ is maximum or minimum if $(a\cos^2\theta+2n\sin\theta\cos\theta+b\sin^2\theta)$ is minimum or maximum, respectively, and this occurs when its derivative is equal to zero. But

$$-2a\cos\theta\sin\theta+2n\cos^2\theta-2n\sin^2\theta+2b\sin\theta\cos\theta=0$$

is equivalent to

$$n\tan^2\theta+(a-b)\tan\theta-n=0. \tag{4}$$

If we let $\tan\theta=m$, then

$$nm^2+(a-b)m-n=0 \tag{5}$$

will produce the slopes of the principal axes of the conic section. Now using the substitution $m=(\lambda-a)/n$ (or $m=n/(\lambda-b)$) equation (5) takes the form

$$(\lambda-a)(\lambda-b)-n^2=0 \tag{6}$$

or

$$\begin{vmatrix}\lambda-a & n\\ n & \lambda-b\end{vmatrix}=0.$$

The roots $\lambda_1$ and $\lambda_2$ of equation (6) are the eigenvalues of the matrix

$$A=\begin{bmatrix} a & n \\ n & b \end{bmatrix}.$$

Hence $A$ is similar to the diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

through some orthogonal matrix (isometric) transformation [1]. This implies that the equation $ax^2+2nxy+by^2+\Delta/(ab-n^2)=0$, which can be written as

$$[x \quad y]\begin{bmatrix} a & n \\ n & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\frac{\Delta}{ab-n^2}=0, \quad \text{would take the form}$$

$$[x \quad y]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\frac{\Delta}{ab-n^2}=0.$$

Thus, $\lambda_1x^2+\lambda_2y^2+\Delta/(ab-n^2)=0$. But (6) implies that $\lambda_1\lambda_2=ab-n^2$. Hence,

$$\lambda_1x^2+\lambda_2y^2+\frac{\Delta}{\lambda_1\lambda_2}=0 \tag{7}$$

is the equation of the conic section referred to its principal axes. And this is exactly the equation we get if we rotate the coordinate axes through an angle $\theta$ of (4). From (7) we conclude that the squares of the radii along the principal axes are

$$-\frac{\Delta}{\lambda_1^2\lambda_2} \quad \text{and} \quad -\frac{\Delta}{\lambda_1\lambda_2^2}$$

and the equations of the corresponding principal axes are

$$y-y_0=m_1(x-x_0) \quad \text{and} \quad y-y_0=m_2(x-x_0),$$

where

$$m_1=\frac{\lambda_1-a}{n} \quad \text{and} \quad m_2=\frac{\lambda_2-a}{n},$$

and $(x_0, y_0)$ is the center of the conic section.

*Example*. Consider the equation $17x^2-12xy+8y^2+12x-16y-12=0$. Here

$$\Delta=\begin{vmatrix} 17 & -6 & 6 \\ -6 & 8 & -8 \\ 6 & -8 & -12 \end{vmatrix}=-2000,$$

and

$$\begin{vmatrix} \lambda-17 & -6 \\ -6 & \lambda-8 \end{vmatrix}=0 \Rightarrow \lambda_1=5 \quad \text{and} \quad \lambda_2=20.$$

The equation of the ellipse referred to its principal axes is

$$5x^2+20y^2-\frac{2000}{5(20)}=0, \quad \text{that is,} \ \frac{x^2}{4}+\frac{y^2}{1}=1.$$

Since the center satisfies $17x-6y+6=0$ and $-6x+8y-8=0$, the center is $(0,1)$.

The slopes of the principal axes are

$$m_1 = \frac{\lambda_1 - a}{n} = \frac{5-17}{-6} = 2 \quad \text{and} \quad m_2 = \frac{\lambda_2 - a}{n} = \frac{20-17}{-6} = -\frac{1}{2}.$$

The equations of the principal axes are

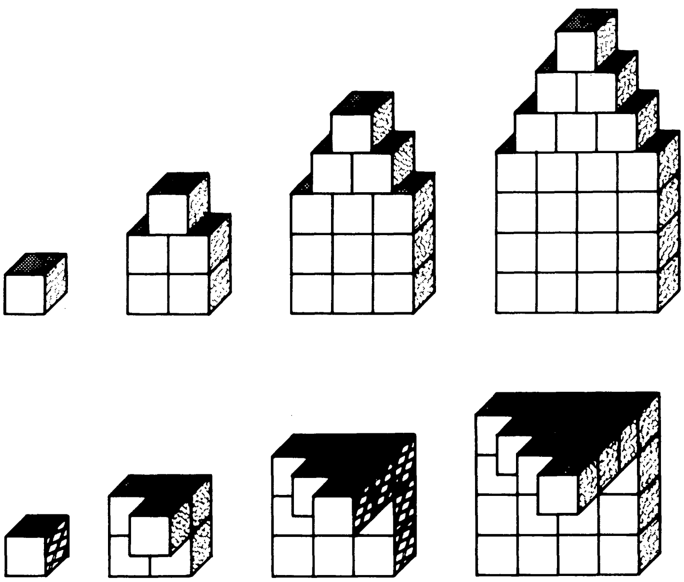
$$y - 1 = 2(x-0) \quad \text{and} \quad y - 1 = -\tfrac{1}{2}(x-0), \quad \text{or equivalently,}$$
$$y = 2x+1 \quad \text{and} \quad y = -\tfrac{1}{2}x + 1.$$

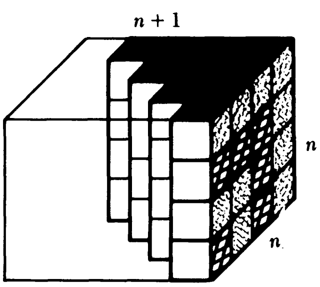
## REFERENCE

1. H. Anton, *Elementary Linear Algebra*, 6th edition, John Wiley & Sons, Inc., New York, 1991, p. 418.

# Proof without Words: Sum of Pentagonal Numbers

$$\frac{1\cdot 2}{2} + \frac{2\cdot 5}{2} + \frac{3\cdot 8}{2} + \cdots + \frac{n(3n-1)}{2} = \frac{n^2(n+1)}{2}$$



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